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SPECIALIZATIONS OF COFINITE SUBALGEBRAS OF A POLYNOMIAL RING

Dedicated to Professor Hirosi Nagao on his sixtieth birthday

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1. Introduction. Let K be a field of characteristic zero and let $R_K := K[x, y]$ be a polynomial ring in two variables over K. A normal K-subalgebra A of R_K is said to be *cofinite* if R_K is a finite A-module with the canonical A-module structure. In the case where K is an algebraically closed field, we know the following results:

(1) If A is regular, A is then a polynomial ring in two variables over K; see [3] and [8].

(2) If A is singular, then there exist a polynomial subalgebra R'_K and a finite group G of linear K-automorphisms of R'_K such that $A=(R'_K)^G$ and G is a small subgroup of $GL(2, K)$; see [4] and [10].

In the present article, we shall show that the structures of normal cofinite subalgebras A of R_K are invariant under specializations, provided the quotient field extension $Q(R_K)/Q(A)$ is a quasi-Galois extension; see Definition 2.2. Our problem is formulated as follows: Let $\mathfrak{O}=k[[t]]$ be a formal power series ring in one variable over an algebraically closed field of characteristic zero and let $R:=\mathfrak{O}[x, y]$ be a polynomial ring in two variables over $\mathfrak{O}$. Let A be an $\mathfrak{O}$-subalgebra of R. We say that A is *cofinite* if R is a finite A-module and that A is *geometrically $\mathfrak{O}$-normal* if $A_K:=A\otimes_{\mathfrak{O}} K$ and $A_k:=A/tA$ are normal domains, where K is the quotient field $Q(\mathfrak{O})$ of $\mathfrak{O}$. If A is a cofinite, geometrically $\mathfrak{O}$-normal subalgebra of R, then A_K and A_k are cofinite normal subalgebras in R_K and R_k, respectively. Let $\bar{K}$ be an algebraic closure of K. We ask whether or not certain properties of a cofinite normal subalgebra $A_{\bar{K}}$ of $R_{\bar{K}}$ are inherited by the cofinite normal subalgebra A_k of R_k. We pose the following

Conjecture 1. *Let $\mathfrak{O}$ and R be as above, and let A be a cofinite, geometrically $\mathfrak{O}$-normal subalgebra of R. Then there exist a cofinite $\mathfrak{O}$-subalgebra R' of R and a finite group G of $\mathfrak{O}$-automorphisms of R' such that*:

(i) *R' is a polynomial ring in two variables over $\mathfrak{O}$ and contains A as an $\mathfrak{O}$-subalgebra*;

(ii) *A is the G-invariant subalgebra $(R')^G$ of R'.*

Our result, though partial, is the following:

Main Theorem. *Let $\mathfrak{O}$, R, K and $\bar{K}$ be as above. Let A be a normal, cofinite $\mathfrak{O}$-subalgebra of R. Suppose that $Q(R)$ is a quasi-Galois extension of $Q(A)$ over K. Let G be the Galois group of the extension $Q(R)\underset{K}{\otimes}\bar{K}/Q(A)\underset{K}{\otimes}\bar{K}$. Then the following assertions hold true:*

(1) *G acts effectively on R, and $A=R^G$. Namely, R is a Galois extension of A with group G in the sense of* [11].

(2) *A is geometrically $\mathfrak{O}$-normal.*

(3) *R_k is a Galois extension of A_k with group G.*

(4) *If $A_{\bar{K}}$ is a polynomial ring in two variables over $\bar{K}$, so is A_k over k.*

We shall see later that Conjecture 1 is reduced to the following:

Conjecture 2. *Let $\mathfrak{O}$ and R be as above. Let A be a normal, cofinite $\mathfrak{O}$-subalgebra of R such that A_K is a polynomial ring over K. Then A_k is a polynomial ring over k; hence A is a polynomial ring over $\mathfrak{O}$* by virtue of a result of Sathaye [14]; see also Kambayashi [6].

Concerning the second conjecture, we can show that Spec A_k has at most one singular point which has necessarily cyclic quotient singularity, provided A is geometrically $\mathfrak{O}$-normal; see Proposition 4.1 below.

2. Representability of a group functor

Let K be a field of characteristic zero, let L be a regular extension of K and let L' be a finite algebraic extension of L. Suppose that L' is a regular extension of K.

Let $\mathcal{C}$ be the category of finite, reduced K-algebras. We define a group functor $\mathbf{Aut}_K(L'/L)$ on the dual category $\mathcal{C}^\circ$ by

$$\mathrm{Spec}(S)\in\mathcal{C}^\circ \mapsto \mathbf{Aut}_K(L'/L)\,(S):=\mathrm{Aut}(L'\underset{K}{\otimes}S/L\underset{K}{\otimes}S)\,,$$

where $\mathrm{Aut}(L'\underset{K}{\otimes}S/L\underset{K}{\otimes}S)$ denotes the group of all $L\underset{K}{\otimes}S$-algebra automorphisms of $L'\underset{K}{\otimes}S$, which is a finite group. We then have the following:

Lemma 2.1. *The functor $\mathbf{Aut}_K(L'/L)$ is representable by a finite group scheme over K.*

Proof. Let X be a projective normal variety defined over K such that $L=K(X)$ and let X' be the normalization of X in L'. Let ν: $X'\to X$ be the normalization morphism. We define a group functor $\mathbf{Aut}_K(X'/X)$ on the category of K-schemes by

$$T\in(\mathrm{Sch}/K)\mapsto \mathbf{Aut}_K(X'/X)(T):=\mathrm{Aut}(X'\underset{K}{\times}T/X\underset{K}{\times}T),$$

where $\mathrm{Aut}(X'\underset{K}{\times}T/X\underset{K}{\times}T)$ denotes the group of all $X\underset{K}{\times}T$-automorphisms of $X'\underset{K}{\times}T$. We claim that the restriction of $\mathbf{Aut}_K(X'/X)$ on the full subcategory $\mathcal{C}^\circ$ of (Sch/K) coincides with the group functor $\mathbf{Aut}_K(L'/L)^\circ$ which is the opposite of $\mathbf{Aut}_K(X'/X)$, i.e., the order of multiplication is reversed.

In fact, let S be a finite, reduced K-algebra. Then S is a direct product $S=\prod_{i=1}^{n}K_i$, where K_i is a finite algebraic extension of K. We have apparently

$$\mathbf{Aut}_K(X'/X)(S)=\prod_{i=1}^{n}\mathrm{Aut}(X'\underset{K}{\otimes}K_i/X\underset{K}{\otimes}K_i),\ \text{and}$$

$$\mathbf{Aut}_K(L'/L)(S)=\prod_{i=1}^{n}\mathrm{Aut}(L'\underset{K}{\otimes}K_i/L\underset{K}{\otimes}K_i).$$

Hence we may (and shall) assume that S is a field. Note that $X\underset{K}{\otimes}S$ is a normal variety and $X'\underset{K}{\otimes}S$ is the normalization of $X\underset{K}{\otimes}S$ in the field $L'\underset{K}{\otimes}S$. Moreover, it is easy to show that the canonical homomorphism

$$\mathrm{Aut}(X'\underset{K}{\otimes}S/X\underset{K}{\otimes}S)^\circ\to\mathrm{Aut}(L'\underset{K}{\otimes}S/L\underset{K}{\otimes}S)$$

is an isomorphism.

Now, applying the representability criterion of Grothendieck [2; 221–19], $\mathbf{Aut}_K(X'/X)$ is representable by a K-group scheme, say $Aut_K(X'/X)$, which is locally of finite type over K[1]. However, since $|\mathbf{Aut}_K(X'/X)(K')|\leqq[L':L]$ for any finite algebraic extension K' of K, $Aut_K(X'/X)$ is a finite K-group scheme. Moreover, since $\mathrm{char}(K)=0$, $Aut_K(X'/X)$ is reduced by a theorem of Cartier (cf. [12]). Therefore we know that $\mathbf{Aut}_K(L'/L)$ is representable by a finite K-group scheme $Aut_K(X'/X)^\circ$. Q.E.D.

We denote $Aut_K(X'/X)^\circ$ by $Aut_K(L'/L)$ or simply by $\mathcal{G}$. Write $\mathcal{G}=\mathrm{Spec}(\mathcal{A})$. Then the identity morphism $id_{\mathcal{G}}$: $\mathcal{G}\to\mathcal{G}$ corresponds to an L-homomorphism

1) Define a functor $\mathbf{Hom}_K(X', X)$ on (Sch/K) by

$$T\in(\mathrm{Sch}/K)\mapsto\mathbf{Hom}_K(X',X)(T)=\mathrm{Hom}_T(X'_T,X_T)$$

(cf. [2], [16]). Then there exists the canonical morphism of functors

$$\phi:\mathbf{Aut}_K(X')\to\mathbf{Hom}_K(X',X)$$

such that, for $T\in(\mathrm{Sch}/K)$ and $\alpha\in\mathrm{Aut}_T(X'_T)$, $\phi_T(\alpha)=\nu_T\cdot\alpha$. Note that both $\mathbf{Aut}_K(X')$ and $\mathbf{Hom}_K(X',X)$ are representable by K-schemes locally of finite type, and hence ϕ is representable by a morphism of K-schemes,

$$f: Aut_K(X')\to Hom_K(X',X)$$

(cf. [15] and [16; Th. 3]). The K-scheme $Hom_K(X',X)$ has a K-rational point ν: $X'\to X$. It is now apparent that $\mathbf{Aut}_K(X'/X)$ is representable by $f^{-1}(\nu)$, which is a K-group scheme locally of finite type.

$$\Delta\colon L' \to L' \underset{K}{\otimes} \mathcal{A},$$

and for any $S \in \mathcal{C}$ and any element $\alpha \in \mathrm{Hom}_{K\text{-alg}}(\mathcal{A}, S) = Aut_K(L'/L)(S)$, the action of α on $L' \underset{K}{\otimes} S$ is given by $(id_{L'} \otimes \alpha)\Delta\colon L' \to L' \underset{K}{\otimes} S$. It is then easy to see that the homomorphism Δ defines an action of $\mathcal{G}$ on Spec L'

$$\sigma\colon \mathcal{G} \times \mathrm{Spec}\, L' \to \mathrm{Spec}\, L'$$

which is a Spec L-morphism. We denote by $(L')^{\mathcal{G}}$ the set

$$(L')^{\mathcal{G}} = \{z \in L' \mid \Delta(z) = z \otimes 1\},$$

which is a subfield of L' containing L.

DEFINITION 2.2. We say that L'/L is a quasi-Galois extension over K if $(L')^{\mathcal{G}} = L$.

Let K' be a finite algebraic field extension of K. Then it is straightforward to show that:

(1) $Aut_{K'}(L' \underset{K}{\otimes} K' / L \underset{K}{\otimes} K') \simeq Aut_K(L'/L) \underset{K}{\otimes} K'$.

(2) The action of $Aut_{K'}(L' \underset{K}{\otimes} K' / L \underset{K}{\otimes} K')$ on Spec $(L' \underset{K}{\otimes} K')$ is given by

$$\Delta \underset{K}{\otimes} K'\colon L' \underset{K}{\otimes} K' \to (L' \underset{K}{\otimes} K') \underset{K'}{\otimes} (\mathcal{A} \underset{K}{\otimes} K'),$$

and we have $(L' \underset{K}{\otimes} K')^{\mathcal{G}'} = (L')^{\mathcal{G}} \underset{K}{\otimes} K'$, where $\mathcal{G}' = \mathcal{G} \underset{K}{\otimes} K'$.

Lemma 2.3. *The following conditions are equivalent*:

(1) *L'/L is a quasi-Galois extension over K.*

(2) *For any finite algebraic field extension K' of K, $L' \underset{K}{\otimes} K' / L \underset{K}{\otimes} K'$ is a quasi-Galois extension over K'.*

(3) *$L' \underset{K}{\otimes} \bar{K} / L \underset{K}{\otimes} \bar{K}$ is a Galois extension, where $\bar{K}$ is an algebraic closure of K.*

Proof. The equivalence of (1) and (2) is clear in view of the preceding observations. (2)$\Rightarrow$(3): There exists a finite algebraic extension K'/K such that $\mathcal{G}' := \mathcal{G} \underset{K}{\otimes} K'$ is a constant K'-group scheme with group $G := \mathcal{G}(K')$. Since $G = \mathrm{Aut}(L' \underset{K}{\otimes} K' / L \underset{K}{\otimes} K')$ and $(L' \underset{K}{\otimes} K')^G = L \underset{K}{\otimes} K'$, $L' \underset{K}{\otimes} K' / L \underset{K}{\otimes} K'$ is a Galois extension with group G. Hence $L' \underset{K}{\otimes} K'' / L \underset{K}{\otimes} K''$ is a Galois extension with group G for any field extension K'' of K with $K'' \supseteq K'$. (3)$\Rightarrow$(1): The condition (3) implies that $L' \underset{K}{\otimes} K' / L \underset{K}{\otimes} K'$ is a Galois extension for some finite algebraic extension K'/K. Since $L \underset{K}{\otimes} K' = (L')^{\mathcal{G}} \underset{K}{\otimes} K'$ as noted above, we have $(L')^{\mathcal{G}} = L$. Namely, L'/L is a quasi-Galois extension over K. Q.E.D.

Corollary 2.4. *L'/L is a quasi-Galois extension over K if and only if* $|\mathcal{G}|$

(:= the rank of K-module $\mathcal{A}$) *is equal to* $[L': L]$.

A quasi-Galois extension is not necessarily a Galois extension as shown by the following trivial

EXAMPLE. Let K be the rational number field $\boldsymbol{Q}$, let $L=K(x)$ with indeterminate x and let $L'=K(y)$, where $y^n=x$ and $n>2$. Then $\mathcal{G}=Aut_K(L'/L) \simeq \text{Spec } \boldsymbol{Q}[\xi]/(\xi^n-1)$ and $\mathcal{G}(\boldsymbol{Q}) \neq \boldsymbol{Z}/n\boldsymbol{Z}$. Hence L'/L is a quasi-Galois extension, but not a Galois extension. In fact, let K' be the extension of $\boldsymbol{Q}$ with all n-th roots of unity adjoined. Then $\mathcal{G}(K') \simeq Z/nZ$ and $L' \otimes_K K'/L \otimes_K K'$ is a Galois extension.

We don't know which conditions on K assure that a quasi-Galois extension L'/L over K is a Galois extension. In the next section, we shall, however, show that this is the case if K is the quotient field of a formal power series ring $k[[t]]$ in one variable over an algebraically closed field k of characteristic zero. We use only the property that $k[[t]]$ is strictly henselian.

3. Constancy of the K-group scheme $\mathrm{Aut}_K(L'/L)$

Let $(\mathfrak{O}, t\mathfrak{O})$ be a discrete valuation ring of equicharacteristic zero, let $K=Q(\mathfrak{O})$ be the quotient field and let k be the residue field. First of all, we shall prove:

Lemma 3.1. *Let A be a finitely generated, normal $\mathfrak{O}$-domain and let $L=Q(A)$. Let L' be a finite Galois extension of L with group G and let A' be the integral closure of A in L'. Then the following assertions hold true*:

(1) *G acts effectively on A'_k, and the canonical injection $A_k \hookrightarrow A'_k$ induces an isomorphism $A_k \simeq (A'_k)^G$.*

(2) *Suppose A'_k is an integral domain. Then $Q(A'_k)$ is a Galois extension of $Q(A_k)$ with group G.*

Proof. Our proof consists of several steps.

(I) Note that A' is a finite A-module (cf. Matsumura [7]). Furthermore, A_k is a subring of A'_k. In fact, we have only to show that $A \cap tA'=tA$. Suppose $a=ta'$ with $a \in A$ and $a' \in A'$. Then $a' \in Q(A)$ and a' is integral over A. Hence $a' \in A$ because A is normal. The Galois group G acts effectively on A' and $A=(A')^G$. Hence G acts on A'_k and $A_k \subseteq (A'_k)^G$.

(II) We shall show that G acts effectively on A'_k. Suppose, on the contrary, that an element $g \in G$ of order $n>1$ acts trivially on A'_k. For any element $a' \in A'$, we have

$$^g a' - a' = ta'_1 \qquad \text{with} \quad a'_1 \in A' .$$

Write $^g a'_1 = a'_1 + ta'_2$ with $a'_2 \in A'$. Inductively, we define $a'_i \in A'$ $(1 \leqq i \leqq n)$ by $^g a'_{i-1} = a'_{i-1} + ta'_i$. Then it is easy to show

$$a' = {}^{g^n}a' = a' + nta'_1 + \cdots + \binom{n}{i} t^i a'_i + \cdots + t^n a'_n .$$

Hence $a'_1 \in tA'$. Namely, we can write ${}^g a' = a' + t^2 a''_1$. This is true for every $a' \in A'$. By the same argument as above with t replaced by t^2, we have $a''_1 \in t^2 A'$. Thus, we can show that ${}^g a' - a' \in \bigcap_{m \geqq 0} t^m A'$. Since A' is a Noetherian integral domain, we have $\bigcap_{m \geqq 0} t^m A' = (0)$ by Krull's intersection theorem (cf. [11]). Namely, ${}^g a' = a'$ for every $a' \in A'$. This is a contradiction.

(III) We shall show that $A_k = (A'_k)^G$. In fact, suppose $\bar{a}' \in (A'_k)^G$, and write

$${}^g a' = a' + tb(g) \qquad \text{with} \quad b(g) \in A' ,$$

where $a' \in A'$ with $\bar{a}' = a' (\text{mod } tA')$. Then we have

$$b(hg) = {}^h b(g) + b(h) \qquad \text{for} \quad g, h \in G .$$

Set $c = (\sum_{g \in G} b(g))/|G|$. Then $b(g) = c - {}^g c$ for any $g \in G$, and $a' + tc \in (A')^G = A$. Hence $\bar{a}' \in A_k$. Namely, we have $A_k = (A'_k)^G$. Now, the assertion (2) is readily ascertained. Q.E.D.

Hereafter, we assume that $\mathfrak{O}$ is a formal power series ring $k[[t]]$ over an algebraically closed field k of characteristic zero. The constancy of the K-group scheme $Aut_K(L'/L)$ is assured by

Lemma 3.2. *Let $\mathfrak{O} = k[[t]]$ be as above and let $K = Q(\mathfrak{O})$. Let L be a regular extension of K and let L' be a quasi-Galois extension of L such that L' is a regular extension of K. Then L'/L is a Galois extension.*

Proof. We have only to prove that the K-group scheme $Aut_K(L'/L)$ is constant. Since the Puiseux field $\bigcup_{n>0} k((t^{1/n}))$ is an algebraic closure of $k((t))$, where $k((t^{1/n}))$ is the quotient field of $k[[t^{1/n}]]$, there exists a cyclic extension $\mathfrak{O}' = k[[\tau]]$ of $\mathfrak{O}$ ($\tau^n = t$) such that $Aut_K(L'/L) \otimes_K K' \simeq Aut_{K'}(L' \otimes_K K' / L \otimes_K K')$ is constant, where $K' = Q(\mathfrak{O}')$. Note that the morphism Spec $\mathfrak{O}' \to$ Spec $\mathfrak{O}$ is a faithfully flat and finite morphism. Let $G = Aut_K(L'/L)(K')$. Then the constant K'-group scheme $G_{K'}$ has apparently a Néron model $G_{\mathfrak{O}'}$, a constant $\mathfrak{O}'$-group scheme with group G. Hence the K-group scheme $Aut_K(L'/L)$ has an $\mathfrak{O}$-Néron model $\mathcal{G}$; see [13] for relevant results. By definition, the group scheme $\mathcal{G}$ is smooth over $\mathfrak{O}$ and satisfies $\mathcal{G} \otimes_{\mathfrak{O}} K \simeq Aut_K(L'/L)$. By virtue of [1; IV (18.10.16)], $\mathcal{G}$ is finite and étale over $\mathfrak{O}$. Therefore $\mathcal{G}$ must be a constant $\mathfrak{O}$-group scheme $H_{\mathfrak{O}}$, where $H \simeq \mathcal{G}(k) = \mathcal{G}(K)$. Since $G = \mathcal{G}(K') \simeq H_{\mathfrak{O}}(K') = H$, we know that $\mathcal{G} \simeq G_{\mathfrak{O}}$. Thus L'/L is a Galois extension with group G. Q.E.D.

Lemma 3.3. *Let the notations and the assumptions be the same as in Lemma 3.1. Assume that L is the quotient field of a finitely generated, normal $\mathfrak{O}$-domain A. Let A' be the normalization of A in L', and let G be the Galois group of the extension L'/L. Then the following assertions hold true:*

(1) *A' is a Galois extension of A with group G.*

(2) *Suppose that A' is geometrically $\mathfrak{O}$-normal. Then, so is A, and A'_k is a Galois extension of A_k with group G.*

Proof. (1) is now clear. As for (2), A'_k is a normal domain by the hypothesis, and $A_k=(A'_k)^G$ by Lemma 3.1. Hence A_k is normal, and A is geometrically O-normal. The remaining assertion is clear by Lemma 3.1. Q.E.D.

Now, Main Theorem except the assertion (4) follows from Lemma 3.3. In fact, set $L:=Q(A)$ and $L':=Q(R)$ with A and R as in Main Theorem. Then R is the normalization of A in L', and R is geometrically $\mathfrak{O}$-normal. So, we can apply Lemma 3.3. We shall prove the assertion (4). Since $A_{\bar{K}}$ is a polynomial ring over $\bar{K}$, A_K is a polynomial ring over K by [5]. We can identify G as a finite subgroup of $GL(2, \bar{K})$, and it is well-known that G is then generated by pseudo-reflections. Recall that an element $g\in GL(2, \bar{K})$ is a pseudo-reflection if and only if the fixed-point locus $\Gamma(g)_{\bar{K}}$ in $\boldsymbol{A}^2_{\bar{K}}:=\operatorname{Spec}\bar{K}[x,y]$ under the action of g has codimension$\leqq 1$. Since g acts on $\boldsymbol{A}^2_{\mathfrak{O}}:=\operatorname{Spec}\mathfrak{O}[x,y]$, let $\Gamma(g)$ be the fixed-point locus in $\boldsymbol{A}^2_{\mathfrak{O}}$ under the action of g. Namely, $\Gamma(g)$ is a closed subscheme of $\boldsymbol{A}^2_{\mathfrak{O}}$ defined by an ideal I, where I is the smallest ideal of $\mathfrak{O}[x,y]$ generated by all elements of the form ${}^g a-a$ with $a\in\mathfrak{O}[x,y]$. Then we know that $\Gamma(g)_{\bar{K}}=\Gamma(g)\otimes_{\mathfrak{O}}\bar{K}$ and that $\Gamma(g)\otimes_{\mathfrak{O}}k$ is the fixed-point locus in $\boldsymbol{A}^2_k:=\operatorname{Spec}k[x,y]$ under the action of g. Hence $\Gamma(g)\otimes k$ has codimension$\leqq 1$ in $\boldsymbol{A}^2_k$. This implies that when one embeds G into $GL(2, k)$ upto conjugation in $\operatorname{Aut}_k k[x,y]$, G is generated by pseudo-reflections. Hence the G-invariant subring A_k of $k[x,y]$ is a polynomial ring over k. This verifies the assertion (4) of Main Theorem.

4. Reduction from Conjecture 1 to Conjecture 2

Let $\mathfrak{O}$, R and A be as in Conjecture 1. Let $Y:=\boldsymbol{A}^2_{\mathfrak{O}}=\operatorname{Spec}R$, let $X:=\operatorname{Spec}A$ and let $\pi\colon Y\to X$ be the canonical finite morphism. For an algebraic closure $\bar{K}$ of $K=Q(\mathfrak{O})$, $A_{\bar{K}}$ is a normal, cofinite $\bar{K}$-subalgebra of $\bar{K}[x,y]$. Note that $X_{\bar{K}}=\operatorname{Spec}A_{\bar{K}}$ has at most one singular point. Let $\bar{Z}'$ be the universal covering space of $X_{\bar{K}}-\operatorname{Sing}(X_{\bar{K}})$. Then $\pi_{\bar{K}}\colon Y_{\bar{K}}-\pi^{-1}(\operatorname{Sing}X_{\bar{K}})\to X_{\bar{K}}-\operatorname{Sing}(X_{\bar{K}})$ factors through $\bar{Z}'$ because $Y_{\bar{K}}-\pi^{-1}(\operatorname{Sing}X_{\bar{K}})$ is simply connected. Let $\bar{Z}$ be the normalization of $X_{\bar{K}}$ in the function field $\bar{K}(\bar{Z}')$ of $\bar{Z}'$. Then $\bar{Z}\simeq\boldsymbol{A}^2_{\bar{K}}$ and $\pi_{\bar{K}}\colon Y_{\bar{K}}\to X_{\bar{K}}$ factors through $\bar{Z}$;

$$\pi_{\bar{K}}\colon Y_{\bar{K}}\xrightarrow{\bar{\alpha}}\bar{Z}\xrightarrow{\bar{\beta}}X_{\bar{K}}\,.$$

See [10] for the relevant results. Choose a K-rational point P of $Y_{\bar{K}}-\pi^{-1}(\mathrm{Sing}\ X_{\bar{K}})$, and let $Q=\bar{\alpha}(P)$. We shall show that $\bar{Z}$ descends down to a K-scheme. Namely, there exist a K-scheme Z and K-morphisms α: $Y\to Z$ and β: $Z\to X$ such that $\bar{Z}=Z\otimes_K\bar{K}$, $\bar{\alpha}=\alpha\otimes_K\bar{K}$ and $\bar{\beta}=\beta\otimes_K\bar{K}$. In fact, for $\sigma\in\mathrm{Gal}(\bar{K}/K)$, let ${}^{\sigma}\bar{Z}=\mathrm{Spec}\ \rho_{\sigma}(\mathcal{O}(\bar{Z}))$, where ρ_{σ}: $\bar{K}[x,y]\to\bar{K}[x,y]$ is $\sigma\otimes id_{K[x,y]}$ and $\mathcal{O}(\bar{Z})$ is the coordinate ring of $\bar{Z}$ which is a $\bar{K}$-subalgebra of $\bar{K}[x,y]$. We denote by ${}^{\sigma}\bar{\alpha}$: $Y_{\bar{K}}\to{}^{\sigma}\bar{Z}$ and ${}^{\sigma}\bar{\beta}$: ${}^{\sigma}\bar{Z}\to X_{\bar{K}}$ the morphisms induced by $\rho_{\sigma}(\mathcal{O}(\bar{Z}))\hookrightarrow\bar{K}[x,y]$ and $A_{\bar{K}}\hookrightarrow\rho_{\sigma}(\mathcal{O}(\bar{Z}))$, respectively. Hence $\pi_{\bar{K}}=({}^{\sigma}\bar{\beta})\cdot({}^{\sigma}\bar{\alpha})$. Let ${}^{\sigma}Q$ be the point of ${}^{\sigma}\bar{Z}$ which corresponds to Q under the canonical isomorphism $\mathrm{Spec}\ \rho_{\sigma}(\mathcal{O}(\bar{Z}))\to\mathrm{Spec}\ \mathcal{O}(\bar{Z})$. Then we have a unique $\bar{K}$-isomorphism ϕ_{σ}: ${}^{\sigma}\bar{Z}\to\bar{Z}$ such that $\rho_{\sigma}({}^{\sigma}Q)=Q$, $\bar{\alpha}=\phi_{\sigma}\cdot{}^{\sigma}\bar{\alpha}$ and ${}^{\sigma}\bar{\beta}=\bar{\beta}\cdot\phi_{\sigma}$. Then it is easy to show that $\phi_{\tau\sigma}=\phi_{\tau}\cdot{}^{\tau}\phi_{\sigma}$ for $\sigma,\tau\in\mathrm{Gal}(\bar{K}/K)$. In fact, this is the case for a finite Galois extension K'/K instead of $\bar{K}/K$. By the faithfully flat descent, we know that there exists a K-scheme Z such that $\bar{Z}=Z\otimes_K\bar{K}$. Then $\bar{\alpha}={}^{\sigma}\bar{\alpha}$ and ${}^{\sigma}\bar{\beta}=\bar{\beta}$ for any $\sigma\in\mathrm{Gal}(\bar{K}/K)$. Therefore $\bar{\alpha}$ and $\bar{\beta}$ descend down to K-morphisms α: $X_K\to Z$ and β: $Z\to Y_K$ such that $\pi_K=\beta\cdot\alpha$. On the other hand, Z is K-isomorphic to $\boldsymbol{A}^2_K$ by virtue of [5]. Identify the coordinate ring $\mathcal{O}(Z)$ with a K-subalgebra of $K[x,y]$ under α. Let B be the normalization of A in the function field $K(Z)$ of Z. Then B is a normal, cofinite $\mathfrak{O}$-subalgebra of R such that $B_K=\mathcal{O}(Z)$ is a polynomial ring over K. The Conjecture 2 then implies that B is a polynomial ring in two variables over $\mathfrak{O}$. Note that $Q(B)$ is a quasi-Galois extension of $Q(A)$ over K. Main Theorem then asserts that Conjecture 1 is affirmative.

As for the Conjecture 2, we know the following:

Proposition 4.1. *Let* $\mathfrak{O}$, R *and* A *be the same as in Conjecture* 2, *and let* $X=\mathrm{Spec}\ A$. *Suppose that* A *is geometrically* $\mathfrak{O}$*-normal. Then* X_k *has at most one singular point which has necessarily cyclic quotient singularity.*

Proof. By the hypothesis, A_K is a polynomial ring $K[u,v]$. Let $\Delta=\dfrac{\partial}{\partial u}$, which is a locally nilpotent K-derivation of A_K. Since A is finitely generated over $\mathfrak{O}$, we find an integer $n\geqq 0$ such that $t^n\Delta(A)\subseteq A$ and $t^n\Delta(A)\nsubseteq tA$. Define a k-derivation δ of A_k by

$$\delta(\bar{a})=t^n\Delta(a)\ (\mathrm{mod}\ tA)\,,$$

where $\bar{a}=a(\mathrm{mod}\ tA)$ with $a\in A$. Then δ is well-defined, and δ is a nontrivial, locally nilpotent k-derivation on A_k. Hence $X_k:=\mathrm{Spec}\ A_k$ is affine-ruled (cf. [9]) and X_k has at most one singular point which has necessarily cyclic quotient singularity (cf. [10]). Q.E.D.

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