Hashimoto, S.
Osaka J. Math.
23 (1986), 307–312

A NOTE ON SOME PERIODICITY OF Ad-COHOMOLOGY GROUPS OF LENS SPACES

SHIN HASHIMOTO

(Received February 12, 1985)

1. Introduction

Let p be an odd prime and $q=p^r$. We choose a positive integer k such that the class of k in $\boldsymbol{Z}/p^2$ generates the group of units $(\boldsymbol{Z}/p^2)^\times$.

Let K^* be the K-cohomology theory and $K^*_{(p)}$ its p-localized theory. The Adams operation ψ^k on K induces a stable operation ψ^k on $K^*_{(p)}$. We denote by $K_{(p)}$ the spectrum which represents the $K^*_{(p)}$-cohomology theory. Since stable operations induce maps of spectra, we have the following cofibration of spectra

$$K_{(p)} \xrightarrow{1-\psi^k} K_{(p)} \longrightarrow C_{1-\psi^k}\,.$$

We define a spectrum Ad as $\Sigma^{-1}C_{1-\psi^k}$ and its associated cohomology theory Ad^*. When k is a prime power, the associated connective theory of Ad^* coincides with the cohomology theory defined by Seymour [9] and Quillen [8].

Let m and n be positive integers. We identify $\boldsymbol{Z}/m$ with the set of m-th root of 1 in C, and S^{2n+1} with the unit sphere in C^{n+1}. The complex vector space structure on C^{n+1} induces a $\boldsymbol{Z}/m$-action on S^{2n+1} and we define the standard Lens space mod m as $S^{2n+1}(\boldsymbol{Z}/m)$. As is well known, the standard Lens space $L^n(m)$ has a CW-complex structure

$$L^n(m) = \bigcup_{i=1}^{2n+1} e^i$$

and we denote its $2n$-skeleton by $L^n_0(m)$. Since the canonical inclusion $C^{n+1}\subset C^{n+2}$ induces a cellular inclusion $L^n(m)\subset L^{n+1}(m)$, we have a CW-complex $L^\infty(m)=\operatorname{colim} L^n(m)$. This space $L^\infty(m)$ is a classifying space $B\boldsymbol{Z}/m$ of $\boldsymbol{Z}/m$. We consider the case $m=p^r$. Main results are the following.

Theorem 1.1. *Let $M(n)=r+[(n-1)/(p-1)]$. For any integers i, j satisfying $i-j\equiv 0 \pmod{(p-1)p^{M(n)-1}}$, there holds the following isomorphisms.*

$$\widetilde{Ad}^{2i}(L^n_0(p^r)) \cong \widetilde{Ad}^{2j}(L^n_0(p^r))$$
$$\widetilde{Ad}^{2i+1}(L^n_0(p^r)) \cong \widetilde{Ad}^{2j+1}(L^n_0(p^r))\,.$$

Theorem 1.2. *We put* $N=r+v_p(i)$. *Let* j *be an integer which satisfies* $v_p(i)=v_p(j)$ *and* $i-j\equiv 0 \pmod{(p-1)p^{N-1}}$. *Then, there holds the following isomorphism*:

$$\widetilde{Ad}^{2i}(L_0^n(p^r)) \cong \widetilde{Ad}^{2j}(L_0^n(p^r)).$$

We used the computer of Osaka City University computer center to calculate the samples of the group structure of $Ad^*(L_0^n(p^r))$.

The author expresses his hearty gratitude to Professors S. Araki, Z. Yosimura and A. Kono.

2. Preliminaries

Let G be a compact Lie group and $R(G)$ be its complex representation ring. The augmentation ideal I is the kernel of the induced ring homomorphism $R(G)\to R(\{1\})$.

Proposition 2.1 (Atiyah, Atiyah-Segal). *There holds the following natural isomorphism*:

$$\alpha\colon R(G)_I^\wedge \to K(BG).$$

In case $G=S^1$, $R(S^1)=\boldsymbol{Z}[\boldsymbol{H}, \boldsymbol{H}^{-1}]$ *where* $\boldsymbol{H}$ *is the canonical representation of* S^1. *Let* x *be* $e(\boldsymbol{H})=\boldsymbol{H}-1$, *the euler class of* $\boldsymbol{H}$. *Since* $I=(x)$, $K(BS^1)=\boldsymbol{Z}[x]_x^\wedge=\boldsymbol{Z}[[x]]$ *and* $K^1(BS^1)=0$. *We consider the case* $G=\boldsymbol{Z}/q$. *There is the canonical inclusion* $i\colon \boldsymbol{Z}/q\subset S^1$, *and we write* $i^*(\boldsymbol{H})=\boldsymbol{H}$ *and* $i^*(x)=x$. *Then* $R(\boldsymbol{Z}/q)=\boldsymbol{Z}[\boldsymbol{H}]/(\boldsymbol{H}^q-1)$ *and* $I=(x)$. *Thus* $K(B\boldsymbol{Z}/q)=(\boldsymbol{Z}[x]/((x+1)^q-1))_x^\wedge$ *and* $K^1(B\boldsymbol{Z}/q)=0$. *We denote* $R=K_{(p)}(B\boldsymbol{Z}/q)=(\boldsymbol{Z}_{(p)}[x]/((x+1)^q-1))_x^\wedge$. *The definition of the completion induces*

Lemma 2.2. *There exists a continuous surjection*

$$\boldsymbol{Z}_{(p)}[x]_x^\wedge/((x+1)^q-1) \to (\boldsymbol{Z}_{(p)}[x]/((x+1)^q-1))_x^\wedge.$$

The K-ring of finite Lens spaces is given by Mahammed [7]. That is

$$(2.3)\qquad \begin{aligned} K(L_0^n(q)) &= \boldsymbol{Z}[x]/((x+1)^q-1,\ x^{n+1}) \\ K^1(L_0^n(q)) &= 0, \end{aligned}$$

where x is the restriction of x, the euler class of the canonical line bundle.

Lemma 2.4. *The ring* $K_{(p)}(L_0^n(q))$ *is isomorphic to* $R/(x^{n+1})$ *and the inclusion* $L_0^n(q)\subset L^\infty(q)$ *induces the projection* $R\to R/(x^{n+1})$.

We denote $L_0^{\infty,n}(q)=L^\infty(q)/L_0^n(q)$ and $L_{0,0}^{m,n}(q)=L_0^m(q)/L_0^n(q)$. Then we have the following exact sequence.

$$(2.5)\qquad 0\to \tilde{K}(L_0^{\infty,n}(q))\to K(L^\infty(q))\to K(L_0^n(q))\to 0.$$

Thus we have

Lemma 2.6. $\tilde{K}(L_0^{\infty,n}(q)) = x^{n+1} R$

$\tilde{K}^1(L_0^{\infty,n}(q)) = 0$.

It is easy to see that $L_{0,0}^{n+1,n}(q)=S^{2n+1}\underset{q}{\cup} e^{2n+2}$. Thus we have

Lemma 2.7. $\tilde{K}(L_{0,0}^{n+1,n}(q))\cong \boldsymbol{Z}/q$

$\tilde{K}^1(L_{0,0}^{n+1,n}(q)) = 0$.

On the other hand, the exact sequence

$$0 \rightarrow \tilde{K}(L_0^{\infty,n+1}(q)) \rightarrow \tilde{K}(L_0^{\infty,n}(q)) \rightarrow \tilde{K}(L_{0,0}^{n+1,n}(q)) \rightarrow 0$$

shows that $\tilde{K}(L_{0,0}^{n+1,n}(q))=x^n R/x^{n+1}$, which is a cyclic group generated by x^n. So we have

Proposition 2.8. *When* $m-n\equiv 0 \pmod{p-1}$, *we put* $t=\min(r, \nu_p(m)+1)$. *Then*

$$\widetilde{Ad}^{2m}(L_{0,0}^{n+1,n}(q))\cong \widetilde{Ad}^{2m+1}(L_{0,0}^{n+1,n}(q))\cong \begin{cases} \boldsymbol{Z}/p^t & \text{if } n-m\equiv 0 \pmod{p-1} \\ 0 & \text{if } n-m\not\equiv 0 \pmod{p-1} . \end{cases}$$

3. Proof of Theorems

We identify $K_{(p)}^{2m}(X)$ with $K_{(p)}(X)$ using the Bott periodicity. We denote the stable operation ψ^k on $K_{(p)}^{2m}(X)$ by $\psi^{k,m}$. Then $\psi^{k,m}=k^{-m}\psi^k$ under this identification.

Lemma 3.1. *Under the above identification, when* $m-m'\equiv 0 \pmod{p-1}$,

$$\mathrm{Im}((1-\psi^{k,m})-(1-\psi^{k,m'}))\subset p^{1+\nu_p(m-m')}\,\mathrm{Im}(\psi^k) .$$

Kobayashi, Murakami and Sugawara [6] have computed the explicit abelian group structure of $K(L_0^n(q))$. One corollary of their results is

Proposition 3.2.

$$p^{r+[(n-1)/(p-1)]}\,\tilde{K}_{(p)}\,(L_0^n(q)) = 0 .$$

Let $M(n)=r+[(n-1)/(p-1)]$ and $M'=(p-1)\,p^{M(n)-1}$. Then

Corollary 3.3. *Under the above identification*

$$1-\psi^{k,m} = 1-\psi^{k,m+M'}\colon K(L_0^n(q)) \rightarrow K(L_0^n(q)) .$$

When $i-j\equiv 0 \pmod{M'}$, we consider the following commutative diagram where horizontal lines are exact:

$$\begin{array}{ccccccccc} 0 \to & Ad^{2i}(L_0^n(q)) & \to & K^{2i}(L_0^n(q)) & \to & K^{2i}(L_0^n(q)) & \to & Ad^{2i+1}(L_0^n(q)) & \to 0 \\ & & & \downarrow \cong & & \downarrow \cong & & & \\ 0 \to & Ad^{2j}(L_0^n(q)) & \to & K^{2j}(L_0^n(q)) & \to & K^{2j}(L_0^n(q)) & \to & Ad^{2j+1}(L_0^n(q)) & \to 0\,, \end{array}$$

where the vertical isomorphism are induced from the Bott periodicity. This completes the proof of Theorem 1.1.

We took k as a generator of $(\boldsymbol{Z}/p^2)^\times$. It is easy to see the following lemmata.

Lemma 3.4. *The class of k in $\boldsymbol{Z}/p^r$ is a generator of $(\boldsymbol{Z}/p^r)^\times$ for every r.*

Lemma 3.5. *Let $W=(p-1)p^{r-1}/2$. Then $k^W \equiv -1 \pmod{p^r}$.*

Let $U=(p-1)/2$. Lemma 3.5 induces

Corollary 3.6.

$$k^{(p-1)p^{r-2}U} \times k^{(p-1)p^{r-2}/2} \equiv -1 \pmod{p^r}\,.$$

DEFINITION 3.7.

$$A = \prod_{i=0}^{p-1} (\boldsymbol{H}^{k^{i(p-1)p^{r-2}}}-1) = \prod_{i=0}^{p-1} (\psi^{k^{(p-1)p^{r-2}}})^i\,(\boldsymbol{H}-1)$$

We compute the action of the Adams operation on the element A, and using Corollary 3.6 we have

Lemma 3.8. *There exists a natural number t such that*

$$(\psi^k)^{(p-1)^{r-2}/2} A = -\boldsymbol{H}^t A\,.$$

The proof is only a computation. The natural number $t=-\sum_{j=1}^{p-1} k^{j(p-1)p^{r-2}}$, but we don't need it. Let $Q=(p-1)p^{r-2}/2$.

Lemma 3.9. *There exists a natural number s such that*

$$(\psi^k)^{(p-1)p^{r-2}/2}(\boldsymbol{H}^s A) = -\boldsymbol{H}^s A\,.$$

Proof. Using Lemma 3.8, we need to solve the following equation in $\boldsymbol{Z}/q$.

$$k^{(p-1)^{r-2}/2} s+t = s\,.$$

Since Q is not a multiple of $p-1$, $\nu_p(1-k^Q)=0$. This implies that $1-k^Q$ is invertible in $\boldsymbol{Z}/p^r$. Thus the above equation has a solution.

DEFINITION 3.10. We put $B=\boldsymbol{H}^s A$ and $C=\sum_{j=0}^{Q-1}(\psi^k)^j B$. This element C is a polynomial of $\boldsymbol{H}$, so we write $C=C(\boldsymbol{H})$. It is easy to see that $\psi^k B=-B$,

$\psi^k C=-C$ and B and C are in the ideal $(\boldsymbol{H}-1)^p$. So we define $D(\boldsymbol{H})=(\boldsymbol{H}-1)^{-p} \times C(\boldsymbol{H})$.

Lemma 3.11. *The integer $D(1)$ is prime to p.*

Proof. Let $f(X)=(X^u-1)/(X-1)$, then $f(1)=u$. Thus

$$\begin{aligned} D(1) &= \sum_{j=0}^{Q-1} k^j \times k^{j+2Q} \times \cdots\cdots \times k^{j+2Q(p-1)} \\ &= k^v\,(1-k^Q)/(1-k^p)\,. \end{aligned}$$

Since $\nu_p(1-k^Q)=0$, the proof is completed.

Lemma 3.12. *The ideal generated by C coincides with the ideal $(\boldsymbol{H}-1)^p$ in R.*

Proof. We write D as a polynomial of $x=\boldsymbol{H}-1$. Then $D=D(1)+$*higher*. So D is an invertible element in $\boldsymbol{Z}_{(p)}[x]^{\wedge}_{x}$. By Lemma 2.2 D is invertible in R. Since $C=(\boldsymbol{H}-1)^p D$, the Lemma is proved.

Proposition 3.13. *When $n\equiv 0,\ -1 \pmod p$, then*

$$\widetilde{Ad}^{2m+1}(L_0^{\infty,n}(q)) = \bigoplus_{j=1}^{r} \boldsymbol{Z}/p^{j+\nu_p(m)}\,.$$

Proof. By Lemma 2.6 $\tilde{K}(L_0^{\infty,n}(q))=x^{n+1}R$. The assumption implies $n=tp-\varepsilon$, where t is a positive integer and ε is 0 or -1. We choose an additive basis of $x^{n+1}R$ as $\{g_n(i,j);\ 1\leqq j\leqq r,\ 1\leqq i\leqq (p-1)p^{j-1}-1\}$ where $g_n(i,j)=(-C)^t \times(\boldsymbol{H}^{p^{r-j}k^i}+\varepsilon-1)$. As same as in computation in [4], we have the required result.

Consider the following exact sequence

$$\begin{aligned} 0 &\to \widetilde{Ad}^{2m}(L_{0;0}^{n;n-1}(q)) \to \widetilde{Ad}^{2m+1}(L_0^{\infty,n}(q)) \\ &\to \widetilde{Ad}^{2m+1}(L_0^{\infty,n-1}(q)) \to \widetilde{Ad}^{2m+1}(L_{0;0}^{n;n-1}(q)) \to 0\,. \end{aligned}$$

By Proposition 2.8, we have

Corollary 3.14. *If $n-m\equiv 0 \pmod{p-1}$, we have an isomorphism:*

$$\widetilde{Ad}^{2m+1}(L_0^{\infty,n}(q)) \cong \widetilde{Ad}^{2m+1}(L_0^{\infty,n-1}(q))\,.$$

Proposition 3.13 and Corollary 3.14 imply

Lemma 3.15. *Let $N(m)=r+\nu_p(m)$. Then*

$$p^{N(m)}\cdot Ad^{2m+1}(L_0^{\infty,n}(q)) = 0\,.$$

The exact sequence

$$0 \to \widetilde{Ad}^{2m}(L_0^n(q)) \to \widetilde{Ad}^{2m+1}(L_0^{\infty,n}(q)) \to \widetilde{Ad}^{2m+1}(L^{\infty}(q))$$

induces

Corollary 3.16. $p^{N(m)} \cdot \widetilde{Ad}^{2m+1}(L_0^n(q))=0$ *for any integer n.*

Lemma 3.1 and Corollary 3.16 imply Theorem 1.2. This completes the proof.

References

[1] J.F. Adams: *On the groups J(X)-II*, Topology **3** (1965), 137–171.

[2] M.F. Atiyah: *Characters and cohomology of finite groups*, Publ. Math. I.H.E.S. **9** (1961), 23–64.

[3] M.F. Atiyah and G.B. Segal: *Equivariant K-theory and completions*, J. Differential Geom. **3** (1969), 1–18.

[4] S. Hashimoto: *Note on Ad*-cohomology groups of Lens spaces*, to appear.

[5] K. Knapp: *Some applications of K-theory to framed bordism: e-invariant and transfer*, Habilitationschrift (1979).

[6] T. Kobayashi, S. Murakami and M. Sugawara: *Note on J-groups of Lens spaces*, Hiroshima Math. J. **7** (1977), 387–409.

[7] N. Mahammed: *A propos de la K-theorie des espaces lenticulaires*, C.R. Acad. Sci. Paris **271** (1970), 639–642.

[8] D. Quillen: *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. **96** (1972), 552–586.

[9] R.M. Seymour: *Vector bundles invariant under the Adams operations*, Quart. J. Math. Oxford (2) **25** (1974), 359–414.

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558, Japan