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GAP THEOREMS FOR CERTAIN SUBMANIFOLDS OF EUCLIDEAN SPACES AND HYPERBOLIC SPACE FORMS

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Introduction

Simons [18] studied minimal submanifolds of spheres and showed, among other things, that a compact minimal submanifold M of the unit n-sphere must be totally geodesic if the square length of the second fundamental form is less than $n/(2-p^{-1})$ ($p=\operatorname{codim} M$) (cf. [13] for the equality discussion). Later, Ogiue [16] and Tanno [20] considered complex submanifolds in the complex projective space and obtained similar results to the Simons' theorem (cf. [17] for other related topics and the references). On the other hand, Greene and Wu [9] have proven a gap theorem for noncompact Riemannian manifolds with a pole (cf. [7] [10] [14]). Roughly speaking, their theorem says that a Riemannian manifold with a pole whose sectional curvature goes to zero in farster than quadratic decay is isometric to Euclidean space if its dimension is greater than two and the curvature does not change its sign. These gap theorems suggest that one could expect similar results for certain open submanifolds of Euclidean space, the hyperbolic space form, the complex hyperbolic space form, etc.. Actually in this note, we shall prove the following theorems.

Theorem A.

(I) *Let M be a connected, minimal submanifold of dimension m properly immersed into Euclidean space $\boldsymbol{R}^n$. Let $\bar{\rho}$ denote the distance in R^n to a fixed point of R^n. Then M is totally geodesic if one of the following conditions holds:*

(A-i) *$m\geqq 3$, M has one end and the second fundamental form α_M of the immersion $M\rightarrow\boldsymbol{R}^n$ satisfies*

$$\limsup \bar{\rho}(x)\,|\,\alpha_M|\,(x)<\kappa_0<1\,,$$

where κ_0 is defined by $\kappa_0\{(1-\kappa_0^2)^{-1}+1\}=\sqrt{2}$.

(A-ii) *$m=2$, M has one end and*

$$\sup \bar{\rho}^2(x)\,|\,\alpha_M|\,(x)<+\infty\,.$$

(A-iii) *$2m>n$, M is imbedded and*

$$\sup \bar{\rho}^{\varepsilon}(x)\,|\alpha_M|(x)<+\infty$$

for some constant $\varepsilon>m$.

(II) *Let* M *be a connected, minimal submanifold of dimension* m *properly immersed into the hyperbolic space form* $H^n(-1)$ *of constant curvature* -1. *Let* $\bar{\rho}$ *denote the distance in* $H^n(-1)$ *to a fixed point of* $H^n(-1)$. *Then* M *is totally geodesic if one of the following conditions holds:*

(A-iv) $m\geqq 3$, M *has one end and the second fundamental form* α_M *of the immersion* $M\to H^n(-1)$ *satisfies*

$$\sup \bar{\rho}^{\varepsilon}(x)e^{\bar{\rho}(x)}\,|\alpha_M|(x)<+\infty$$

for some constant $\varepsilon>1$.

(A-v) $m=2$, M *has one end and*

$$e^{2\bar{\rho}(x)}\,|\alpha_M|(x)\to 0$$

as $x\in M$ *goes to infinity.*

(A-vi) $m=n-1$, M *is imbedded and*

$$e^{m\bar{\rho}(x)}\,|\alpha_M|(x)\to 0$$

as $x\in M$ *tends to infinity.*

Theorem B.

(I) *Let* M *be a connected, noncompact Riemannian submanifold of dimension* m *properly immersed into* R^n. *Suppose that* M *has one end and the second fundamental form* α_M *of the immersion* $M\to R^n$ *satisfies*

$$\sup \bar{\rho}^{\varepsilon}(x)\,|\alpha_M|(x)<+\infty$$

for a constant $\varepsilon>2$. *Then* M *is totally geodesic if* $2m>n$ *and the sectional curvature is nonpositive everywhere on* M, *or if* $m=n-1$ *and the scalar curvature is nonpositive everywhere on* M.

(II) *Let* M *be a connected, noncompact Riemannian submanifold of dimension* m *properly immersed into* $H^n(-1)$. *Suppose that* M *has one end and*

$$e^{2\bar{\rho}(x)}\,|\alpha_M|(x)\to 0$$

as $x\in M$ *goes to infinity. Then* M *is totally geodesic if* $2m>n$ *and the sectional curvature is everywhere less than or equal to* -1 *or if* $m=n-1$ *and the scalar curvature is everywhere less than or equal to* $-m(m-1)$.

(III) *Let* M *be a connected hypersurface of* $H^n(-1)$ *which bounds a totally convex domain* D *of* $H^n(-1)$. *Then* M *is a totally geodesic, provided that*

$$e^{\bar{\rho}(x)}\,|\alpha_M|(x)\to 0$$

as $x \in M$ tends to infinity.

Theorem C. *Let M be a connected, complex submanifold properly immersed into the complex hyperbolic space form $CH^n(-1)$ of constant holomorphic sectional curvature -1. Then M is totally geodesic if the second fundamental form α_M of M satisfies*

$$e^{3\bar{\rho}(x)/2}|\alpha_M|(x) \to 0$$

as $x \in M$ goes to infinity.

REMARKS.

(1) The first part (I) of Theorem A with a stronger condition instead of (A-i) has been proved in [11] and a few examples are given there to illustrate the roles of several hypotheses on M. We should also mention the recent paper of Anderson [3] in which he has investigated complete minimal submanifolds in $\boldsymbol{R}^n$ of finite total scalar curvature. Especially as a consequence derived from his main theorem, which is a generalization of the well known Chern-Osserman theorem on minimal surfaces in R^n of finite total curvature, he shows that a complete minimal submanifold M immersed into R^n is an affine m-space if $m=\dim M \geqq 3$, M has one end and the total scalar curvature: $\int_M |\alpha_M|^m$ is finite, where α_M denotes as before the second fundamental form of M. Moreover the proof of his main theorem suggests that for a complete minimal submanifold M of dimension $m \geqq 3$ immersed into R^n, the immersion is proper and $|\alpha_M| \leqq c/|x|^m$ for some positive constant c if the total scalar curvature is finite. It is easy to see that the total scalar curvature is finite if the immersion is proper and $|\alpha_M| \leqq c/|x|^\varepsilon$ for some constants c and $\varepsilon > 1$ (cf. Section 1).

(2) Recall that $H^n(-1)$ has a natural smooth compactification $\overline{H^n(-1)} = H^n(-1) \cup S(\infty)$ where $S(\infty)$ can be identified with asymptotic classes of geodesic rays in $H^n(-1)$. In [1], Anderson has proved that any closed $(m-1)$-dimensional submanifold $M(\infty)$ of $S(\infty)$ is the asymptotic boundary of a complete, absolutely area-minimizing locally integral m-current M in $H^n(-1)$. As is noted in [1], M is smooth in case $m=n-1 \leqq 6$. It would be interesting to investigate the curvature behavior of his solution M in relation with the 'regularity' of $\bar{M} = M \cup M(\infty)$. One should also consult the recent paper of do Carmo and Lawson [5] for a related result to Theorem A (II).

(3) In [15], Mori constructed a family of complete minimal surfaces μ_λ: $S^1 \times R \to H^3(-1)$. The second fundamental forms α_λ of these embedded surfaces $M_\lambda = \mu_\lambda(S^1 \times R)$ have the property that $|\alpha_\lambda|(x) \sim \exp -2\bar{\rho}(x)$, where $\bar{\rho}$ is the distance in $H^3(-1)$ to a fixed point. Note also that M has two ends.

(4) The last part (III) of Theorem B is concerning the boundary of a totally convex domain in $H^n(-1)$. When we replace R^n for $H^n(-1)$, we have

a similar result (cf. [19]).

(5) Let $\bar{\Sigma}$ be a compact smooth surface with genus>1 and set $\Sigma=\bar{\Sigma}\setminus\{p\}$ ($p\in\bar{\Sigma}$). In the last section of this paper, we shall construct a proper embedding of Σ into R^3 such that the second fundamental form α_Σ satisfies: $|\alpha_\Sigma|(x)\leqq c|x|^{-2}$ and further the Gaussian curvature with respect to the induced metric is everywhere nonpositive (cf. the first part (I) of Theorem B).

(6) In the first part (I) (resp. the second part (II)) of Theorem B, the conditions on the dimension and the curvature of M can be replaced with the following weaker assumption: for every $x\in M$, there is a subspace T of T_xM such that dim $T>n-m$ and the sectional curvature for any plane in T is nonpositive (resp. less than or equal to -1) (cf. the proof of Theorem B in Section 2).

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1. Preliminaries

This section presents a number of lemmas to prove the results stated in Introduction.

1.1. Throughout this section, H denotes a complete, simply connected Riemannian manifold of dimension n whose sectional curvature K_H satisfies

$$-b^2\leqq K_H\leqq -a^2$$

($0\leqq a\leqq b$). We write $S(\infty)$ for the asymptotic classes of geodesic rays in H. Recall that $\bar{H}=H\cup S(\infty)$ has a natural topology which makes H homeomorphic to an n-cell (cf. [6]). In particular, in case of $H=$ the hyperbolic space form $H^n(-a^2)$ of constant negative curvature $-a^2(\neq 0)$ or $H=$the complex hyperbolic space form $CH^n(-a^2)$ of constant holomorphic sectional curvature $-a^2(\neq 0)$, $\bar{H}$ is a natural smooth compactification of H. For any subset A of H, $A(\infty)$ stands for the asymptotic boundary of A, namely, the intersection of the closure $\bar{A}$ in $\bar{H}$ with $S(\infty)$. (As it is pointed out in [5], this concept of asymptotic boundary seems crucial in understanding certain noncompact submanifolds of H.) Observe that for any point o of H, $S(\infty)$ can be naturally identified with the geodesic sphere $S(t)$ of radius t around o through the exponential map at o. Throughout the paper, such identification will be often used to investigate the behavior at infinity of certain noncompact submanifolds of H.

Throughout the paper, we write $J_a(t)$ for the solution of equation: $J_a''-a^2J_a=0$, subject to the initial conditions $J_a(0)=0$, $J_a'(0)=1$, namely, $J_a(t)=t$ if $a=0$, and $J_a(t)=a^{-1}\sinh at$ if $a>0$.

Let us now begin with stating the following well known fact.

Lemma 1 (cf. e.g. [8]). *Fix a point o of H and denote by $\bar{\rho}$ the distance to o. Then the hessian $\bar{\nabla}^2\bar{\rho}$ satisfies*

$$\bar{\nabla}^2\bar{\rho}(X, X) \geqq (J'_a/J_a)\circ\bar{\rho}\{\langle X, X\rangle - \langle\bar{\nabla}\bar{\rho}, X\rangle^2\}$$

on $H-\{o\}$.

The next lemma is an immediate consequence of Toponogov comparison theorem.

Lemma 2. *Consider a geodesic triangle Δ in H with vertices p_1, p_2, p_3. Then*

$$(\sin\Sigma)^2 \leqq \frac{2(\cosh al_1 - 1)}{\sinh al_2 \sinh al_3},$$

where Σ is the angle of Δ at p_1, $l_1 = \mathrm{dis}_H(p_2, p_3)$, $l_2 = \mathrm{dis}_H(p_1, p_3)$ and $l_3 = \mathrm{dis}_H(p_1, p_2)$.

Lemma 3. *Let $c(s)$ be an arc-length parametrized smooth curve in H. Define a curve $\gamma_c(s)$ by $\gamma_c(s) = \exp_{c(s)} - s\dot{c}(s)$, where $\exp_x$ stands for the exponential map of H at $x \in H$. Then*

$$\mathrm{dis}_H(\gamma_c(s), \gamma_c(t)) \leqq \int_s^t J_b(u)\,|\bar{\nabla}_{\dot{c}}\dot{c}|(u)du \qquad (s<t)$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on H.

Proof. Since $|\dot{\gamma}_c(s)| \leqq J_b(s)\,|\bar{\nabla}_{\dot{c}}\dot{c}|(s)$ by the well known comparison theorem on Jacobi fields, we have the above inequality.

1.2. We shall now consider a connected Riemannian submanifold M of dimension m immersed into H. We write α_M: $TM\times TM \to TM^{\perp}$ for the second fundamental form of the immersion $M\to H$ (i.e., $\alpha_M(X, Y) = \bar{\nabla}_X Y - \nabla_X Y$, where ∇ denotes the Levi-Civita connection of M). Fix a point o of H and set $\bar{\rho}(x) := \mathrm{dis}_H(o, x)$, $\rho := \bar{\rho}_{|M}$, $\nu_\rho := \bar{\nabla}\rho - \nabla\rho$, $M(t) := \{x \in M\colon \rho(x) = t\}$, $M^t := \{x \in M\colon \rho(x) \geqq t\}$ and $M_t := \{x \in M\colon \rho(x) \leqq t\}$. In this subsection, some estimates for $|\nu_\rho|$ will be given under certain conditions on H and M.

Lemma 4. *Suppose the gradient $\nabla\rho$ of ρ never vanishes on a connected component $\Gamma(t)$ of $M(t)$. Then:*

$$\begin{aligned}&||\nu_\rho|(x) - |\nu_\rho|(y)| \leqq \\ &\quad 2D(\Gamma(t))\max\{|\alpha_M(X, \nabla\rho)| + |\bar{\nabla}^2\bar{\rho}(X, \nabla\rho)| : X \in T\Gamma(t),\ |X| = 1\}\end{aligned}$$

for any $x, y \in \Gamma(t)$, where $D(\Gamma(t))$ stands for the intrinsic diameter of $\Gamma(t)$.

Moreover $\bar{\nabla}^2\bar{\rho}(X, \nabla\rho)$ *in the above inequality vanishes when* $H=H^n(-a^2)$, *or* $H=CH^n(-a^2)$ *and* M *is a complex submanifold.*

Proof. Let $\gamma\colon (0, d]\to\Gamma(t)$ be an arc-length parametrized smooth curve in $\Gamma(t)$ which joins x to y. Then we have

$$\begin{aligned}\frac{d}{ds}|\nu_\rho|^2(\gamma(s)) &= \frac{d}{ds}[1-|\nabla\rho|^2(\gamma(s))]\\ &= -2\langle\nabla_{\dot\gamma}\nabla\rho, \nabla\rho\rangle\\ &= -2\nabla^2\rho(\dot\gamma, \nabla\rho)\\ &\leqq 2[|\alpha_M(\dot\gamma, \nabla\rho)|\,|\nu_\rho|+\bar{\nabla}^2\bar{\rho}(\dot\gamma, \nabla\rho)|]\end{aligned}$$

This shows the inequality of the lemma.

In what follows, we assume that M is noncompact and the immersion $M\to H$ is proper. Let us take a nonnegative continuous function $k(t)$ such that $|\alpha_M|\leqq k\circ\rho$ on M. Suppose that for some nonnegative constants τ and $c_1<1$,

$$k(t)\leqq c_1 J_a'(t)/J_a(t) \tag{1.1}$$

on $[\tau, \infty)$. Then, since the hessian $\nabla^2\rho$ of ρ satisfies

$$\begin{aligned}\nabla^2\rho(X, X) &= \bar{\nabla}^2\bar{\rho}(X, X)+\langle\alpha_M(X, X), \nu_\tau\rangle\\ &\geqq (J_a'/J_a)\circ\rho\{\langle X, X\rangle-\langle\bar{\nabla}\bar{\rho}, X\rangle^2\}-k\circ\rho|\nu_\rho|\langle X, X\rangle\\ &\geqq (J_a'/J_a)\circ\rho\{(1-c_1)\langle X, X\rangle-\langle\bar{\nabla}\bar{\rho}, X\rangle^2\}\end{aligned}$$

we see that, for a suitable smooth function $F(t)$ (e.g., $F(t)=\int_0^t \exp\int_0^s k(u)+1\,du\,ds$), $F\circ\rho$ is strictly convex on M^τ, so that we may assume $\nabla\rho$ never vanishes on M^τ. Thus M turns out to be diffeomorphic to the interior of an m-dimensional compact manifold N with boundary ∂N. The intersection of a closed regular neighborhood of a component of ∂N with M will be called an *end* of M (which agrees with the usual topological meaning). We shall now define a smooth vector field V_ρ on M^τ by $V_\rho=\nabla\rho/|\nabla\rho|^2$. Then

$$\begin{aligned}V_\rho\cdot|\nu_\rho|^2 &= V_\rho\cdot(1-|\nabla\rho|^2)\\ &= -2\nabla^2\rho(\nabla\rho, \nabla\rho)|\nabla\rho|^{-2}\\ &\leqq -2(J_a'/J_a)\circ\rho|\nu_\rho|^2+2|\alpha_M(\nabla\rho, \nabla\rho)|\,|\nabla\rho|^{-2}|\nu_\rho|\end{aligned}$$

on M^τ. Hence we obtain

$$\begin{aligned}V_\rho\cdot(J_a^2\circ\rho|\nu_\rho|^2) &\leqq 2|\alpha_M(\nabla\rho, \nabla\rho)|\,|\nabla\rho|^{-2}|\nu_\rho|J_a^2\circ\rho\\ &\leqq 2k\circ\rho|\nu_\rho|J_a^2\circ\rho\,.\end{aligned} \tag{1.2}$$

Let x be a point of $M(\tau)$ and $v_x(t)$ ($t\in[\tau, \infty)$) the maximal integral curve of V_ρ

such that $v_x(\tau)=x$. Then it follows from (1.2) that

$$J_a(t)|\nu_\rho|(v_x(t))\leqq J_a(\tau)|\nu_\rho|(x)+\int_\tau^t k(u)J_a(u)du\,.$$

Thus we have the following

Lemma 5.

(i) $(a=0)$ $\limsup_{x\in M\to+\infty}|\nu_\rho|(x)\leqq c$ *if* $\limsup_{x\in M\to+\infty}\rho(x)|\alpha_M|(x)\leqq c$ *for some constant* $c\geqq 0$.

(ii) $(a=0)$ $\sup_M \rho^\delta|\nu_\rho|<+\infty$, $\delta=\min\{\varepsilon-1,1\}$ *if* $\sup_M \rho^\varepsilon|\alpha_M|<+\infty$ *for some constant* ε: $1<\varepsilon<2$ *or* $2<\varepsilon$.

(iii) $(a>0)$ $\sup_M \rho^\varepsilon|\nu_\rho|<+\infty$ *if* $\sup_M \rho^\varepsilon|\alpha_M|<+\infty$ *for some constant* $\varepsilon>0$.

(iv) $(a>0)$ $\sup_M e^{a\rho}|\nu_\rho|<+\infty$ *if* $\sup_M \rho^\varepsilon e^{a\rho}|\alpha_M|<+\infty$ *for some constant* $\varepsilon>1$.

1.3. In this subsection, we shall observe the behavior of $M(t)$ for large t *under assumption* (1.1). Let $v_x(t)$ be as in the preceding subsection and define a map μ_t: $M(\tau)\to M(t)$ by $\mu_t(x)=v_x(t)$ $(t\geqq\tau)$. Then μ_t gives a diffeomorphism from $M(\tau)$ onto $M(t)$. Let $\gamma(s)$ be a smooth regular curve in $M(\tau)$ and set $W(s,t)=\partial v_{\gamma(s)}(t)/\partial s$. Then using Lemma 1, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\log|W|&=|\nabla\rho|^{-2}|W|^{-2}\nabla^2\rho(W,W)\\&=|\nabla\rho|^{-2}|W|^{-2}\{\overline{\nabla}^2\bar\rho(W,W)+\langle\alpha_M(W,W),\nu_\gamma\rangle\}\\&\geqq|\nabla\rho|^{-2}\left\{\frac{d}{dt}\log J_a(t)-|\alpha_M|(v_{\gamma(s)}(t))|\nu_\rho|(v_{\gamma(s)}(t))\right\}\\&\geqq\frac{d}{dt}\log J_a(t)-k(t)|\nu_\rho|(v_{\gamma(s)}(t))\,.\end{aligned}$$

This implies

$$\frac{|W(s,t)|}{|W(s,\tau)|}\geqq\frac{J_a(t)}{J_a(\tau)}\exp\int_\tau^t -k|\nu_\rho|$$

and hence

$$\frac{|\mu_{t*}X|}{J_a(t)}\geqq\frac{|X|}{J_a(\tau)}\exp\int_\tau^t -k|\nu_\rho| \tag{1.3}$$

for any $X\in TM(\tau)$. Similarly, we obtain

$$\frac{|\mu_{t*}X|}{J_b(t)}\leqq\frac{|X|}{J_b(\tau)}\exp\int_\tau^t\frac{(\log J_b)'|\nu_\rho|^2+k|\nu_\rho|}{1-|\nu_\rho|^2} \tag{1.4}$$

for every $X \in TM(\tau)$. As an immediate consequence of (1.4) and Lemma 5, we have the following

Lemma 6. *Suppose $a>0$ and $\rho^{\gamma}|\alpha_M|$ is bounded on M for a constant $\varepsilon>\frac{1}{2}$. Then:*

$$c^{-1}J_a(t) \leqq D(\Gamma(t)) \leqq cJ_b(t) \qquad (t \geqq \tau),$$

where $\Gamma(t)$ is a connected component of $M(t)$ and $c>1$ is a constant independent of t.

Let us now denote by $S(t)$ (resp. Π_t) the geodesic sphere of H around o with radius t (resp. the projection from $S(t)$ onto $S(1)$ along the geodesics joining $S(t)$ to o). We define a family of immersions $\{\phi_t\}_{t\geqq\tau}$ from $M(\tau)$ into $S(1)$ by $\phi_t = \Pi_t \circ \mu_t$. Then for each $x \in M(\tau)$, the smooth curve $\gamma_x \colon t \to \phi_t(x)$ in $S(1)$ satisfies

$$|\dot{\gamma}(t)| \leqq (|\nabla\rho|^{-1}|\nu_\rho|)(\mu_t(x))/J_a(t).$$

Set $\phi_\infty(x) = \lim_{t\to+\infty}\phi_t(x)$ if the limit exists. This is the case for all $x \in M(\tau)$, if $a>0$, or if $a=0$ and $\rho^{\varepsilon}|\alpha_M|$ is bounded for some constant $\varepsilon>1$ (cf. Lemma 5 (ii)). Moreover $\phi_\infty \colon M(\tau) \to S(1)$ is continuous. In fact,

$$\begin{aligned}\mathrm{dis}_{S(1)}(\phi_\infty(x), \phi_\infty(y)) &\leqq \mathrm{dis}_{S(1)}(\phi_\infty(x), \phi_t(x)) + \mathrm{dis}_{S(1)}(\phi_t(x), \phi_t(y)) \\ &\quad + \mathrm{dis}_{S(1)}(\phi_t(y), \phi_\infty(y)) \\ &\leqq \int_t^\infty \eta(s)ds + \frac{J_b(t)}{J_a(t)}\,\mathrm{dis}_{M(\tau)}(x, y),\end{aligned}$$

where η is a positive continuous function such that $t^{\varepsilon}\eta(t)$ (resp. $e^{at}\eta(t)$) is bounded in case of $a=0$ (resp. $a>0$). Observe further that

$$\begin{aligned}\mathrm{dis}_H(\mu_t(x), \exp_o t\phi_\infty(x)) &\leqq J_b(t)\int_t^\infty |\dot{\gamma}_x| \\ &\leqq J_b(t)\int_t^\infty \frac{|\nabla\rho|^{-1}|\nu_\rho|}{J_a},\end{aligned} \tag{1.5}$$

where $\exp_o$ denotes the exponential map of H at o and $S(1)$ is identified with the unit sphere of the tangent space T_oH at o.

REMARK. We see from (1.3) and (1.4) that $\phi_\infty \colon M(\tau) \to S(1)$ is a Lipschitz map if $a=b>0$ and $\rho^{\varepsilon}|\alpha_M|$ is bounded on M for a constant $\varepsilon>\frac{1}{2}$. Moreover making use of the Poincare model or the upper half-plane model for $H^n(-1)$, we can prove the following

Proposition 1. *Let M be a connected, connected, noncompact Riemannian submanifold properly immersed into $H^n(-1)$. Suppose $\rho^{\varepsilon}|\alpha_M|$ is bounded on M*

for a constant $\varepsilon>1$. *Then* $\bar{M}=M\cup M(\infty)$ *is a* C^1*-submanifold immersed into* $\bar{H}^n(-1)=H^n(-1)\cup S(\infty)$ *(a natural smooth compactification of* $H^n(-1)$*).*

The proof of this proposition will be given at the end of Section 4.

1.4. Now we consider the case $H=R^n$ or $H^n(-1)$. We shall compute the second fundamental form α_t of the immersion $\phi_t\colon M(\tau)\to S(1)$ and derive some lemmas to prove Theorems A and B. Observe first that the second fundamental form β_t of the immersion $M(\tau)\to S(t)$ is given as follows:

$$\langle\beta_t(X, Y), \nu\rangle=|\nabla\rho|^{-2}\{\nabla^2\rho(X, Y)+\langle\alpha_M(X, Y), \nu_\rho\rangle\}\langle\nu_\rho, \nu\rangle+\langle\alpha_M(X, Y), \nu\rangle\,,$$

where X, $Y\in TM(t)$, $\nu\in TM(t)^\perp\cap TS(t)$, and $TM(t)^\perp$ denotes the orthogonal complement of $TM(t)$ in TH. This is an immediate consequence of Gauss formula. Suppose now $H=R^n$. Then for any $x\in M(\tau)$ and every X, $Y\in T_xM(\tau)$, we have

$$\begin{aligned}\langle\alpha_t(X, Y), \nu\rangle_x&=\langle\alpha_t(\phi_{t*}X, \phi_{t*}Y), \nu\rangle && \text{at}\quad \phi_t(x)\\ &=\langle\beta_1(\hat{X}, \hat{Y}), \hat{\nu}\rangle && \text{at}\quad \mu_t(x)\\ &=|\nabla\rho|^{-1}\{t^{-1}\langle\hat{X}, \hat{Y}\rangle-\langle\alpha_M(\hat{X}, \hat{Y}), \hat{\nu}\rangle\}\langle\nu_\rho, \hat{\nu}\rangle\\ &\quad+\langle\alpha_M(\hat{X}, \hat{Y}), \hat{\nu}\rangle && \text{at}\quad \mu_t(x)\,,\end{aligned}$$

where $\hat{X}=\mu_{t*}X$, $\hat{Y}=\mu_{t*}Y$ and $\hat{\nu}=t^{-2}\,\Pi_{t*}^{-1}\nu$. This implies that

$$|\alpha_t|(x)\leqq(1+|\alpha_M|)|\nu_\rho|\,|\nabla\rho|^{-2}+t|\alpha_M|\qquad \text{at}\quad \mu_t(x)\ (=\exp_o t\phi_t(x))\,. \tag{1.6}$$

Similarly, in case of $H=H^n(-1)$, we obtain

$$|\alpha_t|(x)\leqq\Big(\frac{\cosh t}{\sinh 1}+|\alpha^M|\Big)|\nu_\rho|\,|\nabla\rho|^{-2}+\frac{\sinh t}{\sinh 1}|\alpha_M| \tag{1.7}$$
$$\text{at}\quad \mu_t(x)\ (=\exp_o t\phi_t(x))\,.$$

Making use of these inequalities (1.6) and (1.7), we can prove the following two lemmas.

Lemma 7. *Let* M *be a connected, noncompact Riemannian submanifold properly immersed into* R^n. *Suppose that* $\bar{\rho}(x)|\alpha_M|(x)$ *goes to* 0 *as* $x\in M\to\infty$, *where* $\bar{\rho}(x)$ *stands for the distance in* R^n *between* $x\in M$ *and a fixed point* $o\in R^n$. *Then for each end* M_j *of* M $(j=1, \cdots, k)$, $\phi_\infty(M_j(\tau))$ *is a totally geodesic* $(m-1)$*-subsphere* S_j^{m-1} *of the unit sphere* $S(1)$ *of* R^n, *where* k *denotes the number of the ends of* M, $m=\dim M$ *and* $M_j(\tau)$ *is the connected component of* $M(\tau)$ *which corresponds to* M_j. *Moreover let* P_j *be the m-plane of* R^n *such that* $P_j\cap S(1)=S_j^{m-1}$. *Then if* $\bar{\rho}^\varepsilon(x)|\alpha_M|(x)$ *is bounded on* M_i *for some constant* $\varepsilon>2$,

$$\mathrm{dis}_{R^n}(x, P_j)\leqq c\bar{\rho}^{2-\varepsilon}(x)$$

on M_j, where c is a positive constant.

Lemma 8. *Let M be a connected, noncompact Riemannian submanifold properly immersed into $H^n(-1)$. Suppose $e^{\varepsilon\bar{\rho}(x)}|\alpha_M|(x)$ tends to 0 as $x\in M$ for some constant $\varepsilon\geqq 2$, where $\bar{\rho}(x)$ denotes the distance to a fixed point o of $H^n(-1)$. Then for any end M_j of M ($j=1, \cdots, k$), $\phi_\infty(M_j(\tau))$ is an $(m-1)$-subphere S_j^{m-1} of the unit sphere $S(1)$ of $H^n(-1)$, where $M_j(\tau)$ is the connected component of $M(\tau)$ corresponding to M_j. Let H_j be the totally geodesic submanifold of $H^n(-1)$ such that $H_j(\infty)=S_j^{m-1}$, where $S(1)$ is identified with the points at infinity $S(\infty)$ of $H^n(-1)$ through the exponential map at o. Then*

$$\mathrm{dis}_{H^n(-1)}(x, H_j)e^{(\varepsilon-1)\bar{\rho}(x)} \to 0$$

as $x\in M_j$ goes to infinity. Moreover there is a distance minimizing geodesic ray $\sigma(s)$ ($s\geqq 0$) of M_j such that

$$\mathrm{dis}_{H^n(-1)}(\sigma(s), H_j)e^{\varepsilon\bar{\rho}(\sigma(s))} \to 0$$

as s tends to infinity.

We shall give only the proof of Lemma 8, the same argument as in which will derive Lemma 7.

Proof of Lemma 8. Let $\Gamma(\tau)$ be a connected component of $M(\tau)$ and set $\Gamma^\tau=\{\mu_t(x): x\in\Gamma(\tau), t\geqq\tau\}$, where μ_t: $M(\tau)\to M(t)$ is as in Subsection 1.3. Let us take a distance minimizing geodesic ray $\sigma(s)$ ($s\geqq\tau$) of Γ^τ such that $\mathrm{dis}_M(\sigma(s), \Gamma(\tau))=s-\tau$. Then we have

$$s\leqq c_1+\rho(\sigma(s))\,, \tag{1.8}$$

where c_1 is a positive constant independent of σ. In fact, let $v_s(t)$ ($t\geqq\tau$) be the maximal integral curve of the vector field $V_\rho=\nabla\rho/|\nabla\rho|^2$ such that $v_s(\rho(\sigma(s)))\sigma=(s)$. Then

$$\begin{aligned} s-\tau &\leqq\int_\tau^{\rho(\sigma(s))} |\dot{v}_s|(u)du \\ &\leqq \rho(\sigma(s))-\tau+\int_\tau^\infty(|\nabla\rho|^{-1}(v_s(u))-1)du \\ &= \rho(\sigma(s))-\tau+\int_\rho^\infty|\nabla\rho|^{-2}(v_s(u))|\nu_\rho|^2(v_s(u))du\,. \end{aligned}$$

This proves (1.8) (cf. Lemma 5 (iii)). Now we define a smooth curve $\gamma_\sigma(s)$ by $\gamma_\sigma(s)=\exp_{\sigma(s)}-s\dot{\sigma}(s)$, where $\exp_{\sigma(s)}$ denotes the exponential map of $H^n(-1)$ at $\sigma(s)$. Then it follows from Lemma 3, the assumption on α_M of Lemma 8 and (1.8) that

$$\begin{aligned}\mathrm{dis}_{H^n(-1)}(\gamma_\sigma(s),\ \gamma_\sigma(t)) &\leqq \int_s^t \sinh u\,|\alpha_M|(\sigma(u))du \\ &\leqq \int_s^t \eta(\rho(\sigma(u)))e^{(1-\varepsilon)\rho(\sigma(u))}du \\ &\leqq \int_s^t \eta(u-c_1)e^{(1-\varepsilon)(u-c_1)}du\,,\end{aligned}$$

where $\eta(t)>0$ is a monotone non-increasing function such that $\eta(t)$ goes to 0 as $t\to+\infty$. Hence $\gamma_\sigma(s)$ converges to a point $\hat{o}$ of $H^n(-1)$ and we have

$$\mathrm{dis}_{H^n(-1)}(\gamma_\sigma(s),\ \hat{o})\leqq\int_s^\infty \eta(u)e^{(1-\varepsilon)u}du\,. \tag{1.9}$$

In what follows, we take this point $\hat{o}$ as a reference point instead of the previous fixed point o. Then considering the geodesic triangle Δ in $H^n(-1)$ with vertices $\sigma(s)$, $\gamma_\sigma(s)$, $\hat{o}$ and applying Lemma 2 to Δ, we obtain by (1.9)

$$|\nu_\rho|(\sigma(s))\leqq c_2e^{-s}\int_s^\infty \eta(u)e^{(1-\varepsilon)u}du\,, \tag{1.10}$$

where c_2 is a positive constant. Now (1.10) and Lemma 4 imply that for any point x of $\Gamma(t)=M(t)\cap\Gamma^\tau$ $(t=\rho(\sigma(s)))$,

$$\begin{aligned}|\nu_\rho|(x)\leqq c_2e^{-s}&\int_s^\infty \eta(u)e^{(1-\varepsilon)u}du \\ &+D(\Gamma(t))\max\{|\alpha_M(X,\nabla\rho)|: X\in T\Gamma(t),\ |X|=1\}\end{aligned}$$

Since $|\rho(\sigma(s))-s|\leqq c_1$ and $D(\Gamma(t))\leqq c_3e^t$ for some positive constant c_3(cf. Lemma 6), we have

$$\max_{x\in\Gamma(t)}|\nu_\rho|(x)\leqq\hat{\eta}(t)e^{(1-\varepsilon)t}\,, \tag{1.11}$$

where $\hat{\eta}(t)>0$ is a monotone non-increasing continuous function such that $\hat{\eta}(t)\to 0$ as $t+\to\infty$. It turns out from (1.11) and (1.7) that the second fundamental form α_t of the immersion ϕ_t: $\Gamma(\tau)\to S(1)$ goes to zero as $t\to+\infty$, where ϕ_t is as in Subsection 1.3 (with respect to $\hat{o}$). That is, $\phi_\infty(\Gamma(\tau))$ is a totally geodesic $(m-1)$-subsphere S^{m-1} of $S(1)$, (This shows the first assertion of Lemma 8.) Let H be the totally geodesic submanifold of $H^n(-1)$ through o such that $H\cap S(1)=S^{m-1}$. Then by (1.5) and (1.11), we see that

$$\begin{aligned}\mathrm{dis}_{H^n(-1)}(x,\ H)&\leqq\sinh\rho(x)\int_{\rho(x)}^\infty\hat{\eta}(u)e^{(1-\varepsilon)u}(\sinh u)^{-1}du \\ &\leqq(\varepsilon-1)^{-1}\hat{\eta}(\rho(x))e^{(1-\varepsilon)\rho(x)}\end{aligned}$$

on Γ^τ. Moreover it follows from (1.5) and (1.10) that

$$\mathrm{dis}_{H^n(-1)}(\sigma(s), H) \leqq c_4 \hat{\eta}(\rho(\sigma(s)))e^{-\varepsilon\rho(\sigma(s))}$$

for some positive constant c_4. This completes the proof of Lemma 8.

1.5. It remains to consider the case: $H=CH^n(-1)$. In this subsection, we shall compute the second fundamental form α_t of the immersion ϕ_t: $M(\tau)\to S(1)$ and show an analogue to Lemmas 7 and 8. We assume that M is a complex submanifold properly immersed into $CH^n(-1)$ satisfying (1.1). Set $\bar{\mathcal{H}}=\{X\in TCH^n(-1)\colon \langle X, \bar{\nabla}\bar{\rho}\rangle=\langle X, J\bar{\nabla}\bar{\rho}\rangle=0\}$ and $\mathcal{H}=\{X\in TM\colon \langle X, \nabla\rho\rangle=\langle X, J\nabla\rho\rangle=0\}$, where J stands for the complex structure of $CH^n(-1)$. Clearly $\mathcal{H}=TM\cap\bar{\mathcal{H}}$. For $X\in TCH^n(-1)$, we denote by X^r (resp. X^h) the $\{\nabla\rho, J\nabla\rho\}$-component of X (resp. the $\mathcal{H}$-component of X), namely, $X^r=\langle X, \nabla\rho\rangle\nabla\rho+\langle X, J\nabla\rho\rangle J\nabla\rho$ and $X^h=X-X^r$. For each $t>0$, we put $\tilde{g}_t(X, Y)=\langle\Pi_{t*}X, \Pi_{t*}Y\rangle$, $\tilde{g}^r_t(X, Y)=\tilde{g}_t(X^r, Y^r)$ and $\tilde{g}^h_t(X, Y)=\tilde{g}_t(X^h, Y^h)$, where X, $Y\in TS(t)$ and Π_t: $S(t)\to S(1)$ is the projection from $S(t)$ onto $S(1)$ along the geodesics issuing at o. Then we have

$$(1.12)\qquad \begin{aligned}\langle X^r, Y^r\rangle &= \left[\frac{\mathrm{sihn}\ t}{\sinh 1}\right]^2 \tilde{g}^r_t(X, Y)\\ \langle X^h, Y^h\rangle &= \left[\frac{\sinh t/2}{\sinh 1/2}\right]^2 \tilde{g}^h_t(X, Y)\,.\end{aligned}$$

Note here that

$$(1.13)\qquad \bar{\nabla}^2\bar{\rho}(X, Y) = \coth t\langle X^r, Y^r\rangle + \frac{1}{2}\coth t/2\langle X^h, Y^h\rangle$$

for any X, $Y\in TS(t)$. Then the Levi-Civita connection $\tilde{\nabla}$ on $S(t)$ with respect to $\tilde{g}_t$ is given by

$$(1.14)\qquad \begin{aligned}\tilde{g}_t(\tilde{\nabla}_X Y, \tilde{\nu}) = {}&\langle\bar{\nabla}_X Y, \nu\rangle\\ &+\frac{1}{2}\coth t/2\left\{\left[\frac{\cosh 1/2}{\cosh t/2}\right]^2-1\right\}\\ &(\langle X, J\nabla\rho\rangle\langle JY, \nu\rangle+\langle Y, J\nabla\rho\rangle\langle JX, \nu\rangle)\end{aligned}$$

where X, Y, $\tilde{\nu}\in TS(t)$ and

$$\nu = \left[\frac{\sinh 1}{\sinh t}\right]^2\tilde{\nu}^r + \left[\frac{\sinh 1/2}{\sinh t/2}\right]^2\tilde{\nu}^h\,.$$

This is an immediate consequence of (1.12), (1.13) and the definition of Levi-Civita connection: the Levi-Civita connection ∇ on a Riemannian manifold (N, g) is given by

$$\begin{aligned}g(\nabla_X Y, Z) = \frac{1}{2}\{&X\cdot g(Y, Z)+Y\cdot g(Z, X)-Z\cdot g(X, Y)\\ &+g(Z, [X, Y])+g(Y, [Z, Y)])-g(X, [Y, Z])\}\,.\end{aligned}$$

Suppose $\tilde{\nu}\in TS(t)$ be orthogonal to $TM(t)$ with respec respect to $\tilde{g}_t$, which implies equivalently that $\nu\in TS(t)$ defined as above is perpendicular to $TM(t)$ with respect to $\langle\ ,\ \rangle$. Then for any $X\in TM(t)$, we obtain

$$\begin{aligned}\langle JX, \nu\rangle &= \langle JX, \nabla\rho/|\nabla\rho|\rangle\langle\nabla\rho/|\nabla\rho|, \nu\rangle \\ &= -|\nabla\rho|^{-2}\langle JX, \nabla\rho\rangle\langle\nu_\rho, \nu\rangle \\ &= |\nabla\rho|^{-2}\langle X, J\nabla\rho\rangle\langle\nu_\rho, \nu\rangle\,.\end{aligned} \tag{1.15}$$

Moreover we have

$$\begin{aligned}\langle\bar{\nabla}_X Y, \nu\rangle = |\nabla\rho|^{-2}\Big\{&\coth t\langle X, J\nabla\rho\rangle\langle Y, J\nabla\rho\rangle+\frac{1}{2}\coth t/2\langle X^h, Y^h\rangle \\ &-\langle\alpha_M(X, Y), \nu_\rho\rangle\Big\}\langle\nu_\rho, \nu\rangle+\langle\alpha_M(X, Y), \nu\rangle\,.\end{aligned} \tag{1.16}$$

Hence it follows from (1.14), (1.15) and (1.16) that

$$\begin{aligned}\tilde{g}_t(\tilde{\nabla}_X Y, \tilde{\nu}) = &\frac{(2(\cosh 1/2)^2-1)\langle X, J\bar{\nabla}\bar{\rho}\rangle\langle Y, J\bar{\nabla}\rho\rangle\langle\nu_\rho, \nu\rangle}{\sinh t|\nabla\rho|^2} \\ &-\frac{\langle\alpha_M(X, Y), \nu_\rho\rangle\langle\nu_\rho, \nu\rangle}{|\nabla\rho|^2}+\frac{\coth t/2\langle X^h, Y^h\rangle}{2|\nabla\rho|^2} \\ &+\langle\alpha_M(X, Y), \nu\rangle\,.\end{aligned}$$

This shows that for some positive constant c, we have

$$\begin{aligned}|\alpha_t|(x)\leqq c(e^{t/2}|\nabla\rho|^{-2}|\nu_\rho|+e^{3t/2}|\nabla\rho|^{-2}|\alpha_M|\,|\nu_\rho|^2+e^{3t/2}|\alpha_M|) \\ \text{at}\quad \mu_t(x)\ (=\exp_o t\phi_t(x))\end{aligned} \tag{1.17}$$

on $M(\tau)$. Then (1.17) and the same argument as in the proof of Lemma 8 prove the following

Lemma 9. *Let M be a connected, complex submanifold of complex dimension m properly immersed into $CH^n(-1)$. Suppose that $e^{3\bar{\rho}(x)/2}|\alpha_M|(x)$ goes to 0 as $x\in M\to+\infty$, where $\bar{\rho}$ denotes the distance in $CH^n(-1)$ to a fixed point of $CH^n(-1)$. Then for each end M_j of M, there exists a totally geodesic complex submanifold CH_j of complex dimension m in $CH^n(-1)$ such that $M_j(\infty)=CH_j(\infty)$.*

1.6. In order to prove a part of Theorem A, we shall need

Lemma 10. *Let M be a minimal submanifold of dimension m properly immersed into H. Then:*

$$\frac{\mathrm{Vol}_m(M_t)}{\mathrm{Vol}_m(B_a(t))}\leqq\frac{\mathrm{Vol}_{m-1}(M(t))}{\mathrm{Vol}_{m-1}(S_a^{m-1}(t))}$$

for any $t>0$. In particular, the function

$$\frac{\mathrm{Vol}_m(M_t)}{\mathrm{Vol}_m(B_a(t))}$$

is monotone non-decreasing in t. Here M_t *and* $M(t)$ *are as before and* $B_a(t)$ *denotes the metric ball of radius t in* $H^m(-a^2)$ $(a \geqq 0)$ *and* $S_a^{m-1}(t) = \partial B_a(t)$.

Although this lemma is well known (cf. e.g., [1]), we shall prove it in a more general form for the convinience of the readers.

Proposition 2. *Under the same assumption as in Lemma* 10, *let f be a non-negative subharmonic function on M. Then:*

$$\frac{\int_{M_t} f}{\mathrm{Vol}_m(B_a(t))} \leqq \frac{\int_{M(t)} f}{\mathrm{Vol}_{m-1}(S_a^{m-1}(t))}$$

for any $t > 0$. *In particular, the function*

$$\frac{\int_{M_t} f}{\mathrm{Vol}_m(B_a(t))}$$

is monotone non-decreasing in t.

The proof of this proposition will be given at the end of Section 4.

1.7. REMARK. Let H be a complete, simply connected Riemannian manifold whose sectional curvature is bounded above by a nonpositive constant $-a^2$ and let $M \to H$ be an isometric immersion from a complete, connected, non-compact Riemannian manifold M into H. Suppose that the second fundamentəl form α_M of the immersion satisfies

$$\limsup_{M \ni x \to \infty} (J_a/J_a')(\mathrm{dis}_M(x, o))\,|\alpha_M|(x) < 1\,,$$

where $\mathrm{dis}_M(x, o)$ stands for the distance in M between x and a fixed point o of M. Then the immersion turns out to be *proper*. Actually, taking a suitable smooth function $F(t)$ with $F' > 0$, we see that $F \circ \bar{p}$ $(\bar{p} = \mathrm{dis}_H(o, *))$ is strictly convex outside a compact subset of M. Suppose that the immersion would not be proper. Then we can find a geodesic ray $\sigma: [0, \infty) \to M$ such that $\bar{p} \circ \sigma$ is bounded, and hence $F \circ \bar{p}(\sigma(t))''$ is bounded away from 0 for large t. This contradicts the boundedness of $F \circ \bar{p}(\sigma(t))$.

2. Proofs of Theorems A, B, and C

In this section, we keep the notations in the preceding sections.

Proof of Theorem A. We shall begin with proving the second part (II) of the theorem. Let M be a connected, minimal submanifold of dimension m properly immersed into $H^n(-1)$. Suppose first that M satisfies the condition (A-v). Then it turns out from Lemma 8 that there is a 2-dimensional totaly geodesic submanifold H such that $\bar{\rho}_H(x):=\mathrm{dis}_{H^n(-1)}(x, H)$ goes to 0 as $x\in M\to +\infty$. On the other hand, $\bar{\rho}_H$ is subharmonic on M, and hence $\bar{\rho}_M$ must vanish identically on M. This implies that $M=H$. Suppose next that M satisfies the condition (A-iv). Then the sectional curvature K_t of $M(t)$ for large t has a lowre estimate:

$$K_t \geqq (\sinh t)^{-2} - c_1(|\alpha_M|^2 + |\alpha_M|^2|\nu_\rho|^2 + |\alpha_M||\nu_\rho|)\,, \tag{2.1}$$

where c_1 is a positive constant. Actually the sectional curvature $K_t(\pi)$ of $M(t)$ for a plane $\pi\subset TM(t)$ is given by

$$\begin{aligned} K_t(\pi) &= -1+\langle\alpha_M(X, X), \alpha_M(Y, Y)\rangle-\langle\alpha_M(X, Y), \alpha_M(X, Y)\rangle \\ &\quad +|\nabla\rho|^{-2}\{\nabla^2\rho(X, X)\nabla^2\rho(Y, Y)-(\nabla^2\rho(X, Y))^2\} \\ &= (\sinh t)^{-2}+\langle\alpha_M(X, X), \alpha_M(Y, Y)\rangle-\langle\alpha_M(X, Y), \alpha_M(X, Y)\rangle \\ &\quad +(|\nabla\rho|^{-2}-1)(\cosh t)^2+|\nabla\rho|^{-2}\{\langle\alpha_M(X, X), \nu_\rho\rangle\langle\alpha_M(Y, Y), \nu_\rho\rangle \\ &\quad +\cosh t\langle\alpha_M(X, X)+\alpha_M(Y, Y), \nu_\rho\rangle-\langle\alpha_M(X, Y), \nu_\rho\rangle^2\}, \end{aligned}$$

where $\{X, Y\}$ is an orthogonal basis of π. Now we apply Lemma 5 (iv) to (2.1) and obtain

$$K_t \geqq (\sinh t)^{-2}(1-\eta(t))\,,$$

where $\eta(t)$ goes to 0 as $t\to+\infty$. Then Rauch comparison theorem (as a special case of the Bishop comparison theorem for Ricci curvature) derives the following volume estimate:

$$\frac{\mathrm{Vol}_{m-1}(M(t))}{\omega_{m-1}(\sinh t)^{m-1}} \leqq (1-\eta(t))^{-(m-1)/2}\,,$$

and hence

$$\limsup_{t\to+\infty} \frac{\mathrm{Vol}_{m-1}\,(M(t))}{\omega_{m-1}(\sinh t)^{m-1}} \leqq 1\,,$$

where ω_{m-1}=the volume of the unit sphere of R^m and we have used the assumption that M has one end. Since we may assume that M contains the fixed point, it turns out from Lemma 10 that

$$\frac{\mathrm{Vol}_m(M_t)}{\mathrm{Vol}_m(B_1(t))} \equiv 1$$

for any $t>0$. Now it is easy to see that M is totally geodesic (cf. the proof of Proposition 2 in Section 4).

It remains to prove that M is totally geodesic if it satisfies the condition (A-vi). This is an immediate consequence of Lemma 8 and the following

Lemma 11. *Let M be a connected minimal submanifold of dimension m properly immersed into $H^n(-1)$ such that the second fundamental form α_M satisfies*

$$e^{m\bar{\rho}(x)}|\alpha_M|(x) \to 0$$

as $x \in M$ tends to infinity, where $\bar{\rho}(x)=\mathrm{dis}_{H^n(-1)}(x, o)$ and o is a fixed point of $H^n(-1)$. Suppose there is an open ball B of $S(\infty)$ (the points at infinity of $H^n(-1)$) such that $B \cap M(\infty)=\phi$ and for some end Γ of M, $\Gamma(\infty)$ is contained in the boundary S_B of B. Then M must lie in the totally geodesic hypersurface H of $H^n(-1)$ which corresponds to S_B (i.e., $H(\infty)=S_B$).

Proof. Let D^+ (resp. D^-) be the open domain of $H^n(-1)$ which bounds H and $S(\infty)\backslash B$ (resp. H and B) in $\overline{H^n(-1)}$. Define a function δ on M by

$$\delta(x)=\begin{cases} \mathrm{dis}_{H^n(-1)}(x, H) & \text{if } x \in D^+ \cap M \\ -\mathrm{dis}_{H^n(-1)}(x, H) & \text{if } x \in D^- \cap M . \end{cases}$$

Then δ is smooth and satisfies

$$\Delta_M \delta = \tanh \delta(m-|\nabla\delta|^2)$$

on M. In particular, we have

$$\Delta_M \delta \leqq m\delta \tag{2.2}$$

on M. Since $M(\infty) \cap B=\phi$, we see by the maximum principle that $\delta \geqq 0$ on M. Now suppose that δ would not vanish identically. Then by (2.2),

$$\delta>0 \tag{2.3}$$

on M. Observe here that on Γ,

$$\delta(x)>c_1 e^{-m\rho(x)} \tag{2.4}$$

for some positive constant c_1. In fact, take a bounded smooth function $f(t)$ and a sufficiently large number T such that

$$(-f''+2mf'+f'^2+m^2)\circ\rho|\nabla\rho|^2-\coth \rho(f'+m)\circ\rho(m-|\Delta\rho|^2) \geqq m$$

on $\Gamma^T=\{x \in \Gamma: \rho(x) \geqq T\}$ (cf. Lemma 5 (iv)). Define a function F by $F=\exp-(m\rho+f\circ\rho)$. Then F satisfies

$$\Delta_M F \geqq mF \tag{2.5}$$

on Γ^T. After multiplying a positive constant with F, we may assume by (2.3) that

$$F<\delta \tag{2.6}$$

on $\partial\Gamma^T$. Then it follows from (2.2), (2.5) and (2.6) that $F<\delta$ on Γ^T. This implies (2.4). However it contradicts the inequality ($\varepsilon=m$) in Lemma 8. Thus we have shown that δ identically vanishes on M, i.e., $M=H$. This completes the proof of Lemma 11.

As for the first part (I) of Theorem A, it was proved in [11] when M satisfies the conditions (A-ii) or (A-iii). Let us now suppose the condition (A-i) holds. Then it follows from (1.6) that the sectional curvature K_t of $M(t)$ for large t has a lower estimate:

$$K_t \geqq t^{-2}\left[1-\frac{1}{2}\kappa_M^2\{(1-\kappa_M^2)^{-1}+1\}^2-\eta(t)\right]$$

where $\kappa_M := \limsup |x|\,|\alpha_M(x)|$ and $\eta(t)$ goes to 0 as $t\to+\infty$. Since $1-\frac{1}{2}\kappa_M^2\{(1-\kappa_M^2)^{-1}\}^2+1>0$ by the assumption, we have

$$\mathrm{Vol}_{m-1}(M(t)) \leqq c_1 t^{m-1}$$

for large t, where c_1 is a positive constant depending only on m and κ_M. Then it turns out from the proof of Theorem 1.1 in [2] that for some divergent sequence $\{t_n\}$, $\dfrac{1}{t_n}M$ converges to a minimal cone Σ_∞ of $\boldsymbol{R}^N$, where the convergence is smooth in $\boldsymbol{R}^N\setminus\{o\}$. Moreover the second fundamental form α_∞ of the minimal submanifold $\Sigma_\infty\cap S(1)$ in the unit sphere $S(1)$ satisfies

$$|\alpha_\infty| \leqq \kappa_M\{(1-\kappa_M^2)^{-1}+1\}<2$$

(Cf. (1.6)). Thus we see from the Simons' theorem cited in the introduction that α_∞ vanishes, that is, Σ_∞ consists of m-planes. Since M is assumed to have one end, M must be totally geodesic (cf. the proof of Proposition 2 in Section 4).

Proof of Theorem B. Initially, we shall prove the first part (I) of the theorem. Let M be a connected, noncompact Riemannian submanifold properly immersed into R^n. Suppose that M has one end and $\bar{\rho}^\varepsilon(x)\,|\alpha_M|(x)$ is bounded on M for some constant $\varepsilon>2$. Then by Lemma 7, there is an m-plane P of R^n such that $\mathrm{dis}_{R^n}(x, P)$ goes to 0 as $x\in M\to+\infty$. Therefore if M does not coincide with P, we can find a point $x\in M$ and a sufficiently large ball B of R^n such that M is tangent to ∂B at x from the inside of B, which implies that

$$\langle\alpha_M(X, X), \nu_B\rangle \geqq c\langle X, X\rangle$$

for any tangent vector $X(\neq 0)$ of T_xM, where c is a positive constant and ν_B denotes the outer unit normal of ∂B. Thus it follows from Otsuki's lemma (cf. [13: p. 28]) that M must coincide with P if $2m>n$ and the sectional curvature of M is nonpositive, or $m=n-1$ and the scalar curvature of M is nonpositive. This proves the first part of Theorem B.

The second part (II) of the theorem follows from the same argument as above.

Finally, we shall prove the last part (III). Let M and D be as in Theorem B (III). It suffices to prove the assertion in the case: $m=n-1=2$, since for any totally geodesic 3-subspace H^3 of $H^n(-1)$ which is tangent to a normal vector of M, $H^3\cap D$ is a totally convex region of H^3. Let us take a point o of M as a fixed point. Set $D(t)=D\cap S(t)$, $M(t)=M\cap S(t)$ and $H^2=\exp_o T_oM$, where $\exp_o$ stands for the exponential map of $H^3(-1)$ at o. We claim that $M=H^2$. Suppose it would not be the case. Then it follows from the convexity of D that there are positive constants c_1 and c_2 such that

$$\frac{\mathrm{Vol}_2(D(t))}{\mathrm{Vol}_2(S(t))}<c_1<\frac{1}{2} \tag{2.7}$$

for any $t\geqq c_2$. We write ν_M for the unit inner normal of M. Let $\gamma_t(s)$ $(0\leqq s\leqq L_t)$ be an arc-length parametrization of $M(t)$ $(=\partial D(t))$ and ν_t the inner unit normal of $M(t)$ in $S(t)$. Then we have

$$\begin{aligned}\langle\overline{\nabla}_{\dot\gamma_t}\dot\gamma_t, \nu_M\rangle&=\langle\overline{\nabla}_{\dot\gamma_t}\dot\gamma_t, \overline{\nabla}\rho\rangle\langle\overline{\nabla}\rho\,\nu_M\rangle+\langle\overline{\nabla}_{\dot\gamma_t}\dot\gamma_t, \nu_t\rangle\langle\nu_t, \nu_M\rangle\\&\geqq\langle\overline{\nabla}_{\dot\gamma_t}\dot\gamma_t, \nu_t\rangle\langle\nu_t, \nu_M\rangle\\&\geqq\frac{1}{2}\langle\overline{\nabla}_{\dot\gamma_t}\dot\gamma_t, \nu_t\rangle\end{aligned}$$

for large t, becaue because $\langle\overline{\nabla}_{\dot\gamma_t}\dot\gamma_t, \overline{\nabla}\rho\rangle=-\coth t$, $\langle\overline{\nabla}\rho, \nu_M\rangle\leqq 0$ and $\langle\nu_t, \nu_M\rangle$ converges to 1 as $t\to+\infty$ (cf. Lemma 5). Hence by (2.7) and Gauss-Bonnet theorem, we obtain

$$\begin{aligned}\int_0^{L_t}\langle\nabla_{\dot\gamma_t}\dot\gamma_t, \nu_M\rangle(\gamma_t(s))\,ds&>\frac{1}{2}\int_0^{L_t}\langle\nabla_{\dot\gamma_t}\dot\gamma_t, \nu_t\rangle\\&=\frac{1}{2}\left\{2\pi\chi(D(t))-\frac{\mathrm{Vol}_2(D(t))}{(\sinh t)^2}\right\}\\&\geqq(1-2c_1)\pi>0\,.\end{aligned}$$

On the other hand, since

$$\begin{aligned}\int_0^{L_t}\langle\nabla_{\dot\gamma}\gamma_t, \nu_M\rangle(\gamma_t(s))\,ds&\leqq\int_0^{L_t}|\alpha_M|(\gamma_t(s))ds\\&\leqq L_t\max_{M(t)}|\alpha_M|\\&\leqq c_3e^t\max_{M(t)}|\alpha_M|\end{aligned}$$

for some positive constant c_3 (cf. Lemma 6), we have

$$c_3 e^t \max |\alpha_M| \geqq (1-2c_1)\pi > 0\,.$$

This contradicts the assumption on $|\alpha_M|$. Thus we have seen that M must be totally geodesic. This completes the proof of Theorem B.

Proof of Theorem C. We can apply Lemma 9 to M. Let CH_j $(j=1, \cdots, k)$ be as in Lemma 9, where k denotes the number of the ends of M. For each j, we can find $(n-m)$ bounded holomorphic functions $\{h_{j,\alpha}\}_{\alpha=1,\cdots,n-m}$ on $CH^n(-1)$ such that $CH_j=\{z\in CH^n(-1)\colon h_{j,1}(z)=\cdots=h_{j,n-m}(z)=0\}$. Define a function P on M by $P(x)=\prod_{j=1}^{k}(\sum_{\alpha=1}^{n-1}|h_{j,\alpha}|^2(x))$. Then P is a bounded plurisubharmonic function on M such that $P(x)$ goes to 0 as $x\in M\to+\infty$ (cf. Lemma 9). This implies that $P=0$ on M, that is, $M\subset\bigcup_{j=1}^{k} CH_j$. Since M is assumed to be connected, M must be contained in some CH_j. Thus we have shown that M is a totally geodesic complex submanifold of $CH^n(-1)$.

3. A gap theorem for noncompact Riemannian manifolds

In this section, we shall prove a supplementary result to the gap theorems for noncompact Riemannian manifolds due to Greene and Wu [9: esp. Theorem 4 and Theorem 5]. Our proof is more elementary than theirs, but the basic idea is due to them.

Theorem 1. *Let H be a complete Riemannian manifold with a pole o (i.e., $\exp_o\colon T_oH\to H$ induces a diffeomorphism between T_oH and H). Set $\bar{k}(t)=$the maximum of the sectional curvature of H on $S(t)$ and $\underline{k}(t)=$the minimum of the sectional curvature of H on $S(t)$, where $S(t)$ denotes the geodesic sphere around o of radius t. Suppose the dimension n of H is greater than or equal to 3. Then H is isometric to $H^n(-a^2)$ $(a\geqq 0)$ if (and only if) either of the following two conditions holds:*

$$\underline{k}(t)\leqq\bar{k}(t)\leqq -a^2\,,\quad \limsup_{t\to\infty}(a^{-1}\sinh at)^2(\underline{k}(t)+a^2)=0\,, \tag{3.1}$$

$$-a^2\leqq\underline{k}(t)\leqq\bar{k}(t)\quad \liminf_{t\to\infty}(a^{-1}\sinh at)^2(\bar{k}(t)+a^2)=0\,. \tag{3.2}$$

Here we understand $H^n(-a^2)=R^n$ and $a^{-1}\sinh at=t$ when $a=0$.

Proof. For the sake of simplisity, we shall prove the theorem in case of $a=1$. Define a metric $\tilde{g}_t$ on $S(t)$ by $\tilde{g}_t(X, Y)=(\sinh t)^{-2}\langle X, Y\rangle$. Then the sectional curvature $\tilde{K}_t(\pi)$ of $\tilde{g}_t$ for a plane π in $TS(t)$ is given by

$$\begin{aligned}\tilde{K}_t(\pi) = {} & (\sinh t)^2(K_H(\pi)+1)+1\\ & +(\sinh t^2)(E_t(X, X)E_t(Y, Y)-E_t(X, Y)^2)\\ & +\sinh t\cosh t(E_t(X, X)+E_t(Y, Y))\,,\end{aligned}$$

where $\{X, Y\}$ is an orthonormal basis of π with respect to the induced metric $\langle\, ,\rangle$ on $S(t)$ and we have set $E_t(X, Y)=\nabla^2\bar{\rho}(X, Y)-\coth t\langle X, Y\rangle$ ($\bar{\rho}=\mathrm{dis}_H(o, *)$). Suppose first condition (3.1) holds. Then E_t is positive semi-definite (cf. Lemma 1) and hence we have

$$\tilde{K}_t(\pi)\geqq(\sinh t)^2(\underline{k}(t)+1)+1\ .$$

By (3.1), we can take a sequence $\{t_i\}_{i=1,2,\dots}$ such that $\underline{\varepsilon}(t_i)=(\sinh t_i)^2(\underline{k}(t_i)+1)$ goes to 0 as $t_i\to+\infty$. This implies that

$$\limsup_{t_i\to\infty}\mathrm{Vol}_{n-1}((S(t_i), g_{t_i}))\leqq\omega_{n-1}\ ,$$

or equivalently

$$\limsup_{t_i\to\infty}\frac{\mathrm{Vol}_{n-1}(S(t_i))}{\omega_{n-1}(\sinh t_i)^{n-1}}\leqq 1\ .$$

On the other hand, we know that for any $t>0$,

$$\frac{\mathrm{Vol}_{n-1}(S(t))}{\omega_{n-1}(\sinh t)^{n-1}}\geqq 1$$

and if $\limsup_{t\to\infty}\mathrm{Vol}_{n-1}(S(t))/\omega_{n-1}(\sinh t)^{n-1}=1$, then H is isometric to $H^n(-1)$ (cf. [9: Lemma 2] or the proof of Proposition 2).

Suppose next that condition (3.2) holds. Since $\nabla^2\bar{\rho}(X, X)\geqq-\langle X, X\rangle$ (cf. the proof of Lemma 5 in [9]), we see that

$$\begin{aligned}\bar{K}_t(\pi)&\leqq(\sinh t)^2(K_H(\pi)+1)+1\\ &\leqq(\sinh t)^2(\bar{k}(t)+1)+1\ .\end{aligned}$$

By the assumption, we can take a sequence $\{t_i\}_{i=1,2,\dots}$ such that $\bar{\varepsilon}(t_i)=(\sinh t_i)^2\cdot(\bar{k}(t_i)+1)$ goes to 0 as $t_i\to+\infty$. Moreover it follows from Rauch comparison theorem that a diffeomorphism ϕ_t from the unit sphere $S^{n-1}(1)$ in T_oH onto $S(t)$ defined by $\phi_t(v)=\exp_o tv$ satisfies: $\phi_t^*\tilde{g}_t\geqq g_0$, where g_0 is the metric on $S^{n-1}(1)$ of constant curvature 1. Then the next lemma and the above argument show that H is isometric to $H^n(-a^2)$.

Lemma 12. *Let (S^n, g_0) be the standard sphere of constant curvature* 1 *and g a Riemannian metric on S^n. Suppose the curvature of (S^n, g) is bounded above by $1+\varepsilon$ for a constant ε: $0\leqq\varepsilon<3$, and suppose there is a diffeomorphism ϕ: $S^n\to S^n$ such that $\phi^*g\leqq g_0$. Then the injectivity radius* $\mathrm{Inj}(S^n, g)$ *of (S^n, g) is greater than or equal to $2\pi\sqrt{1+\varepsilon}-\pi$.*

Proof. It suffices to consider the case: $\mathrm{Inj}(S^n, g)<\pi/\sqrt{1+\varepsilon}$. Then there is a closed geodesic γ: $[0, l]\to(S^n, g)$ with $|\dot\gamma|_g=1$ such that the length $L_g(\gamma)$ is equal to $2\,\mathrm{Inj}\,(S^n, g)$. Define a closed curve C: $[0, l]\to(S^n, g_o)$ by $C=\phi^{-1}\circ\gamma$. We consider first the case: the antipodal point $-C(0)$ of $C(0)$ does not lie on C. Then for any $t\in[0, l]$, let us denote by $C_t(s)$ $(0\leqq s\leqq 1)$ the unique geodesic in (S^n, g_o) which joins $C(0)$ to $C(t)$. Set $\gamma_t=\phi\circ C_t$ and write for the (unique) lift of γ_t to $B(R_o)\subset T_{\gamma(0)}S^n$ (i.e., $\exp_{\gamma(0)}\circ\tilde\gamma_t=\gamma_t$) such that $\tilde\gamma_t(0)=0$ if it exists, where $B(R_o)$ denotes the ball of $T_{\gamma(0)}S^n$ with the maximum rank radius R_o $(\geqq\pi/\sqrt{1+\varepsilon})$ and $\exp_{\gamma(0)}$: $T_{\gamma(0)}S^n\to(S^n, g)$ is the exponential map of (S^n, g) at $\gamma(0)$. Observe that if t is close to 0 or l, $\tilde\gamma_t$ exists and further if $\tilde\gamma_t$ exists and t' is close to t, then so does $\tilde\gamma_{t'}$. Since γ is a closed geodesic with $L_g(\gamma)=2\,\mathrm{Inj}\,(S^n, g)$, there is a number $t_o\in(0, l)$ such that $\tilde\gamma_{t_o}$ exists and $|\tilde\gamma_{t_o}(s_o)|=\pi/\sqrt{1+\varepsilon}$ for some $s_o\in(0, 1)$. We may assume that $t_o\in(0, \frac{1}{2}l]$, by changing the orientation of γ if necessary. Then we have

$$\begin{aligned}\pi+\frac{1}{2}l&\geqq L_{g_o}(C_{t_o})+t_o\\ &\geqq L_g(\gamma_{t_o})+t_o\\ &=L_g(\gamma_{t_o}|[0,s_o])+L_g(\gamma|[0,t_o]\cup\gamma_{t_o}|[s_o,1])\\ &\geqq 2\pi/\sqrt{1+\varepsilon}.\end{aligned}$$

Here we have applied Gauss lemma to the last inequality. Thus we have obtained a lower estimate of $\mathrm{Inj}\,(S^n, g)$: $\mathrm{Inj}\,(S^n, g)\geqq 2\pi/\sqrt{1+\varepsilon}-\pi$. Now we consider the case that for any $t\in[0, l]$, $-C(t)$ lies on C. Take the number t_1 such that $-C(0)=C(t_1)$. Then for any $t\in[0, t_1)\cup(t_1, l]$, we denote again by C the unique geodesic joining $C(0)$ to $C(t)$ and by C_{t_1-0} (resp. C_{t_1+0}) the limit of C_t as $t\to t_1-0$ (resp. $t\to t_1+0$). After connecting C_{t_1-0} with C_{t_1+0} by a one-parameter family of geodesics joining $C(0)$ to $C(t_1)$ and using the same argument as above, we have a lower estimate: $\mathrm{Inj}\,(S^n, g)\geqq 2\pi\sqrt{1+\varepsilon}-\pi$. This completes the proof of Lemma 12.

Let us now show a slight generalization of the Greene and Wu's result in the case of nonpositively curved manifolds.

Theorem 2. *Let H be a complete, connected Riemannian manifold of nonpositive curvature. Suppose the curvature K_H satisfies*

$$\mathrm{dis}_H(x, o)^2K_H(x)\to 0$$

as $x\in H\to\infty$. Then H is isometric to Euclidean space if H is simply connected at infinity.

Here a noncompact manifold H is said to be *simply connected at infinity* if for any compact set $K\subset H$, there is a compact set $\tilde K$ with $K\subset\tilde K\subset H$ and with

$H\backslash\tilde{K}$ (connected and) simply connected.

Theorem 2 is an immediate consequence of Theorem 1 and the following

Lemma 13. *Let H be a complete, connected and noncompact Riemannian manifold of nonpositive curvature. Suppose there is a compact set K of H such that the fundamental group of a noncompact connected component Ω of $H\backslash K$ is finite. Then H is simply connected.*

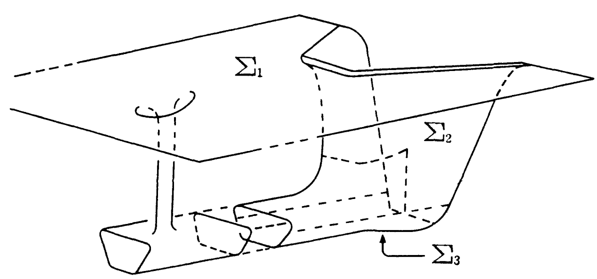
Proof. Let $\pi: \tilde{H}\to H$ be the universal covering of H and $\tilde{\Omega}$ a connected component of $\pi^{-1}(\Omega)$. Then the restriction of the projection π to $\tilde{\Omega}$ gives rise to a finite covering of Ω, and hence the boundary of $\tilde{\Omega}$ in H is compact. This implies that $\tilde{\Omega}=\pi^{-1}(\Omega)$ and further $\pi: \tilde{H}\to H$ is a finite covering. Thus it turns out that H must be simply connected.

REMARKS. (1) The proof of Theorem 1 indicates a more general version of the theorem and some relations between the curvature and the volume growth rate of the metric balls of H (cf. [12]).

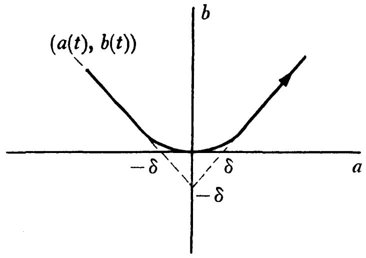
(2) In Theorem B, we can delete the condition that M has one end if we assume the sectional curvature of M is everywhere nonpositive (cf. Lemma 13 and the proof of Theorem B).

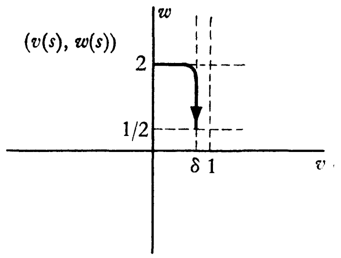
4. Surfaces stated in Remark (5) of Introduction and the proofs of Propositions 1 and 2

In the first part of this section, we shall construct a nonpositively curved surface M_g of genus $g>1$ in R^3 such that it has one end and the second fundamental form decays in quadratic order. Our M_g can be obtained by gluing two copies of the following surface:

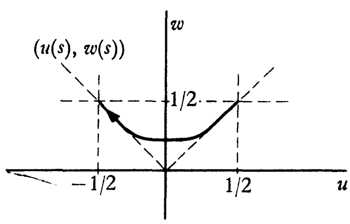


We shall now explain the main parts Σ_1, Σ_2 and Σ_3 of the above surface. Σ_1 is a part of a plane. Σ_2 is a part of a surface $\Sigma_2=\{(a(t)u(t), b(t)+v(s), w(s)): t\in R, s\in[s_0, s_1], u(s)=s_1+\frac{1}{2}-s\}$. Here $a(t)$, $b(t)$, $v(s)$ and $w(s)$ are given as follows:





Here δ: $0<\delta<1$, $a'(t)^2+b'(t)^2=1$, and $v'(s)^2+w'(s)^2=1$. Finally Σ_3 is given by $\Sigma_3=\{(tu(s)/\sqrt{2}, t/\sqrt{2}, w(s)): t\in[t_o, \infty)\ (t_o<\sqrt{2}\delta), s\in[s_2, s_3]\}$. Here $u(s)$ and $w(s)$ are defined as follows:



In the rest of this section, we shall prove Propositions 1 and 2.

Proof of Proposition 1. We shall use the Poincare model for $H^n(-1)$ where it is viewed as

$$B^n(1)=\{x=(x_1, \cdots, x_n)\in R^n: |x|<1\}$$

with the metric

$$g=\frac{4g_o}{(1-|x|^2)^2}, \quad g_o=dx_1^2+\cdots+dx_n^2 .$$

We set $r(x)=|x|$ for $x\in M$ and write Dr for the gradient of r on M with the induced metric $g_{o|M}$. Then $\rho(x)=\log(1+r(x))/(1-r(x))$ and $g_o(Dr, Dr)=g(\nabla\rho, \nabla\rho)$. Hence by the assumption on α_M, we may assume that Dr never vanishes on $\{x\in M: r(x)\geqq\tau_o\}$ for some $\tau_o>0$. Define as before a smooth vector field V_r on $\{x\in M: r(x)\geqq\tau_o\}$ by $V_r=\mathrm{D}r/g_o(\mathrm{D}r, \mathrm{D}r)$. Let $\gamma(s)$ be any regular curve in $\{x\in M: r(x)=\tau_o\}$ and, for any s, denote by $v(s; t)$ $(\tau_o\leqq t\leqq 1)$ the maximal integral curve of V_r such that $v(s; \tau_o)=\gamma(s)$. Then after direct computations, we see that for any unit vector Z of R^n,

$$(4.1)\qquad \begin{aligned} &\left|\frac{\partial}{\partial t}g_o(\partial v(s; t)/\partial t, Z)\right|\leqq c_1|A_M|_o \\ &\left|\frac{\partial}{\partial t}g_o(\partial v(s; t)/\partial s, Z)\right|\leqq c_1|A_M|_o \end{aligned}$$

where c_1 is an absolute positive constant and $|A_M|_o$ stands for the length of the second fundamental form A_M of M with respect to g_o. Moreover by the assumption on α_M and Lemma 5 (iii), we have

$$|A_M|_o \leqq \frac{c_2}{(1-r)|\log(1-r)|^{\varepsilon}} \tag{4.2}$$

for some positive constant c_2. Hence it follows from (4.1) and (4.2) that $\partial v(s;t)/\partial t$ and $\partial v(s;t)/\partial s$, respectively, have the limits $V_1(s)$ and $V_2(s)$ as $t \to 1$ and they satisfy

$$|\partial v(s;t)/\partial t - V_1(s)| \leqq \frac{c_1 c_2}{(\varepsilon-1)|\log(1-t)|^{\varepsilon-1}}$$

$$|\partial v(s;t)/\partial s - V_2(s)| \leqq \frac{c_1 c_2}{(\varepsilon-1)|\log(1-t)|^{\varepsilon-1}}.$$

This implies that both $V_1(s)$ and $V_2(s)$ are continuous in s. Thus we have shown that $\bar{M} = M \cup M(\infty)$ is a C^1-submanifold immersed into $\overline{B^n(1)} = B^n(1) \cup S^{n-1}(1)$. This completes the proof of Proposition 1.

Proof of Proposition 2. Define a function $F_{a,r}(t)$ $(t \geqq 0)$ by

$$F_{a,r}(t) = -\int_t^r \frac{\mathrm{Vol}_m(B_a(u))}{\mathrm{Vol}_{m-1}(S_a^{m-1}(u))}\,du = -\int_t^r \frac{\int_0^s J_a^{m-1}(u)\,du}{J_a^{m-1}(s)}\,ds$$

Then it follows from the minimality of M and Lemma 1 that

$$\begin{aligned}
\Delta_M F_{a,r}\circ\rho &= F_{a,r}''\circ\rho|\nabla\rho|^2 + F_{a,r}'\circ\rho\Delta_M\rho \\
&\geqq F_{a,r}''\circ\rho|\nabla\rho|^2 + F_{a,r}'\circ\rho(\log J_a)'\circ\rho(m-|\nabla\rho|^2) \\
&\geqq \min\left\{1, \frac{mJ_a'\circ\rho\int_0^\rho J_a^{m-1}(u)\,du}{J_a^{m-1}\circ\rho}\right\} \\
&\geqq 1,
\end{aligned}$$

since

$$\begin{aligned}
\frac{mJ_a'(t)\int_0^t J_a^{m-1}(u)\,du}{J_a^m(t)} &= \frac{\int_0^t (J_a^m)'(u)(J_a'(t)/J_a'(u))\,du}{J_a^m(t)} \\
&\geqq \frac{\int_0^t (J_a^m)'(u)\,du}{J_a^m(t)} \\
&= 1.
\end{aligned}$$

Therefore for any nonnegative subharmonic function f, we have

$$\begin{aligned}\int_{M_r} f &\leqq \int_{M_r} f \Delta_M F_{a,r} \circ \rho \\ &\leqq \int_{M_r} f \Delta_M F_{a,r} \circ \rho - F_{a,r} \circ \rho \Delta_M f \\ &= \frac{\mathrm{Vol}_m(B_a(r))}{\mathrm{Vol}_{m-1}(S_a^{m-1}(r))} \int_{m(r)} f * d\rho \\ &\leqq \frac{\mathrm{Vol}_m(B_a(r))}{\mathrm{Vol}_{m-1}(S_a^{m-1}(r))} \int_{M(r)} f .\end{aligned}$$

This proves Proposition 2.

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