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ON THE VOLUME OF POSITIVELY CURVED KÄHLER MANIFOLDS

Dedicated to Professor Shingo Murakami on his 60th birthday

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1. Introduction

In Riemannian geometry, it is a fundamental question to ask how the geometric invariants of Riemannian manifolds are influenced by curvature restrictions.

The volume of Riemannian manifolds and of geodesic balls in them is one of the most basic invariants, for which Bishop-Gromov's comparison theorem is well known (see [3], [5]).

In this note, we prove Bishop-Gromov type comparison theorem for the class of complete connected Kähler manifolds of complex dimension n whose Ricci curvature and holomorphic sectional curvature satisfy

$$\mathrm{Ric} \geqq 2(n+1)\,\delta^2$$

and

$$K_H \geqq 4\,\delta^2\,,$$

respecitvely, where δ is a positive real number (Theorem 1). Moreover, we characterize the complex projective space with the canonical metric by its volume among this class (Theorem 2).

In Section 2 we prepare notations and some preliminary results. Here we reduce our problem to the estimation of the mean curvature of geodesic spheres, which is carried out in Section 3. In Section 4 we estimate the volume element with respect to polar coordinates and prove our main theorems.

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2. Notations and preliminaries

Let M be a complete connected Riemannian manifold of dimension $m(m\geqq$

2). We denote the tangent space of M at p by T_pM and the inner product of tangent vectors u and v in T_pM by $\langle u, v\rangle$. The length or norm of v is denoted by $\|v\|$ and the unit sphere in T_pM by U_pM.

Let $B(p, r)$ and $S(p, r)$ denote the geodesic ball and the geodesic sphere of radius r centered at p, respectively, that is,

$$B(p, r) = \{\exp_p tv : v \in U_pM,\ 0 \leqq t < r\}$$

and

$$S(p, r) = \{\exp_p rv : v \in U_pM\}\ ,$$

where $\exp_p : T_pM \to M$ stands for the exponential map at p.

For $p \in M$ and $v \in U_pM$, let $c_{p,v} : [0, \infty) \to M$ denote the geodesic parametrized by arc length emanating from p with the initial direction v. We denote by $R_p(v)$ (resp. $\varphi_p(v)$) the cut value (resp. the smallest conjugate value) of p in the direction v. We write simply $R(v)$ and $\varphi(v)$ when no confusion is likely to occur.

If u and v are two orthogonal unit vectors at a point of M, then we denote by $K(u, v)$ the sectional curvature of the plane section determined by u and v. Moreover, let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of T_pM and let v be a unit tangent vector at p. Then the quantity $\mathrm{Ric}(v) = \sum_{i=1}^m K(v, e_i)$ is called the Ricci curvature of v.

Let λ be a real number. We write $\mathrm{Ric} \geqq \lambda$ if the inequality $\mathrm{Ric}(v) \geqq \lambda$ holds for every unit tangent vector v of M.

Let $p \in M$, $v \in U_pM$ and $r \in (0, \varphi(v))$. Then the geodesic sphere $S(p, r)$ is locally a hypersurface near around the point $c_{p,v}(r)$ and the velocity vector $\dot{c}_{p,v}(r)$ is a unit normal vector to $S(p, r)$. Let $\eta_{p,v}(r)$ denote the mean curvature of $S(p, r)$ with respect to $\dot{c}_{p,v}(r)$.

We choose an orthonormal basis $\{e_1, \cdots, e_{m-1}\}$ in the tangent space of $S(p, r)$ at $c_{p,v}(r)$. There exist Jacobi fields $Y_1(t), \cdots, Y_{m-1}(t)$ along $c_{p,v} | [0, r]$ satisfying

$$Y_i(0) = 0,\ Y_i(r) = e_i\ , \quad 1 \leqq i \leqq m-1\ .$$

Then $\eta_{p,v}(r)$ can be expressed as follows (see [6]):

$$\begin{aligned}(1)\qquad \eta_{p,v}(r) &= -1/(m-1) \textstyle\sum_{i=1}^{m-1} \langle Y_i'(r), Y_i(r)\rangle \\ &= -1/(m-1) \textstyle\sum_{i=1}^{m-1} \displaystyle\int_0^r [\|Y_i'(t)\|^2 \\ &\qquad -\langle R(\dot{c}_{p,v}, Y_i)\, \dot{c}_{p,v}, Y_i\rangle (t)]\, dt\ ,\end{aligned}$$

where $Y_i'(t)$ denotes the covariant derivative of $Y_i(t)$ along $c_{p,v}$ and R denotes the Riemannain curvature tensor of M.

Now fix $p \in M$. We define a smooth map Θ_p from $U_pM \times [0, \infty)$ onto M by

$$\Theta_p(v, t) = \exp_p tv\,, \quad v \in U_pM\,, \quad t \in [0, \infty)\,.$$

We choose an orientation for T_pM and define an orientation for U_pM so that for each $v \in U_pM$, $\{e_1, \cdots, e_{m-1}\}$ is a positively oriented basis of $T_v U_p M$ if and only if $\{e_1, \cdots, e_{m-1}, v\}$ is a positively oriented basis of T_pM, where $T_v U_pM$ is considered as a subspace of T_pM. Let $d\sigma$ denote the volume element on U_pM with respect to this orientation. Let dp be the parallel translate of the volume element on T_pM along the geodesics emanating from p. We now define a smooth function $\theta_p(v, t)$ on $U_pM \times [0, \infty)$ by

$$\Theta_p^* \, dp = \theta_p(v, t)\, d\sigma \wedge dt\,.$$

Then it is easy to verify that for each $v \in U_pM$

$$\theta_p(v, 0) = \theta_p(v, \varphi(v)) = 0\,,$$

and

$$\theta_p(v, t) > 0\,, \quad 0 < t < \varphi(v)\,.$$

Choose a basis $\{e_1, \cdots, e_{m-1}\}$ of $T_v U_p M$. There exist Jacobi fields $Y_1(t)$, $\cdots$, $Y_{m-1}(t)$ along $c_{p,v}$ satisfying $Y_i(0)=0$, $Y_i'(0)=e_i$, $1 \leqq i \leqq m-1$. We extend e_1, $\cdots$, e_{m-1} to the parallel vector fields $e_1(t), \cdots, e_{m-1}(t)$ along $c_{p,v}$, respectively. Since $d\,\Theta_{p(v,t)}\,(e_i) = Y_i(t)$, $1 \leqq i \leqq m-1$, we have the following expression of the function $\theta_p(v, t)$ (see [1], [3]):

$$\theta_p(v, t) = (Y_1(t) \wedge \cdots \wedge Y_{m-1}(t))/(e_1(t) \wedge \cdots \wedge e_{m-1}(t))\,. \tag{2}$$

Using $\theta_p(v, t)$, the volume of the geodesic ball $B(p, r)$ can be expressed as follows. If we set $s(v) = \min(r, R(v))$ for each $v \in U_pM$, then

$$\mathrm{vol}(B(p, r)) = \int_{U_pM} d\sigma \int_0^{s(v)} \theta_p(v, t)\, dt\,. \tag{3}$$

This formula is an immediate consequence of Fubini's Theorem, which is permissible because the functions concerned are all continuous.

We have the following relation between the functions $\theta_p(v, t)$ and $\eta_{p,v}(t)$:

Lemma 1. *Let $p \in M$ and $v \in U_pM$. Then we have*

$$d/dt \log \theta_p(v, t) = -(m-1)\, \eta_{p,v}(t)\,, \quad 0 < t < \varphi(v)\,. \tag{4}$$

Proof. Fix $t_0 \in (0, \varphi(v))$ arbitrarily. In the formula (2) we may assume that $Y_1(t_0), \cdots, Y_{m-1}(t_0)$ are orthonormal. We define a matrix-valued function $A(t) = (a_i^j(t))_{1 \leq i,j \leq m-1}$ by

$$Y_i(t) = \textstyle\sum_{j=1}^{m-1} a_i^j(t)\, e_j(t)\,, \quad 1 \leqq i \leqq m-1\,.$$

Then we have $\theta_p(v, t) = \det A(t)$ by (2).

We set $b_{ij}(t)=\langle Y_i(t), Y_j(t)\rangle$, $B(t)=(b_{ij}(t))_{1\leq i,j\leq m-1}$, $p_{ij}=\langle e_i, e_j\rangle$ and $P=(p_{ij})_{1\leq i,j\leq m-1}$. Then $b_{ij}(t)=\sum_{k,l=1}^{m-1} a_i^k(t)\, p_{kl}\, a_j^l(t)$, so that $\det B(t)=(\det A(t))^2 \det P$. Hence differentiating the both sides, we obtain

$$
\begin{aligned}
(5) \qquad d/dt \log (\det B(t)) &= 2\, d/dt \log (\det A(t)) \\
&= 2\, d/dt \log \theta_p(v, t)\,.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(6) \qquad d/dt \log (\det B(t))\,|_{t=t_0} &= 2(\det B(t_0))^{-1} \\
&\quad \times \textstyle\sum_{i=1}^{m-1}\langle Y_i'(t_0), Y_i(t_0)\rangle \\
&= -2(m-1)\, \eta_{p,v}(t_0)
\end{aligned}
$$

by (1). Combining (5) with (6), we conclude that

$$d/dt \log \theta_p(v, t)\,|_{t=t_0} = -(m-1)\, \eta_{p,v}(t_0)\,.$$

Since t_0 is arbitrary, this completes the proof.//

EXAMPLE 1. Let S_ρ^m be the m-dimensional sphere with the canonical metric of constant sectional curvature $\rho^2 (\rho>0)$. Then $\eta_{p,v}$ and θ_p are independent of particular choices of p and v, and

$$\eta_{p,v}(t) = -\rho \cot \rho t$$

and

$$\theta_p(v, t) = (\sin \rho t/\rho)^{m-1}\,.$$

EXAMPLE 2. Let $P_\delta^n(\boldsymbol{C})$ be the complex projective space of complex dimension n with the canonical metric of constant holomorphic sectional curvature $4\delta^2$ $(\delta>0)$. In this case $\eta_{p,v}$ and θ_p are also independent of p and v, and

$$\eta_{p,v}(t) = \eta_{\delta,n}(t) = -\delta(\cot \delta t - 1/(2n-1) \tan \delta t)$$

and

$$\theta_p(v, t) = \theta_{\delta,n}(t) = (\sin \delta t/\delta)^{2n-1} \cos \delta t\,.$$

These expressions follows from the formulas (1) and (2), respectively, since the Jacobi fields on $P_\delta^n(\boldsymbol{C})$ are well-known (see [1]).

3. Estimate of $\eta_{p,v}$

Let M be a Kähler manifold. Let J stand for the almost complex structure of M. For a unit tangent vector v of M, the sectional curvature $K(v, Jv)$ is called the holomorphic sectional curvature of v and denoted by $K_H(v)$.

Let λ be a real number. We write $K_H \geqq \lambda$ if the inequality $K_H(v) \geqq \lambda$ holds for every unit tangent vector v of M.

In the rest of this paper, we consider a complete connected Kähler manifold of complex dimension n which satisfies the following conditions for a positive real number δ:

$$\mathrm{Ric} \geqq 2(n+1)\,\delta^2 \quad \text{and} \quad K_H \geqq 4\delta^2 .$$

It is well-known that such a Kähler manifold is simply connected and compact, and in fact its diameter is less than or equal to $\pi/2\delta$ (see [8], [9]).

Lemma 2. *Let M be as above. Then we have*

$$\eta_{p,v}(t) \geqq \eta_{\delta,n}(t),\ 0 < t < \min(\varphi(v), \pi/2\delta)\ , \tag{7}$$

for any unit tangent vector v at any point p of M.

Proof. Fix $t \in (o, \min(\varphi(v), \pi/2\delta))$ arbitrarily. We take unit vectors $e_1, \cdots, e_{2n-2}$ at $c_{p,v}(t)$ so that $e_1, \cdots, e_{2n-2}, J\cdot\dot{c}_{p,v}(t) = e_{2n-1}$ form an orthonormal basis of $T_{c_{p,v}(t)}\,S(p, t)$, and extend them to the parallel vector fields $e_1(s), \cdots, e_{2n-1}(s)$ along $c_{p,v}|_{[0,t]}$. Note that we have $e_{2n-1}(s) = J\cdot\dot{c}_{p,v}(s)$. Let $Y_1(s), \cdots, Y_{2n-1}(s)$ be Jacobi fields along $c_{p,v}|_{[0,t]}$ satisfying

$$Y_i(0) = 0\ , \quad Y_i(t) = e_i\ , \quad 1 \leqq i \leqq 2n-1\ .$$

Moreover, we define vector fields $Z_1(s), \cdots, Z_{2n-1}(s)$ along $c_{p,v}|_{[0,t]}$ by

$$Z_i(s) = \begin{cases} (\sin \delta s/\sin \delta t)\ e_i(s)\ , & 1 \leqq i \leqq 2n-2\ , \\ (\sin 2\delta s/\sin 2\delta t)\ e_{2n-1}(s)\ , & i = 2n-1\ . \end{cases}$$

Let $I_0^t(X)$ denote the index form of a vector field X along $c_{p,v}|_{[0,t]}$. Then by the minimization theorem (see [3]), we have $I_0^t(Y_i) \leqq I_0^t(Z_i)$, $1 \leqq i \leqq 2n-1$. Hence

$$\begin{aligned}
-(2n-1)\,\eta_{p,v}(t) &= \textstyle\sum_{i=1}^{2n-1} I_0^t(Y_i) \leqq \sum_{i=1}^{2n-1} I_0^t(Z_i) \\
&= \int_0^t [(2n-2)\,(\delta\cos\delta\, s/\sin\delta\, t)^2 - (\sin\delta\, s/\sin\delta\, t)^2 \textstyle\sum_{i=1}^{2n-2} K(\dot{c}_{p,v}(s), e_i(s)) \\
&\quad + (2\delta\cos 2\delta\, s/\sin 2\delta\, t)^2 - (\sin\delta\, s/\sin\delta\, t)^2\,(\cos\delta\, s/\cos\delta\, t)^2\, K_H(\dot{c}_{p,v}(s))]\, ds \\
&= \int_0^t [(2n-2)\,(\delta\cos\delta\, s/\sin\delta\, t)^2 - (\sin\delta\, s/\sin\delta\, t)^2\, Ric(\dot{c}_{p,v}(s)) \\
&\quad + (2\delta\cos 2\delta\, s/\sin 2\delta\, t)^2 - (\sin\delta\, s/\sin\delta\, t)^2\, \{(\cos\delta\, s/\cos\delta\, t)^2 - 1\} \\
&\quad \times K_H(\dot{c}_{p,v}(s))]\, ds \\
&\leqq \int_0^t [(2n-2)\,(\delta\cos\delta\, s/\sin\delta\, t)^2 - (\sin\delta\, s/\sin\delta\, t)^2\,(2n+2)\,\delta^2 \\
&\quad + (2\delta\cos 2\delta\, s/\sin 2\delta\, t)^2 - (\sin\delta\, s/\sin\delta\, t)^2\, \{(\cos\delta\, s/\cos\delta\, t)^2 - 1\}\ 4\delta^2]\, ds \\
&= \int_0^t [(2n-2)\,(\delta\cos\delta\, s/\sin\delta\, t)^2 - (\sin\delta\, s/\sin\delta\, t)^2\,(2n-2)\,\delta^2
\end{aligned}$$

$+(2\delta\cos 2\delta\, s/\sin 2\delta\, t)^2-(\sin 2\delta\, s/\sin 2\delta\, t)^2\, 4\delta^2]\, ds$
$=-(2n-1)\,\eta_{\delta,n}(t)$. This completes the proof.//

4. Estimate of θ_p and comparison theorems for the volume of geodesic balls

Lemma 3. *Let M be a complete connected Kähler manifold of complex dimension n which satisfies*

$$Ric\geqq 2(n+1)\,\delta^2 \quad \text{and} \quad K_H\geqq 4\delta^2$$

for a positive real number δ.
Then we have

$$\theta_p(v, t')/\theta_p(v, t)\leqq\theta_{\delta,n}(t')/\theta_{\delta,n}(t)\,, \quad 0<t\leqq t'<\varphi(v) \tag{8}$$

for any unit tangent vector v at any point p of M.

Proof. We put $a=\min(\varphi(v), \pi/2\delta)$. Let us consider in the interval $(0, a)$. For simplicity we set

$$\theta(s)=\theta_p(v, s) \quad \text{and} \quad \theta_0(s)=\theta_{\delta,n}(s)\,.$$

By Lemma 1 and Lemma 2 we have

$$(\log\theta(s))'\leqq(\log\theta_0(s))'\,,$$

where "$\,'\,$" stands for the differentiation with respect to the variable s. Let $0<t\leqq t'<a$. By integrating the both sides from t to t' we obtain

$$(\log(\theta(t')/\theta(t))\leqq\log(\theta_0(t')/\theta_0(t))\,.$$

Hence the inequality (8) follows and

$$\theta(t')/\theta_0(t')\leqq\theta(t)/\theta_0(t)\,. \tag{9}$$

Since the right-hand side goes to 1 as t tends to 0, we obtain

$$\theta(t')\leqq\theta_0(t')\,.$$

From this inequality, it can be easily seen that the smallest conjugate value $\varphi(v)\leqq\pi/2\delta$. This completes the proof.//

In the last of the proof, we have obtained

Corollary 1. *Let M be as in Lemma 3. Then the smallest conjugate value satisfies*

$$\varphi(v)\leqq\pi/2\delta \tag{10}$$

for any unit tangent vector v of M.

REMARK. From (10), we know that the inequality (7) in Lemma 2 holds in the interval $(0, \varphi(v))$.

From Lemma 3, we obtain the following Bishop-Gromov type comparison theorem.

Theorem 1. *Let M be a complete connected Kähler manifold of complex dimension n which satisfies*

$$Ric \geqq 2(n+1)\delta^2 \quad and \quad K_H \geqq 4\delta^2$$

for a positive real number δ. Then for any $p \in M$ and for any $r' \geqq r > 0$, we have

$$vol(B(p, r'))/vol(B(p, r)) \leqq v_{\delta,n}(r')/v_{\delta,n}(r), \tag{11}$$

where $v_{\delta,n}(r)$ denote the volume of the geodesic ball of radius r in $P^n_\delta(\boldsymbol{C})$.

Proof. By Lemma 3 we know that the function

$$t \mapsto \theta_p(v, t)/\theta_{\delta,n}(t), \quad 0<t<\varphi(v)$$

is nonincreasing for any $p \in M$ and for any $v \in U_pM$.

We set

$$\bar{\theta}(v, t) = \begin{cases} \theta_p(v, t), & 0<t<R(v), \\ 0, & t \geqq R(v), \end{cases}$$

$$\bar{\theta}_0(t) = \begin{cases} \theta_{\delta,n}(t), & 0<t<\pi/2\delta, \\ 0, & t \geqq \pi/2\delta, \end{cases}$$

and

$$f(t) = \begin{cases} \bar{\theta}(v, t)/\bar{\theta}_0(t), & 0<t<R(v), \\ 0, & t \geqq R(v). \end{cases}$$

Note that $f(t)$ is nonincreasing

By (3) we have

$$vol(B(p, r))/v_{\delta,n}(r) = (1/vol(S_1^{2n-1}))\int_{U_pM}\left[\int_0^r \bar{\theta}(v, t)\, dt \Big/ \int_0^r \bar{\theta}_0(t)\, dt\right] d\sigma.$$

Setting $d\mu(t) = \bar{\theta}_0(t)\, dt$, we have

$$\int_0^r \bar{\theta}(v, t)\, dt \Big/ \int_0^r \bar{\theta}_0(t)\, dt = \left(1\Big/\int_0^r d\mu(t)\right) \times \int_0^r f(t)\, d\mu(t).$$

Since $f(t)$ is nonincreasing, the right-hand side is also nonincreasing (see [5]).

Hence $\mathrm{vol}(B(p, r))/v_{\delta,n}(r)$ is a nonincreasing function of r. This completes the proof.//

Since the quantity $\mathrm{vol}(B(p, r))/v_{\delta,n}(r)$ goes to 1 as r tends to 0, we have

Corollary 2. *Let M be as in Theorem 1. Then for any $p\in M$ and for any $r>0$, we have*

$$vol(B(p, r))\leqq v_{\delta,n}(r)\,. \tag{12}$$

Theorem 2. *Let M be as in Theorem 1. Then*

$$vol(M)\leqq vol(P_{\delta}^{n}(\boldsymbol{C}))\,.$$

The equality holds if and only if M is holomorphically isometric to $P_{\delta}^{n}(\boldsymbol{C})$.

REMARK. Let M be as in the theorem. Then by Bishop's comparison theorem (see [3]), we have

$$vol(M)\leqq vol(S_{\rho}^{2n})\,,$$

where $\rho=\sqrt{2(n+1)/(2n-1)}\,\delta$. The inequality in Theorem 2 is sharper than this one.

Bishop-Goldberg [4] obtained the similar result using the circle bundle method. Our method here is quite different from theirs and the assumption of the theorem seems more reasonable.

Proof. The first assertion follows if we set $r=\pi/2\delta$ in (12).

To prove the second assertion, we assume that $\mathrm{vol}(M)=\mathrm{vol}(P_{\delta}^{n}(\boldsymbol{C}))$ and fix $p\in M$ arbitrarily. Then we can easily verify that for any $v\in U_pM$

$$\theta_p(v, t) = \theta_{\delta,n}(t)\,, \quad 0\leqq t\leqq\varphi(v) = \pi/2\delta\,.$$

Hence we have for any $v\in U_pM$

$$\eta_{p,v}(t) = \eta_{\delta,n}(t)\,, \quad 0<t<\pi/2\delta\,.$$

Then all the inequalities in the proof of Lemma 2 must be equalities. In particular, we have the equality sign in the last inequality and

$$K_H(\dot{c}_{p,v}(s)) = 4\delta^2\,, \quad 0\leqq s\leqq t\,.$$

Since p and v were taken arbitrarily, we know that the holomorphic sectional curvature of M is constant and equal to $4\delta^2$. Hence M is holomorphically isometric to $P_{\delta}^{n}(\boldsymbol{C})$ (see [1], [7]).//

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