In the previous papers [6] and [7], we gave several rings generalized from Nakayama rings. We shall study the same problem, in this note, following those methods.

In the first two sections, we shall consider some right artinian rings with properties (*, 1) and (*, 2), respectively (see §1), and we shall give the complete types of US-4 algebras with \( J^3 = 0 \) over an algebraically closed field in the third section. In the final section, we shall give a structure of US-4 algebras with (*, 1).

1. (*, 1). In this paper, we shall study a right artinian ring \( R \) with identity, and every \( R \)-module is assumed a unitary right \( R \)-module. We denote the Jacobson radical and the socle of an \( R \)-module \( M \) by \( J(M) \) and \( \text{Soc}(M) \), respectively. Occasionally, we write \( J = J(R) \). \( |M| \) means the length of a composition series of \( M \).

We have studied the following condition in [5]:

\((*, n) \) Every maximal submodule of a direct sum of any \( n \) hollow modules is also a direct sum of hollow modules.

In [7] we have given characterizations of a right artinian ring with (*, 3). We shall study, in this section, a right artinian ring \( R \) with (*, 1). If \( R \) satisfies (*, 1), \( eJ = A_1 \oplus A_2 \oplus \cdots \oplus A_n \) for each primitive idempotent \( e \), where the \( A_i \) are hollow. Since \( A_i \) is hollow, \( J(A_i) \) is a unique maximal submodule of \( A_i \) and \( J(A_i) = \sum_{k=1}^{n_i} A_{ik} \) where the \( A_{ik} \) are hollow. Hence we obtain the diagram (see [4] for the meaning of the diagram).
and \( ef^i \) is a direct sum of hollow modules. Let \( M \) be a hollow module. Then \( M \cong eR/A \) for some primitive idempotent \( e \) and some right ideal \( A \) in \( eR \). If \( R \) satisfies \((\ast, 1)\), \( J(M) \cong eJ/A \) is a direct sum of hollow modules. Hence we have the following proposition:

**Proposition 1.** \( R \) satisfies \((\ast, 1)\) if and only if \( eJ/A \) is a direct sum of hollow modules for any primitive idempotent \( e \) and any right ideal \( A \) in \( eJ \).

Following Proposition 1, we may study conditions under which \( eJ/A \) is a direct sum of hollow modules. Assume that an \( R \)-module \( D \) is a direct sum of two submodules \( E_1 \) and \( E_2 \), i.e., \( D = E_1 \oplus E_2 \). Let \( \pi_i : D \to E_i \) be the projection. For any submodule \( B \) of \( D \) we put \( B^{(i)} = \pi_i(B) \) and \( B^{(\overline{i})} = B \setminus E_i \) (\( i = 1, 2 \)).

The following lemma is well known:

**Lemma 2.** \( B^{(1)}|B^{(i)} \cong B^{(2)}|B^{(2)} \) and \( B = \{ (e_1 + B^{(1)}) + (f(e_1) + B^{(2)}) \mid e_1 \in B^{(1)} \} \).

By \( B^{(1)}(f)B^{(2)} \) we denote the \( B \) in Lemma 2. If \( B^{(2)} = 0 \), \( B^{(1)}(f)B^{(2)} \) is nothing but the graph of \( B^{(1)} \) with respect to \( f' : B^{(1)}|B^{(1)} \to B^{(2)} \) and we denote it by \( B^{(1)}(f') \). We call \( B \) a standard submodule of \( D \) provided \( B = B^{(1)} \oplus B^{(2)} \).

**Lemma 3.** Assume that \( D = A \oplus C \), where \( C \) is semi-simple. Let \( B \) be a submodule of \( D \). Then \( D|B \cong A|B' \oplus C' \), where \( B' \) is a submodule of \( A \) and \( C' \) is semi-simple.

**Proof.** \((\overline{D} = D/(B^{(1)} \oplus B^{(2)}) \cong A/B^{(1)} \oplus C/B^{(2)} \). Since \( C \) is semi-simple, \( C = C_1 \oplus B^{(2)} \). Hence \( \overline{D} = A/B^{(1)} \oplus C \supset B/(B^{(1)} \oplus B^{(2)}) \) \((\equiv \overline{B}) \). Put \( C_1 = C_1 \oplus B^{(2)} \). Since \( f : B^{(2)}|B^{(2)} \cong B^{(3)}|B^{(3)} \), \( \overline{B} = B^{(2)}|B^{(2)}(f) \). Therefore \( \overline{D} = A/B^{(1)} \oplus \overline{B} \oplus C_2 \), and so \( D/B \cong \overline{D}/\overline{B} = A/B^{(1)} \oplus C_2 \).

**Remark.** In order to study \((\ast, 1)\), it is sufficient from Lemma 3 that we find a direct decomposition of \( A|B' \) whose direct summands are hollow, when \( D = eJ \).

**Theorem 4.** Let \( R \) be a right artinian ring. Then \( R \) satisfies \((\ast, 1)\) for any hollow module if and only if the following two conditions are fulfilled:

\[
\begin{align*}
eR & \quad eJ \ (1) \\
A_1 & \quad A_2 \quad \ldots \ldots \quad A_n \\
A_{11} & \quad A_{12} \quad \ldots \ldots \quad A_{1n} \\
A_{21} & \quad A_{22} \quad \ldots \ldots \quad A_{2n} \\
& \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \\
A_{n1} & \quad A_{n2} \quad \ldots \ldots \quad A_{nn}
\end{align*}
\]
1) \[ eJ = \sum_{i=1}^{n} \oplus A_i \text{, where } e \text{ is any primitive idempotent in } R \text{ and the } A_i \text{ are hollow.} \]

2) Let \( C_i \supset D_i \) be two submodules of \( A_i \) such that \( C_i / D_i \) is simple. If \( f: C_i / D_i \approx C_j / D_j \) for \( i \neq j \), \( f \) or \( f^{-1} \) is extendible to an element in \( \text{Hom}_{R}(A_i / D_i, A_j / D_j) \) or \( \text{Hom}_{R}(A_j / D_j, A_i / D_i) \).

Proof. Assume that \((\ast, 1)\) is satisfied. Then \( eJ = \sum_{i=1}^{n} \oplus A_i \) as 1) by assumption. Assume \( f: C_i / D_i \approx C_j / D_j \). We shall consider \( eJ / D_i = eJ / D_j \), and hence we may assume that \( D_i = D_j = 0 \) and \( C_i, C_j \) are simple. Put \( C = C_1(-f) \oplus \sum A_j \) and consider \( eJ / C = (A_1 \oplus A_2) / (C(f) C_2) \). Then \( eJ / C = \sum_{i=1}^{n} \oplus D_i \) by assumption, where the \( D_i \) are hollow. We obtain the following exact sequence:

\[ 0 \rightarrow C_1(-f) \rightarrow A_1 \oplus A_2 \rightarrow D_1 \oplus \cdots \oplus D_t \rightarrow 0 \]

We may assume that \( C_i = A_i \) for \( i = 1, 2 \), and so \( C \subseteq \{A_1 \oplus A_2\} \), and \( t = 2 \). Let \( \rho_i: eJ / C \rightarrow D_i \) be the projection. We may assume that \( A_1 \) and \( A_2 \) are submodules of \( eJ / C \). Since \( D_i \) is hollow and \( A_1 \subseteq \{eJ / C\} \), \( D_1 = \rho_1(A_1) \) or \( \rho_2(A_2) \) and \( \rho_1(A_1) = D_1 \) or \( \rho_2(A_2) = D_2 \) for \( i = 1, 2 \). Assume that \( |A_1| \geq |A_2| \). We note that \( |eJ / C| = |A_1| + |A_2| - 1 \). We shall show that \( \rho_i(A_i) \) is an isomorphism for some \( i \) and \( j \). Contrarily assume that no-one of \( \rho_i / A_j \) is an isomorphism. Now we have the following two cases from the remark above:

1) \( \rho_i(A_i) = D_1 \) and \( \rho_2(A_2) = D_2 \) (cf. [8], Lemma 2.1).

2) \( \rho_i(A_i) = D_1 \) and \( \rho_2(A_2) = D_2 \), we have a contradiction as above. Hence some \( \rho_i / A_j \) is an isomorphism. If so is \( \rho_i / A_i \), \( eJ / C = A_1 \oplus D_2 \). Then \( f^{-1} \) is extendible to \( g / A_2 \), where \( g: eJ / C \rightarrow A_1 \) is the projection (cf. [2], p. 771). We obtain a similar result for the remainder. Conversely, let \( S \) be any simple submodule of \( eJ = \sum_{i=1}^{n} \oplus A_i \). We shall show that \( eJ / S = \sum_{i=1}^{n} \oplus B_i \) such that \( \{B_i\}_{i=1}^{n} \) fulfills the same condition as \( \{A_i\}_{i=1}^{n} \). We may assume that \( S \) is of the form \( \{s_1 + f_2(s_1) + \cdots + f_i(s_1) | s_1 \in \text{Soc}(A_i) \text{ and } f_i \in \text{Hom}_{R}(\text{Soc}(A_i), \text{Soc}(A_i))\} \). From the assumption, there exists \( p \leq l \) such that \( f_k f_{k-1} \) is extendible to an element \( g_k \) in \( \text{Hom}_{R}(A_p, A_p) \) for all \( k \leq l \). Put \( A' = \{g_1(a) + \cdots + g_p(a) + a + g_{p+1}(a) + \cdots + g_l(a) | a \in A_1\} \) (cf. [1], p. 787). Then \( eJ = \sum_{i=1}^{n} \oplus A_i \oplus A_1 \oplus A_2 \oplus \cdots \oplus A_n \) and \( A_i \approx A'_i \) for the desired direct decomposition. Hence \( eJ / S \) has the desired direct decomposition. Let \( B \) be any submodule of \( eJ \). Take a simple submodule \( S_i \) in \( B \). Then \( B / S_i \) is a submodule of \( eJ / S_i \), which has the same direct decomposition as \( eJ \). Iterating this procedure, we know that \( eJ / S_i \) is a direct sum of hollow modules.
REMARK. We assume in 1) of Theorem 4 that all $A_i$ are uniserial. Then, since $A_i/D_i$ is uniform, we can show easily and similarly to the proof of Theorem 4 that either $A_1$ or $A_2$ is a direct summand of $eJ/C$ without $|C_i/D_i|=1$. Hence if 2) of Theorem 4 is satisfied, we have the same without $|C_i/D_i|=1$, provided all $A_i$ are uniserial.

Corollary 5. Assume that $eJ=\sum_{i=1}^{n(e)} A_i$ for each $e$, where the $A_i$ are hollow. Further assume that any sub-factor module of $A_i$ is not isomorphic to any one of $A_j$ for any pair $i, j (i\neq j, i, j=1, 2, \ldots, n(e))$ and for any $e$, then $R$ satisfies $(\ast, 1)$.

Proposition 6. If $J^2=0$, $R$ satisfies $(\ast, 2)$ and hence $(\ast, 1)$.

Proof. This is clear from [4], Proposition 3.

Example.

$$R = \begin{array}{ccc|ccc}
K & K & K & K & K & K \\
K & K & K & 0 & 0 & 0 \\
0 & K & 0 & K & 0 & 0 \\
& & & & & \\
0 & 0 & 0 & & & \\
0 & 0 & 0 & & & \\
\end{array}$$

Put $A=(0, K, K, K, 0, 0, 0)$ and $B=(0, 0, 0, 0, K, K, K)$. Then $eJ=A\oplus B$. Any subfactor module of $A$ is not isomorphic to any one of $B$. Hence $R$ satisfies $(\ast, 1)$ from Corollary 5, but $R$ does not satisfy $(\ast, 3)$ and $R$ is not US-3 by [3], Theorem 1 and [5], Lemma 1, since $J(A)$ is a direct sum of two simple modules. Further we know from Theorem 18 in §4 that $R$ does not satisfy $(\ast, 2)$.

2. $(\ast, 2)$. We have shown in Proposition 6 that if $J^2=0$, $R$ satisfies $(\ast, 2)$. In this section, we shall give a relationship between rings with $(\ast, 2)$ and ones with $(\ast, 3)$.

If $R$ is an algebra over an algebraically closed field $K_0$, then $eRe/eJe=K_0\bar{e}$. Hence $R$ satisfies

Condition II" [4]. $eRe/eJe=\bar{e}K'$ for each primitive idempotent $e$, where $K'$ is a field contained in the center of $R$.

In this case every unit element $x$ in $eRe$ is of the form

$$ek+j,$$

where $k\in K'$ and $j\in eJe$. Further $K'=\Delta(A)$ for any right ideal $A$ in $eJ$ (see [4]).
We always assume in this section that \( R \) satisfies Condition II". If \( R \) satisfies further \((*, 3)\), then
\[
e J = A_1 \oplus B_1 \quad \text{such that} \quad A_1 | (A_1) \cong B_1 | (B_1),
\]
where \( A_1, B_1 \) are hollow from [3], Proposition 26 and [4], Theorem 1. Next assume that \( R \) satisfies \((*, 2)\) and the above condition \((3)\) for each \( e \). Then every proof given in [4], Lemmas 3~18 is valid under \((3)\). Hence \( eR \) has the structure given in [4], Theorem 1. Therefore \( R \) satisfies \((*, n)\) for any \( n \) from [7], Theorems 2 and 3. Thus we obtain

**Theorem 7.** Let \( R \) be an algebra over a field \( K \) with condition II". Then the following conditions are equivalent:
1) \( R \) satisfies \((*, 2)\) and \((3)\).
2) \( R \) satisfies \((*, n)\) for all \( n \).

Put
\[
R = \begin{pmatrix} K & K \oplus K \\ 0 & K \end{pmatrix} \quad \text{and} \quad A_1 = (0, K \oplus 0), \quad B_1 = (0, 0 \oplus K).
\]

Then \( eJ = A_1 \oplus B_2, \quad A_1 \cong B_2 \). Hence \((3)\) is not fulfilled and \( R \) satisfies \((*, 2)\) but not \((*, 3)\) from Proposition 6 and Theorem 7.

3. **US-4 algebras with \( J^2 = 0 \).** We have studied US-3 rings with \((*, 1)\) or \((*, 2)\) in [7]. We shall observe US-4 algebras over an algebraically closed field. We have defined in [6]
\[
(**, n) \quad \text{Every maximal submodule of a direct sum} \ D \ \text{of any} \ n \ \text{hollow modules contains a non-trivial direct summand of} \ D.
\]

We call a ring \( R \) (right) \( US-n \) provided that \((**, n)\) holds for any \( D \) [6]. Let \( K \) be a field, and put
\[
R_n = \begin{pmatrix} K & K & K & \cdots & K \\ 0 & K & 0 & 0 & \cdots \\ 0 & 0 & \ddots & & \\ \end{pmatrix}
\]

Then \( R_n \) is a US-\( m \) algebra, but not US-(\( m-1 \)) algebra from some \( m \) from [5], Corollaries 1 and 2 of Theorem 2.

**Proposition 8.** Let \( R \) be an algebra of finite dimension over a field. Then the number of isomorphism classes of hollow modules is finite if and only if \( R \) is a (right) US-\( n \) algebra for some integer \( n \). Hence an algebra of finite representation type is a US-\( n \) algebra.

Proof. Assume that \( R \) is a US-\( n \) algebra. Let \( \{A_1, A_2, \ldots, A_m\} \) be a set
of $\mathcal{E}$-modules in $\mathcal{E}\mathcal{R}$ such that $|\mathcal{E}\mathcal{R}/\mathcal{A}_i|=t$ for all $i$, where $t$ is a fixed integer. If $m\geq n+1$, from [5], Corollary 2 of Theorem 2 there exists a unit $x$ in $\mathcal{E}\mathcal{R}$ such that $x\mathcal{A}_i\subseteq \mathcal{A}_j$ for some pair $(i,j)$. Since $|\mathcal{E}\mathcal{R}/\mathcal{A}_i|=|\mathcal{E}\mathcal{R}/\mathcal{A}_j|$, $\mathcal{E}\mathcal{R}/\mathcal{A}_i\cong \mathcal{E}\mathcal{R}/\mathcal{A}_j$. Therefore there exist at most $n$ pairwise non-isomorphic, hollow modules $\mathcal{E}\mathcal{R}/\mathcal{B}$ with $|\mathcal{E}\mathcal{R}/\mathcal{B}|=t$. Accordingly, $\mathcal{R}$ being artinian, the number of isomorphism classes of hollow modules is finite. Conversely, we assume that $\mathcal{R}$ is as above. Let $m$ be the number of isomorphism classes of hollow modules $\mathcal{E}\mathcal{R}/\mathcal{B}$ for a fixed primitive idempotent $e$. Since $[\mathcal{R}: \mathcal{K}]=p<\infty$, $[\Delta: \Delta(\mathcal{B})]\leq p$, where $\Delta=\mathcal{E}\mathcal{R}/\mathcal{J}e$ (see [4]). Let $D$ be a direct sum of $(p+1)m$ hollow modules $\mathcal{E}\mathcal{R}/\mathcal{A}_j$. Then there exists some direct summand $\mathcal{E}\mathcal{R}/\mathcal{A}_j$, which appears at least $(p+1)$-times in $D$. Hence a direct sum of $p$-copies of $\mathcal{E}\mathcal{R}/\mathcal{A}_j$ satisfies (**, $p+1$) by [5], Corollary 1 of Theorem 2. Hence $D$ satisfies (**, $(p+1)m$) by [5], Theorem 2. Repeating this argument for each primitive idempotent $e$, we know that $\mathcal{R}$ is a US-$n$ algebra for some $n$.

From now on, we always assume, unless otherwise stated, that $\mathcal{R}$ is an algebra over a field with Condition II*.

Employing the similar argument given in the proof of [6], Proposition 1, we have

**Lemma 9.** If $\mathcal{R}$ is a US-$n$ algebra, $|\mathcal{E}\mathcal{J}_i/\mathcal{E}\mathcal{J}^{i+1}|\leq n-1$ and every intermediate submodules between $\mathcal{E}\mathcal{J}_i$ and $\mathcal{E}\mathcal{J}^{i+1}$ are characteristic for each primitive idempotenti $e$.

The following proposition is an immediate consequence of [5], Corollaries 1 and 2 of Theorem 2 for any right artinian ring.

**Proposition 10.** Let $\mathcal{R}$ be a right artinian ring. Then $\mathcal{R}$ is a US-$4$ if and only if for any set of four submodules $\{\mathcal{A}_i\}_{i=1}^4$ of $\mathcal{E}\mathcal{J}$, there exists a pair $(i, j)$ such that $\mathcal{A}_i\sim \mathcal{A}_j$. Further $\mathcal{R}$ is the algebra mentioned in the beginning, then there exist at most three maximal submodules in $\mathcal{E}\mathcal{J}_i$ and they are characteristic.

First we shall give the lattices of submodules in $\mathcal{E}\mathcal{J}$, provided that $\mathcal{R}$ is a US-4 algebra which satisfies Condition II*.

Assume that $|\mathcal{E}\mathcal{J}_i/\mathcal{E}\mathcal{J}^{i+1}|=3$ and $\mathcal{E}\mathcal{J}_i/\mathcal{E}\mathcal{J}^{i+1}=\bar{D}_1\oplus \bar{D}_2\oplus \bar{D}_3$, where the $\bar{D}_i$ are intermediate submodules between $\mathcal{E}\mathcal{J}_i$ and $\mathcal{E}\mathcal{J}^{i+1}$ such that the $\bar{D}_i$ are simple. Since every maximal submodule in $\mathcal{E}\mathcal{J}_i$ is characteristic by Proposition 10, so is $D_i$ and $D_i\sim D_j$ for $i\neq j$. Put $C_1=D_1+D_2$, $C_2=D_2+D_3$ and $C_3=D_1+D_3$. Then the $C_i$ are characteristic, and $C_i\sim C_j$ for $i\neq j$. The set $\{C_i, D_i\}_{i,j=1}^3$ is the set of all intermediate submodules between $\mathcal{E}\mathcal{J}_i$ and $\mathcal{E}\mathcal{J}^{i+1}$. Assume that no one of the $D_i$ is hollow. Let $E_i$ be a maximal submodule of $D_i$ not equal to $\mathcal{E}\mathcal{J}^{i+1}$. If $x\mathcal{E}_3=E_i$ for some $x$ in $\mathcal{E}\mathcal{R}$, $E_i=x\mathcal{E}_3\subseteq D_i\cap D_3=\mathcal{E}\mathcal{J}^{i+1}$. Hence $E_i\sim E_{k'}$ for $k\neq k'$, a contradiction by Proposition 10. Therefore one of the $D_i$ is hollow. Further, since the $C_i$ and the $D_i$ are characteristic and $D_i\sim D_j\sim D_3$,
the $C_i$ contains exactly two maximal submodules $D_i$. Thus we obtain the following diagram of submodules:

![Diagram](image)

where each factor module given from connected modules is simple, (cf. [6]). Next assume that $|eJ^i/eJ^{i+1}|=2$. Consider a diagram:

![Diagram](image)

Lemma 11. $B_1 \cap F \neq B_1 \cap B_2$ in (5).

Proof. This is clear from the fact: $|B_1/(B_1 \cap F)| = 1$ and $|B_1/(B_1 \cap B_2)| = 2$.

Lemma 12. If $|F|=1$ in the above, either $B_1$ or $B_2$ does not exist.

Consider a diagram:

![Diagram](image)

Similarly to Lemma 11, we have

Lemma 13. Assume that $G$ is a waist. Then $E_1$ and $E_3$ are hollow.

We observe the following:

![Diagram](image)
$D_1$ and $C'_1$ are characteristic from Proposition 10. Put $D_1 = D_1/C'_1 = H_1 \oplus \mathfrak{a} J^{i+1}$. We have two cases a) $H_1 \cong \mathfrak{a} J^{i+1}$ and b) $H_1 \cong \mathfrak{a} J^{i+1}$ (see $R_3$ in Example in [6]). There exists no $H_2$ in the case a). On the other hand, there exist many $H_2$ in the case b). If all $H_2$ are characteristic, we have a contradiction from Proposition 10. Hence we may assume that $H_1$ is not characteristic. If $e J H_1 \subset C'_1$, $H_1$ is characteristic by (2), and so there exists $j$ in $e J e$ such that $j H_1 \subset C'_1$. Since $H_1$ is simple, $H_1 = h K + C'_1$ for $H_1 = \mathfrak{a} J (H_1) \subset C'_1$ (we may assume that $R$ is basic). Let $j \bar{h} = \bar{h} k + \bar{x}$, where $k \in K$ and $\bar{x} \in e J^{i+1}$. Since $j$ is nil, $k = 0$. Hence $j \bar{h}$ is a generator of $e J^{i+1}$. Let $\bar{h}_2$ be a generator of $H_2$. Then $\bar{h}_2 = \bar{h} k + j \bar{h}_2$, and so $\bar{h}_2 k^{-1} = (k^{-1} k_i + j) \bar{h}$. Hence $H_2 = (k^{-1} k_i + j) H_1$, since $C'_1$ is characteristic. Thus we obtain

**Lemma 14.** $H_1 \sim H_2$ if $e J^{i+1} | C'_1 \cong H_1 / C'_1$, and $H_2$ does not exist if $e J^{i+1} | C'_1 \cong H_1 / C'_1$.

Now we shall give all lattices of the submodules in $e R$ for each primitive idempotent $e$, provided $R$ is a right US-4 algebra and $J^2 = 0$ ($J^2 \neq 0$).

1  
\begin{align*}
1 & e R \\
e & e J \\
e & e J^2 \\
0 & 0
\end{align*}

(Use Lemmas 12 and 14)

2  
\begin{align*}
1 & e R \\
e & e J \\
e & e J^2 \\
0 & 0
\end{align*}

(Use Lemmas 11, 12, 13 and 14)

3  
\begin{align*}
1 & e R \\
e & e J \\
e & e J^2 \\
0 & 0 \\
or & e J^2
\end{align*}

(Use Lemmas 11, 12 and 14)

4  
\begin{align*}
1 & e R \\
e & e J \\
0 & 0
\end{align*}

(Use Lemmas 11, 12 and 14)
We shall give an example for each case. The types 1) and 2) are the cases of US-1 and US-2. 4): $e_j = A_i \oplus A_j$, where the $A_i$ are uniserial such that any sub-factor module of $A_i$ is not isomorphic to one of $A_j$.

3): Let $R$ be a vector space with basis $\{e_1, 1x_1, 1y_1, 1y_2, 1y_3, e_2, 2w_1, e_3, 3x_1, e_4\}$. Define $e_i e_j = \delta_{ij} e_i$ and $1y^2$ means $e_1 1y^2 e_2 = 1y^2$, and so on. Put $y_2w = x, y_3z = x$ and other products are zero. Then

$$e_1 R$$

$$\langle x, y_1, y_2, y_3 \rangle$$

$$\langle x, y_1, y_2, y_3 \rangle$$

$$\langle x, y_1, y_2, y_3 \rangle$$

$$\langle x, y_1, y_2, y_3 \rangle$$

$$\langle x, y_1 \rangle$$

$$\langle x, y_2 \rangle$$

$$\langle x, y_3 \rangle$$

$$\langle y_1 \rangle$$

$$\langle y_2 \rangle$$

$$\langle y_3 \rangle$$

$$\langle x \rangle$$

$$\langle 0 \rangle$$

7): $R = \langle e_1, 1y_1, 1y_2, 1y_3, 1x_1, 1x_2, 1x_3, e_2, 2w_1, 2w_2, 2w_3, e_3, 3x_1, 3x_2, 3x_3, e_4, 4y_1, 4y_2, 4y_3, 4z_1 \rangle$ is a vector space. Define $y_1y_2 = x, y_1y_3 = x_3, y_2y_1 = x_1$ and other products are zero. Then we have the following lattice given in the next page.

4. **US-4 algebras with $(\ast, n)$, $n=1, 2$.** We shall study, in this section, a US-4 algebra $R$ over an infinite field $K$, which satisfies Condition II' and $(\ast, 1)$.

**Lemma 15.** Let $R$ be a US-$n$ algebra over $K$ as above. Assume that $e^J = \sum_{i=1}^n A_i$ with $A_i$ hollow. Then, for any submodule $B_j$ of $A_j, A_i/B_i \cong A_j/B_j$, provided $i \neq j$ and $A_i \neq B_i$.

Proof. Assume that $f: A_i/B_i \cong A_2/B_2$ and $f \neq 0$. We may assume $|A_i/B_i| = 1$. We shall show that there exist no units $x$ in $eRe$ such that $A_i(f) A_j = x(A_i(g)A_j)$ for any $(f \neq 0) g: A_i/B_i \rightarrow A_2/B_2$. Assume that there exists $x$ as above. Let $a$ be a generator of $A_i$ and $x = k + j$ as (2). Since $A_i(f) A_2 = x(A_i(g)A_2)$,

$$a + f(a) = (k + j)(a' + g(a') + b)$$

where $a' \in A_2, b \in B_2 \oplus B_2$ and $f(a), g(a')$ are fixed representative elements of $f(a + A_i)$ and $g(a' + A_1)$, respectively. Let $\pi_i: \sum_{j=1}^n A_j \rightarrow A_i$ be the projection. Then
\[ a = \pi_1(a + f(a)) = ka' + \pi_1(j(a' + g(a') + b) + kb) \]
\[ f(a) = \pi_2(a + f(a)) = kg(a') + \pi_2(kb + j(g(a') + b)). \]  
\[ (9) \]

Hence, since \( A_i \) is hollow and \( jej^i \subset e{j}^{i+1} \),
\[ a = ka' \quad \text{and} \quad f(a) = kg(a') = g(a) \]  
\[ (10) \]

where \( a \) is the class of \( a \), which is a contradiction. Since \( K \) is infinite, there exist infinitely many isomorphisms \( g = kf \). However, \( R \) is US-\( n \), and hence we obtain a contradiction from [5], Corollaries 1 and 2 of Theorem 2 (cf. Proposition 10). Therefore \( A_1 = B_1 \).
Now we assume that $R$ is a US-4 algebra with $(\ast, 1)$. Then $|eJ|/|eJ^{i+1}| \leq 3$ by Lemma 9, and so $eJ^i = A_1 \oplus A_2 \oplus A_3$ from the diagram (1), where the $A_i$ are hollow. Assume that all $A_i$ are non-zero. First we assume that $J(A_i) = 0$, $J(A_2) = 0$. Then $\{A_1, A_2, A_3, J(A_2)\}$ gives a contradiction to Proposition 10, since every unit in $eRe$ is of the form (2). Hence we may assume that $A_2$ and $A_3$ are simple. If $A_1$ is not uniserial, there exist two submodules $B_1, B_2$ with $B_1 \sim B_2$ from the fact: $J^i = A_1 J^i$ and Lemma 9. Then $\{A_2, A_3, B_1, B_2\}$ gives a contradiction to Proposition 10. Therefore $A_1$ is uniserial and $A_2, A_3$ are simple, i.e.,

$$
\begin{array}{ccc}
A_1 & A_2 & A_3 \\
\vdots & 0 & 0 \\
D_i & & \\
\vdots & & \\
D_n & & \end{array}
$$

(11)

We shall show

$$D_i/D_{i+1} \cong A_2 \text{ for } i < n$$

(12)

Assume contrarily that $f: D_i/D_{i+1} \cong A_2$. Since $eJ e A_2 \subseteq eJ^{i+1} = A_1 J$ and $A_2$ is simple, $eJ e A_2 \subseteq \text{Soc}(eJ^{i+1}) = D_n$. Put $g = k f$ ($k \neq 0 \in K$), then $f = g$. Assume that there exists a unit $x$ in $eRe$ such that $D_i(f) = x D_i(g)$. Using (2), we know that $j(g(a') + a') \subseteq D_n + D_{i+1} = D_{i+1}$ ($i < n$), and so $(a + D_{i+1}) = k(a' + D_{i+1})$ and $f(a + D_i) = g(k a' + D_{i+1}) = g(a + D_{i+1})$, which is a contradiction. Hence $D_i(f) \not\sim D_i(g)$, provided $k \neq 0$. Since $K$ is infinite, we obtain a contradiction from Proposition 10.

Next assume

$$f: D_n \cong A_2.$$  

(13)

Let $g$ be a non-zero element in $\text{Hom}_R(A_2, D_n)$ and assume $A_2(f) \sim x A_2(g)$ for some unit $x$ in $eRe$. Since $eJ e D_n = 0$ and $eJ A_2 \subseteq D_n$, from (8) we have

$$a = k a' \quad \text{and} \quad (f - g)(a) = ja$$

(14)

where $a, a'$ are in $A_2$. Now $A_1 \not\sim A_2$. Assume $A_2(g) \sim A_2$. Then $f = j_1$ (left multiplication of $j$) by putting $g = 0$ in (14). If $A_2(f) \not\sim A_2$, put $g = k f$ ($k \neq 0, 1$). Since $A_2(f) \not\sim A_1$ from (2), $A_2(g)$ is related to one of $\{A_1, A_2, A_2(f)\}$ with respect to $\sim$ by Proposition 10. If $A_2(g) \sim A_2$, $f = k g = (-k) j_1$ by replacing $f$ with 0 in (14). Finally assume $A_2(g) \sim A_2(f)$ (note $A_2(g) \not\sim A_1$). Then $f - g = (1 - k) f = j_1$ from (14). Hence $f = ((1 - k)^{-1}) j_1$. In any cases $f$ is given by a left multi-
plication of an element in $eJ$. Thus we obtain

**Lemma 16.** If $R$ is a US-4 algebra with $(\ast, 1)$ and $|eJ^i|eJ^{i+1}| = 3$, then $eJ^i = A_1 \oplus A_2 \oplus A_3$ such that

1) $A_1$ is uniserial, and $A_2$, $A_3$ are simple and they are not isomorphic to one another.

2) $D_i/D_{i+1} \not\cong A_2$ (or $A_3$) provided $D_i \not\cong \text{Soc}(A_1)$.

3) If $D_n = \text{Soc}(A_1) \cong A_2$ (or $A_3$), this isomorphism is given by a left multiplication of an element in $eJ$.

Next assume $|eJ^i|eJ^{i+1}| = 2$ and $eJ^i = A_1 \oplus A_2$. If $A_1$ is uniserial and $A_2$ is simple, we have the same property as in Lemma 16 for $A_1$ and $A_2$. Assume that neither $A_1$ nor $A_2$ is uniserial. Then there exist $C_1 \oplus C_2'$ in $A_1$ and $D_1 \oplus D_2'$ in $A_2$ such that $\{C_1', D_1\}$ are not related one another with respect to $\sim$ from the diagram (1) and (2). Hence either $A_1$ or $A_2$ is uniserial. First we assume that both $A_1$ and $A_2$ are uniserial, i.e.,

\[
\begin{array}{cccc}
A_1 & A_2 & eJ^i \\
D_2 & E_2 & eJ^{i+1} \\
D_3 & & \\
& & E_m \\
D_n & 0 & eJ^n \\
0 & & \\
\end{array}
\]

We may assume $n \geq m$. Then any one of $\{A_1, D_2 \oplus E_m, D_3 \oplus E_{m-1}, \ldots, A_2\}$ is not related to another one with respect to $\sim$ and hence $m \leq 2$, i.e.,

\[
\begin{array}{cccc}
A_1 & A_2 & eJ^i \\
D_2 & E_2 & eJ^{i+1} \\
D_3 & & \\
& & 0 \\
D_n & & eJ^n \\
0 & & \\
\end{array}
\]

(15)

In this case $\{A_1, A_2, D_2 \oplus E_2\}$ are not related to one another with respect to $\sim$. If $f: E_2 \cong D_i/D_{i+1}$, $f$ is extendible to $f': A_2 \rightarrow A_i/D_i$. Hence $A_2 \cong D_i/D_{i+1}$. Therefore, if a sub-factor module of $A_2$ is isomorphic to one of $A_2$, we may assume $f$: $A_2/E_2 \cong D_i/D_{i+1}$. Put $B = A_2(f)D_i$. Then $A_2 \cong B$ by
Lemma 15 and Proposition 10, i.e., \( xA_2 \subset B \) for a unit \( x = e + j \) in \( eRe \); \( j \in eFe \). Let \( A_2 = a_2R \) and \( \pi_j : e^{j+1} \to A_2 \), the projection. Then \( xa_2 = (e + j)a_2 = a_2f(a_2) + d_{i+1} + e_2 : r \in R, \ d_{i+1} \in D_{i+1}, \ e_2 \in E_2 \) and \( f(a_2 + E_2) = f(a_2) + D_{i+1} \). Hence

\[
\begin{align*}
\pi_2(ja_2) & = a_2 + \pi_2(ja_2) \\
\pi_1(ja_2) & = f(a_2) + d_{i+1}.
\end{align*}
\]

Since \( jA_2 \subset D_{n-1} \oplus E_2 \) (note \( |A_2| = 2 \)), \( a_2 \equiv a_2 \) (mod \( E_2 \)) and \( f(a_2) \in D_{i+1} \), provided \( i + 1 \leq n - 1 \). Hence \( i \geq n - 1 \). Now assume \( i = n \), and consider \( \{A_1, D_n \oplus A_2, D_2 \oplus E_2, B\} \). Then \( D_n \oplus A_2 \sim B \) by Proposition 10. Since \( |D_n \oplus A_2| = |B| \), \( x(A_2(f)D_n) = D_n \oplus A_2 \). Similarly, we can show that every submodule in \( e^j \) is isomorphic to a standard submodule in \( e^j \) via \( x_1 \). We can express the above situation as the form (14). For instance, assume \( f : A_2/E_2 \cong D_n \). Then \( A_2(f) = xA_2 \) as above, \( (x = k + j) \). Hence for some \( a_2' \) in \( A_2 \)

\[
\begin{align*}
a_2' &= ka_2' + \pi_3(ja_2') \\
f(a_2') &= \pi_1(ja_2') = ja_2' - \pi_2(ja_2') \\
&= j(k^{-1}a_2 - k^{-1} \pi_2(ja_2')) - \pi_2(ja_2')
\end{align*}
\]

On the other hand, for an element \( e_2 \) in \( E_2 \)

\[
e_2 = e_2 + f(e_2) = (k + j) a_2'; \ a_2' \in A_2
\]

Hence \( a_2' \in E_2 \). Since \( jE_2 \subset D_n \), \( ja_2' = 0 \) from (17). \( E_2 = a_2'R \) implies \( jE_2 = 0 \). In (16), \( \pi_2(ja_2') \in E_2 \), and so \( f(a_2) = jk^{-1}a_2 - \pi_2(ja_2) \). Hence

\[
f(a_2) \equiv j' a_2 \pmod{E_2} \quad \text{and} \quad j'E_2 = 0
\]

Next consider a diagram

\[
\begin{array}{ccc}
A_1 & & A_2 \ \\
\downarrow & & \downarrow e^{j+1} \\
E_2 & & E_2 \\
\downarrow & & \downarrow 0 \\
C_1 & & C_2 \ \\
\end{array}
\]

Consider \( \{A_1, A_2, C_1 \oplus E_2, C_2 \oplus E_2\} \). Since \( R \) is US-4, \( C_1 \oplus E_2 \sim C_2 \oplus E_2 \), and hence \( C_1 \oplus E_2 \approx C_2 \oplus E_2 \). Therefore \( C_1 \approx C_2 \). However \( e^{j+k} \) contains at most three maximal submodules by Lemma 9 and Proposition 10. If \( C_1 \approx C_2 \), we can construct many submodules, which is a contradiction.

Further consider a diagram:
If $F_1 \neq 0$ and $F_2 \neq 0$, \{A_2, C_1, C_2, F_1 \oplus F_2\} gives a contradiction. Hence either $F_1$ or $F_2$ is zero. Hence we obtain

$$
\begin{array}{c}
A_1 & A_2 & eF^j \\
\vdots & \vdots & \vdots \\
C_1 & C_2 & eF^{i+j} \\
F_1 & F_2
\end{array}
$$

Considering \{G_1, G_2, C_2, A_2\}, we know $G_2 = 0$. Therefore

$$
\begin{array}{c}
A_1 & A_2 & eF \\
\vdots & \vdots & \vdots \\
C_1 & C_2 & eF \\
D_n & 0
\end{array}
$$

(19)

It is clear from the previous argument that $A_2$ (resp. $C_2$) is not isomorphic to a sub-factor module of $A_1$ except $A_2 \cong C_2$, $A_2 \cong D_n$ (resp. $C_2 \cong D_n$). Further those isomorphisms are given by left multiplications of elements in $eF$. Conversely, we assume that the lattice of $eR$ has the structure given in (11), (15) and (19). If $j$ is in $eFe$, $e+j$ gives an isomorphism, and hence every submodule of $eF$ may be assumed standard except (15). Assume $f : (A_2 \to) A_2 | E_2 \cong D_n$ such that (18) holds. Then $(e+j')A_2 \subset A_2(f) + E_2 = A_2(f)$, since $f(E_2) = 0$, and so $(e+j')A_2 = A_2(f)$.

Summarizing the above we obtain

**Theorem 17.** Let $R$ be an algebra over a field satisfying Condition II*. Then $R$ is a US-4 algebra with $(\ast, 1)$ if and only if, for each primitive idempotent $e$, $eR$ has one of the following structures:
1) $eR$

2) $eR$

$eJ$

$D_i/D_{i+1} \cong A_j$ for $i=1, \ldots, n$; $j=2, 3$ $(D_1=A_1)$ and $D_n$ is isomorphic to $A_j$ via $f$ if and only if $j=j_i; j \in eJ$.

3) $eR$

$eJ$

$A_1 \quad A_2 \quad eJ^i$

$D_2 \quad E_2 \quad eJ^{i+1}$

$\vdots \quad 0 \quad eJ^{i+2}$

$D_n \quad eJ^a$

$0$

1) Any $g: E_2 \cong D_n$ is extendible to $g': A_2 \cong D_{n-1}$.

2) Every submodule in $eJ^i$ is isomorphic to a standard submodule in $eJ^i$ via $x_i$; $x$ is a unit in $eRe$. In this case no sub-factor modules of $A_2$ are isomorphic to any one of $A_1/D_{n-1}$.

4) $eR$

$eJ$

$A_1 \quad A_2 \quad eJ^i$

$C_1 \quad C_2 \quad eJ^m$

$\vdots \quad 0 \quad eJ^n$

$E_2 \quad eJ^n$

$0$
$A_2$ (resp. $C_2$) is not isomorphic to any sub-factor module of $A_1$ except $C_2$ and $E_1$ (resp. $E_1$). If they are isomorphic, those isomorphisms are $j_1$, where $j_1$ is the left multiplication of an element $j$ in $eJ$.

We give an example.

$$R = \begin{pmatrix}
K & K & K & K & K \\
K & K & K & K & K \\
K & 0 & 0 & 0 & 0 \\
0 & K & K & K & K \\
0 & K & K & K & K
\end{pmatrix}$$

is of the type 4).

**Theorem 18.** Let $R$ be as in Theorem 17. Then $R$ is a US-4 algebra with $(\ast, 2)$ if and only if, for each primitive idempotent $e$, $eR$ has one of the structures 1)~3) replaced $ej$ with $eJ$ in Theorem 17.

Proof. Assume that $R$ is a US-4 algebra with $(\ast, 2)$. Then $(\ast, 1)$ is fulfilled. First assume that $J_3 = 0$ and $eR$ has the structure 4) replaced $ej$ with $eJ$ in Theorem 17. Then $C_1$ and $C_2$ are simple. Since $A_1/A_1J \cong A_2/A_2J = A_2$, we can use the same argument in the first paragraph of [4], p. 87, and obtain $C_2 = 0$. Similarly we can show that $eR$ has one of the structures 1)~3) replaced $ej$ with $eJ$, provided $J_3 = 0$. In general case we assume that $eR$ has one of the structures 2)~4) with $i \neq 1$. Taking $R/J^{i+1} = \tilde{R}$, we know that there exists an $\tilde{R}$-hollow module of the following structure:

They are also $R/J^2$-hollow modules. However, there do not exist those hollow modules as shown in the beginning. Hence $i = 1$ (cf. the proof of Lemma 12 in [4]). Conversely assume the above structure. Making use of Theorem 17, we can compute all types of maximal submodules of direct sum of two hollow modules, for example, non trivial maximal submodules of $(eR/(D_i \oplus A_2) \oplus eR/(D_j \oplus A_2))$ is isomorphic to $eR/D_j \oplus A_1/D_i$, where the $A_i$ and $D_k$ are in the structure 2) and $i \leq j$. Hence $(\ast, 2)$ is fulfilled.
References


Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558
Japan