We have studied left serial rings with \((*, 1)\) or \((*, 2)\) in [7] and [8] as a generalization of Nakayama ring (generalized uniserial ring).

In this note, we shall replace the assumption “left serial” to “hereditary”, and give, in Sections 2~5, characterizations of an artinian hereditary ring with \((*, n)\) in terms of the structure of \(R; n \leq 3\). In Section 6, we shall study another type of hereditary algebras over an algebraically closed field, i.e., right US-n hereditary algebras.

1. Hereditary rings

Throughout this paper we assume that a ring \(R\) is a left and right artinian ring with identity. We shall use the notations and terminologies given in [2]~[8].

First we recall the definition of \((*, n)\).

\((*, n)\) Every maximal submodule of a direct sum of \(n\) hollow modules is also a direct sum of hollow modules [2] and [4].

In this case we may restrict ourselves to a direct sum of hollow modules of a form \(eR/K\), where \(e\) is a primitive idempotent and \(K\) is a submodule of \(eR\) [4].

Let \(R\) be an artinian hereditary ring. Then \(R\) is isomorphic to the ring of generalized tri-angular matrices over simple rings [1]. We are interested in a hereditary ring with \((*, n)\), and so we may assume that \(R\) is basic. Then

\[
R \approx \begin{pmatrix}
\Delta_1 & M_{12} & \cdots & M_{1n} \\
\Delta_2 & M_{23} & \cdots & M_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \Delta_n
\end{pmatrix}
\]

(1)

where the \(\Delta_i\) are division rings and the \(M_{ij}\) are left \(\Delta_i\) and right \(\Delta_j\) modules. It is clear that \(M_{ij} = e_iRe_j\) (\(e_i = e_{ii}\) matrix units).
Lemma 1. Let \( R \) be a hereditary ring as above. Then for any \( t \),
\[ \sum_{j \leq t} \oplus Re_j \text{ (resp. } \sum_{j < t} \oplus Re_j) \]
is an ideal and \( R[\sum_{j \neq t} \oplus e_j R] \) is also
hereditary.

Proof. This is clear from [1], Theorem 1.

Lemma 2. Every non-zero element in \( \text{Hom}_R(e_i R, e_j R) \) \( (i \leq j) \) is a monomorphism.

Proof. Since \( e_i R \) is indecomposable and \( f(e_i R) \) is projective for \( f \in \text{Hom}_R \)
(\( e_i R, e_j R \)), this is clear.

Let \( R \) be a ring as (1). We may study hollow modules \( e_i R/\mathcal{A} \) by the initial
remark. Put \( e = e_i \) and \( H = \{h | M_{ih} \neq 0\} \), \( J = \{j | M_{ij} = 0\} \), and further put
\( E_i = \sum_{h \in H} e_h R \), \( R_i = e_i R E_i \) and \( X_i = \sum_{j} e_j R \oplus \sum_{h \in H} e_h R \). Since \( R \) is hereditary, \( e_i R e_j = 0 \)
for \( h \in H \) and \( j \in J \) (cf. [1]), and so \( X_i \) is a two sided ideal in \( R \) by Lemma
1 and \( R_i X_i = 0 \). If \( e_p R e_q \neq 0 \) for \( p \in H \), then \( 0 = e_i R e_p R e_q \subseteq e_i R e_q \) by [1], and so
\( q \in H \). Hence \( e_p R = e_p R E_i \) and
\[
R_i = E_i R \quad \text{and} \quad R_i X_i = 0 .
\]

It is clear that \( R = R_i \oplus X_i \) as \( R \)-modules and \( R_i \) is hereditary (cf. [1]).
Hence every \( R_i \)-submodule in \( R_i \) is nothing but an \( R \)-submodule in \( R_i \) from
(2). Further let \( h_1 < h_2 < \cdots < h_q \) \( (h_1 \in H) \), then we note that \( e_{h_q} R e_{h_i} = 0 \) for all
\( q \). Therefore we obtain

Lemma 3. Let \( R \) be a hereditary ring as in (1) and let \( R_i \) be as above.
Then \( (*, n) \) holds for any \( n \) hollow modules if and only if, for any \( i \), the same holds
on any \( R_i \)-modules. Further \( R_i \) satisfies \( e_{h_q} R e_{h_i} = 0 \) for all \( h_q > h_i \).

Next we shall observe a construction of hereditary (basic) rings. In order
to make the observation clear, we shall first give an example.

Let
\[
R = \begin{pmatrix}
K_{11} & 0 & K_{13} & 0 & K_{16} & 0 & K_{18} \\
K_{22} & 0 & K_{24} & 0 & K_{26} & 0 & K_{28} \\
K_{33} & K_{34} & 0 & 0 & 0 & 0 \\
K_{44} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_{55} & K_{56} & 0 & K_{58} \\
0 & 0 & 0 & 0 & K_{66} & 0 & K_{68} & K_{77} & K_{79} \\
0 & 0 & 0 & 0 & K_{77} & K_{79} \\
0 & 0 & 0 & 0 & K_{77} & K_{79} & K_{88}
\end{pmatrix},
\]

where \( K_{ij} = K \) is a field.
We take non-zero entries in $e_1 R$ and put

$$R_1 = \begin{pmatrix} K_{11} & K_{12} & K_{14} & K_{16} & K_{18} \\ K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & K_{44} & 0 & 0 & 0 \\ 0 & 0 & K_{66} & K_{68} & K_{88} \\ K_{88} & K_{88} & K_{88} & K_{88} & K_{88} \end{pmatrix}$$

Since $K_{22}$ does not appear in $R_1$ (since $M_{12} = 0$), we take

$$R_2 = \begin{pmatrix} K_{22} & K_{24} & K_{26} & K_{28} \\ K_{44} & 0 & 0 & 0 \\ 0 & K_{66} & K_{68} & K_{88} \\ K_{88} \end{pmatrix}$$

Since $K_{55}$ does not appear in $R_1$ and $R_3$, put

$$R_3 = \begin{pmatrix} K_{55} & K_{56} & K_{58} \\ 0 & K_{66} & K_{68} \\ K_{88} \end{pmatrix}$$

Similarly to the above, we put

$$R_7 = \begin{pmatrix} K_{77} & K_{78} \\ 0 & K_{88} \end{pmatrix}$$

Then

$$A_{12} = \begin{pmatrix} K_{44} & 0 & 0 \\ 0 & K_{66} & K_{68} \\ 0 & 0 & K_{88} \end{pmatrix}$$

is the common components between $R_1$ and $R_2$. Similarly we can define

$$A_{16} = A_{26} = \begin{pmatrix} K_{66} & K_{66} \\ 0 & K_{88} \end{pmatrix}.$$

$$A_{17} = A_{27} = A_{57} = (K_{88}).$$

We note that the products in $R$ of two components in $R_i$ and $R_j$ not contained in $A_{ij}$ are zero. Now $R_2$ and $R_4$ are of right local type (see §5) and $R_3$ and $R_4$ are right serial. Further we know from the above note that $R$ is the subring of $R_1 \oplus R_2 \oplus R_3 \oplus R_7$ given by identifying elements in the same $K_{ij}$, namely in $A_{ij}$. If we carefully observe the above constructions, we know that only some right ideals contained in $(1_i - e_1^{(i)}) R_i$ are identified, where $1_i$ is the identity of $R_i$ and
is the matrix unit in \( R_i \).

We shall study the above fact in general. Let

\[
R = \begin{pmatrix}
M_{i1} & M_{i2} & \cdots & M_{in} \\
M_{n1} & M_{n2} & \cdots & M_{nn} \\
0 & & & \\
& & & \\
& & & 
\end{pmatrix}
\]

where \( M_{ii} = \Delta_i \) are division rings. We define \( R_i \) as before Lemma 3 and express \( R_i \) as

\[
R_i = \begin{pmatrix}
M^{(i)}_{11} & M^{(i)}_{12} & \cdots & M^{(i)}_{1n} \\
0 & & & \\
& & \ddots & \\
& & & M^{(i)}_{nn} 
\end{pmatrix}
\]

where \( M^{(i)}_{ij} \) is equal to some \( M_{im} \) in (3) \( (M^{(i)}_{ii} = M_{ii} \) in (3)) and \( M^{(i)}_{ij} = 0 \) for all \( k \).

We note first the following fact: Assume \( M_{ab} \neq 0 \) for some \( a \) and \( b \). Put \( I_a = \{ x | M_{ax} = 0 \} \) and \( I_b = \{ y | M_{by} = 0 \} \). Since \( M_{ab}R = e_bR \) (direct sum of \( m \)-copies of \( e_bR \)),

\[
I_a \subseteq I_b
\]

Starting with \( R_i (= R_{ih} \), from the initial observation we can construct \( R_{ih} \) so that \( M^{(i)}_{ij} \) does not appear on the diagonal of \( R_{ih} \) for all \( t_h < i \leq t_h \) and so that each component \( M_{pq} \) in (3) appears at least once in some \( R_{ih} \). Take \( R_i \) and \( R_j \) \( (t_h < i < j = t_{h'} \) ), and assume that \( M^{(i)}_{k} = M^{(j)}_{k} \) \( (= M_{pq} \) in (3)) are common components between \( R_i \) and \( R_j \). Then \( M^{(i)}_{k} = M^{(j)}_{k} = M_{pp} \) in (3)) are also common ones between \( R_i \) and \( R_j \) by the definition of \( R_{ih} \) and \( R_{ih'} \). We shall consider those components in (3). It is clear from (5) that

\[
e^{(i)}_b R_i = e_p R = e^{(j)}_i R_j
\]

Now let

\[
e^{(i)}_k R_i = (0 \cdots M_{kh}^{(i)} 0 \cdots M_{kh}^{(i)} 0 \cdots M_{kh}^{(i)} 0) = e^{(j)}_k R_j; \quad M^{(i)}_{kh} = 0
\]

Then \( e^{(i)}_k R_i = e^{(j)}_k R_j \) for all \( l \leq t \) from (5). By \( A_{ij} \) we shall denote the right ideal whose components appear in \( R_i \) and \( R_j \). Let \( I_i \) and \( I_j \) be as before (5) where \( i = t_h \) and \( j = t_{h'} \) and put \( I_i \cap I_j = \{ \pi_1 < \pi_2 < \cdots < \pi_s \} \). Then we know from the argument above that

\[
\begin{align*}
i) & \quad A_{ij} = \sum \oplus e_{\pi_p} R, \\
ii) & \quad A_{ij} e_p R = 0 \quad \text{for} \quad p \notin \{ \pi_1, \ldots, \pi_s \}, \\
& \quad \text{and so}
\end{align*}
\]

\[
\begin{align*}
iii) & \quad \text{the lattice of right } R \text{-modules of } A_{ij} \text{ is equal to the lattice of right } A_{ij} \text{-modules of } A_{ij}.
\end{align*}
\]
Finally we assume for some $b$ (1 $\leq b \leq n$) that $(M_{a_k}$ in (3)) = $M_{i,j}^{(b)} \neq 0$ and $(M_{k_i}$ in (3)) = $M_{i,j}^{(b)} \neq 0$. Then $b \in I_i \cap I_j$ and so $M_{i,j}^{(b)} \subset A_{i,j}$ from (7)-i) and ii). Hence the product in $R$ of an entry of $R_i$ and one of $R_j$ is zero if the latter (and hence two of them) is not contained in $A_{i,j}$. Thus we can find a set $\{R_{i,j}\}$ of hereditary rings such that $e^{(i)_k}R_ie^{(i)_k} \neq 0$ for all $k$ and a set $\{A_{i,j}\}$ of right ideals as (7), and $R$ is the subring of $\Sigma \oplus R_i$ such that the entries in $A_{i,j}$ of $R_i$ are equal to the entries in $A_{i,j}$ of $R_i$. Conversely, let $\{R_{i,j}\} \subseteq \{R_i\}$ be a set of hereditary (basic) rings and $\{A_{i,j}\}$ a set of right ideals in $R_i$ and $R_j$ which satisfy (7) where we replace $R$ with $R_i$ and $R_j$. Then we can easily show that the subring of $\Sigma \oplus R_i$ whose components in $A_{i,j}$ are identified for all $i,j$ is a hereditary ring. We shall call such a ring the patched ring of $\{R_i\}$ with respect to (briefly w.r.t.) $\{A_{ij}\}$, (the name comes from the following examples).

We shall give some examples of the patched ring. In the following examples, tri-angules and squares mean tri-angular matrices and matrices over a field $K$, respectively and straight lines do vector spaces over $K$.

1. $R_1 = \begin{pmatrix} a & b \\ c & \end{pmatrix} \quad R_2 = \begin{pmatrix} a' & b' \\ c' & \end{pmatrix} \quad \text{and } A_{i1} = \begin{pmatrix} 0 \\ \end{pmatrix}$

Then $\begin{pmatrix} a & 0 & b \\ a' & b' & c' \end{pmatrix}$ is the patched ring of $R_1$ and $R_2$ w.r.t $A_{i1}$.

2. $R_3 = \begin{pmatrix} a & b & 0 \\ c & d & e \end{pmatrix} \quad R_4 = \begin{pmatrix} a' & b' & 0 \\ a'' & b'' & c'' \end{pmatrix} \quad \text{and } B_{i1} = \begin{pmatrix} d \\ \end{pmatrix}$

Then $\begin{pmatrix} a & 0 & b & 0 \\ e' & 0 & a'' & b'' \end{pmatrix}$ is the patched ring of $R_3$ and $R_4$ w.r.t $B_{i1}$.
We note that $R_1$ and $R_2$ are left and right serial, but $R$ is not left serial. $R_3$ and $R_4$ are of right local type, but $R$ is not and $(\ast, 3)$ holds (see §§4 and 5). We shall show in §5 that every hereditary (basic) algebra over an algebraically closed field with $(\ast, 3)$ is obtained as the patched ring of $R_1$'s and $R_2$'s above.

Thus we obtain

**Proposition 1.** Let $R$ be a hereditary (basic) ring. Then $R$ is the patched ring of hereditary rings $\{R_i\}$ such that $e_k^{(i)}r_0e_k^{(i)} = 0$ for all $k$, where $e_k^{(i)}$ is the matrix unit $e_{pp}$ in $R_i$.

**Remark 1.** Let $R$ be a hereditary ring which is one of $R_i$ given in Proposition 1. Since $e_iRe_j \neq 0$, $e_jR$ is monomorphic to $e_jR$. Hence, if the structure of $e_jR$ is known as right $R$-modules, then we can see those of $e_iR$ (cf. Theorem 2).

2. Hereditary rings with $(\ast, 1)$

We shall first give some remarks on $(\ast, 1)$. If $R$ satisfies $(\ast, 1)$, for $eJ/C$ $eJ/C = \sum A_i$ with $A_i$ hollow. Since $A_i$ is hollow, $A_iJ = \sum B_{ij}$ with $B_{ij}$ hollow by $(\ast, 1)$. Hence $eJ/C = \sum A_iJ = \sum \sum B_{ij}$. By induction

$$eJ/C$$

is a direct sum of hollow modules.

In general, we assume that a module $M$ is a direct sum of submodules $M_i$. For submodules $N_i$ of $M_i$, we call $\sum N_i$ a standard submodule of $M$ (with respect to the decomposition $\sum M_i$).

**Proposition 2.** Let $N$ be a finitely generated $R$-module. Then the following are equivalent:
1) $N$ is a direct sum of hollow modules.
2) Let $P$ be a projective cover of $N$ ($P \twoheadrightarrow N$). Then $\ker f$ is a standard submodule of $P$ with respect to a suitable direct decomposition of indecomposable modules.
3) Let $P'$ be projective and $f' : P' \twoheadrightarrow N$ an epimorphism. Then $\ker f'$ is a standard submodule of $P'$ as 2).

Proof. Every hollow module is of a form $eR/A$. Hence 1)$\iff$2) and 3)$\implies$2) are clear.
2)$\implies$3) Let

$$0 \rightarrow K' \rightarrow P' \rightarrow N \rightarrow 0$$
be exact with $P'$ projective. Since $P$ is a projective cover of $N$, there exist $g: P \to P'$ and $h: P' \to P$ such that $hg = 1_P$. Let $P = \sum P_i$ and $\ker f = K = \sum K_i$ by 2), where the $P_i$ are indecomposable and $K_i \subset P_i$. It is clear that $g(K) \oplus h^{-1}(0) = \sum g(K_i) \oplus h^{-1}(0) \subset \ker f'$ and $P' = g(P) \oplus h^{-1}(0)$. Hence $f' = \sum g(K_i) \oplus h^{-1}(0) \subset \sum g(P_i) \oplus h^{-1}(0) = P'$.

We shall study, in this section, a hereditary ring with $(\ast, 1)$ as a right $R$-module. Hence we may assume that $R$ is basic. We shall give a characterization of a hereditary ring with $(\ast, 1)$.

In the following, $\alpha, \beta, \cdots$ mean indices and $|i, \alpha, \beta, \cdots, \eta|$ means a natural number related with the index $(i, \alpha, \beta, \cdots, \eta)$. If $R$ is a basic hereditary ring,

$$J(eR) = e_i J = N(i, \alpha) \oplus N(i, \beta) \oplus N(i, \gamma) \oplus \cdots,$$

where $N(i, \alpha) \approx e_{i, i, \alpha} R$, $N(i, \beta) \approx e_{i, i, \beta} R$, $\cdots$, \hfill (9)

$$J(N(i, \alpha)) = N(i, \alpha, \alpha_i) \oplus N(i, \alpha, \alpha_i') \oplus \cdots,$$

where $N(i, \alpha, \alpha_i) \approx e_{i, i, \alpha_i} R$, $N(i, \alpha, \alpha_i') \approx e_{i, i, \alpha_i'} R$, \hfill and so on. It is clear that $i < |i, \alpha| < |i, \alpha, \alpha_i| < |i, \alpha, \alpha_i, \alpha_{i_2}|$ and so on, and \hfill (10)

$$e_i Re_j = M_{ij} = \sum_{|i, \gamma| = j} N(i, \cdots, \gamma) e_j.$$

**Theorem 1.** Let $R$ be a hereditary (basic) ring and $N(i, \cdots, \gamma)$ be as in (9). Then the following conditions are equivalent:

1) $(\ast, 1)$ holds for any hollow right $R$-module.

2) The following conditions are satisfied.

i) Let $i < k = |i, \alpha| \leq j = |i, \beta| (\alpha \neq \beta)$, i.e., $e_i J$ contains two direct summands isomorphic to $e_k R$ and $e_j R$, respectively. If $N(i, \alpha, \cdots, \gamma)$ and $N(i, \beta, \cdots, \gamma')$ with $|i, \alpha, \cdots, \gamma| = |i, \beta, \cdots, \gamma'| = h$ appear in (9), i.e., for some $h$, simultaneously $e_k Re_h \neq 0$ and $e_j Re_h \neq 0$, then $e_i R$ is uniserial, and hence $[M_{ij}: \Delta_q] < 1$ for $q > j$. Further if we denote exactly $N(i, \alpha, \cdots, \gamma)$ as $N(i, \alpha, \alpha_2, \cdots, \alpha_i = \gamma)$, there exists a (unique) $s$ such that $|i, \alpha, \alpha_2, \cdots, \alpha_s| = j$.

ii) If $M_{ij} = x\Delta_q (q > j)$, there exists an isomorphism $\sigma$ of $\Delta_q$ onto $\Delta_i$ such that $x\delta = \sigma(\delta)x$ for all $\delta$ in $\Delta_q$.

3) For any submodule $A$ in $e_i J^k$ for any $k$, there exists a direct decomposition $e_i J^k = \sum \oplus P_\alpha$ such that $A = \sum \oplus A_\alpha; A_\alpha \subset P_\alpha$ and $P_\alpha$ is indecomposable, i.e., $A$ is a standard submodule of $e_i J^k$ with respect to the decomposition $\sum \oplus P_\alpha$.

4) For any submodule $A$ in $e_i J$, there exists a direct decomposition $e_i J = \sum \oplus N(i, \alpha')$ such that $A = \sum \oplus A_\alpha; A_\alpha \subset N(i, \alpha')$ and $N(i, \alpha') \approx N(i, \alpha)$, i.e., $A$ is a standard submodule of $e_i J$ with respect to the decomposition $\sum \oplus N(i, \alpha')$.

**Proof.** 1) $\to$ 2) Assume $(\ast, 1)$ and $i = 1$ from Lemma 1. Put $i_1 = |1, \alpha|$ and $i_2 = |1, \beta|$. Assume $N(1, \alpha, \cdots, \gamma)$ and $N(1, \beta, \cdots, \gamma')$ appear in (9) for
$k = |1, \alpha, \ldots, \gamma| = |1, \beta, \ldots, \gamma'|$. Then $M_{i_1} \neq 0$, $M_{i_2} \neq 0$ and $M_{i_3} \neq 0$. First we shall show $e_{i_2}R$ is monomorphic to $e_{i_1}R$ and $[M_{i_2} : \Delta_k] = 1$. If we can show that $e_{i_2}R$ contains a non-zero element $y$ in $M_{i_2}$, $e_{i_2}R \ni yR \subset e_{i_1}R$ ($e_{i_2} \to y$) is a monomorphism from Lemma 2. Hence we may assume $\Delta_{i_1} = \cdots = \Delta_{k} = 0$ from Lemma 1. We shall identify $N(1, \alpha)$ with $e_{i_1}R$ (resp. $N(1, \beta)$ with $e_{i_2}R$). From the above assumption let $M_{i_2} = \sum_{j \geq 2} A_j$; the $A_j$ are simple $R$-modules and $[A_j : \Delta_k] = 1$. Since $e_{i_1}R \supset M_{i_1} \ni N(1, \alpha, \ldots, \gamma) \neq 0$, there exists a natural homomorphism

$$f : M_{i_2} \cong \sum_{j \geq 2} A_j \ni A \to M_{i_1}.$$ 

From the assumption (*, 1), $f$ is extendible to an element $h'$ in $\text{Hom}_R(e_{i_2}R/\sum A_j, e_{i_1}R)$ by [6], Theorem 4 (note that $\text{Hom}_R(e_{i_1}R, e_{i_2}R/\sum A_j) = 0$ by Lemma 2 in case of $i_1 = i_2$ and $j \geq 2$ and that we identify $e_{i_2}R$ and $e_{i_1}R$ with $N(1, \alpha)$ and $N(1, \beta)$, respectively). Consider a homomorphism

$$h : e_{i_2}R \to e_{i_1}R/\sum A_j \ni e_{i_1}R.$$ 

Since $h \neq 0$ is a monomorphism by Lemma 2, $M_{i_2} = A_1$. Therefore

$$e_{i_2}R \text{ is monomorphic to } e_{i_1}R \text{ and } [M_{i_2} : \Delta_k] = 1, \text{ provided } M_{i_2} \neq 0.$$ 

We shall show similarly to (11) that $e_{i_3}R$ is uniserial. Put $e_{i_3} = e$ and $e^{j+1} = \sum e_{i_2}R$ for some $t$, since $R$ is hereditary. Let $B$ be a simple submodule of $e_{i_3}R$. Then we obtain a monomorphism of $(B \oplus \sum e_{i_2}R) \ni \sum e_{i_2}R \cong B$ to $e_{i_2}R$ (see (11)). From the argument before (11), $\sum e_{i_2}R \supset 0$, and so $e^{j+1} = e_{i_2}R$ is simple. Therefore $eR$ is uniserial. Next assume $M_{i_3} = x\Delta_k$ and we show ii). Hence we may assume $\Delta_{i+1} = \cdots = \Delta_k = 0$ from Lemma 1. For any $\delta$ in $\Delta_k$, define an endomorphism $\phi$ of $M_{i_3}R$ by setting $\phi(x\delta') = x\delta\delta'$. We may regard $\phi$ as an isomorphism of $M_{i_3}R$ onto $N(1, \alpha, \ldots, \gamma)$ ($|1, \alpha, \ldots, \gamma| = k$). Further, for an extension $g$ (in $\text{Hom}_R(eR, e_{i_3}R) \subset \text{Hom}_R(eR, e_{i_3}R))$ of $\phi$ by [6], Theorem 4, $g(eRe_{i_3}R) = e_{i_3}R \subset M_{i_3}R = \sum N(1, \alpha, \ldots, \gamma) e_{i_3}R$. Noting the structure (9) and $g(M_{i_3}R) = \phi(M_{i_3}R) = N(1, \alpha, \ldots, \gamma)$, we obtain

$$e_{i_3}R \text{ is monomorphic to } e_{i_3}R \text{ and } [M_{i_3} : \Delta_k] = 1, \text{ provided } M_{i_3} \neq 0.$$ 

Therefore $\phi$ is extendible to an element in $\text{Hom}_R(eR, eR) = \Delta_{i_3}$ (take the projection to $N(1, \alpha, \cdots, \epsilon)$), which implies that there exists $\delta*$ in $\Delta_{i_3}$ such that $\delta* x = x\delta$. It is clear that the mapping: $\delta \to \sigma(\delta) = \delta*$ is a monomorphism. We shall show that $\sigma$ is an isomorphism. Let $\delta**$ be an element in $\Delta_{i_3}$. Since
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\[ M_{i\alpha} = x\Delta_{i} \] is a left \( \Delta_{i} \)-module, \( \delta**x = x\delta'' \) for some \( \delta'' \) in \( \Delta_{i} \). Hence \( \delta** = \sigma(\delta'') \).

The last part of ii) is clear from (12) and its argument.

2)→1) Assume that i) and ii) are satisfied. We shall show that the condition ii) of [6], Theorem 4 is fulfilled, and so we may study a case \( e = e_{i} \) by Lemma 1.

Let

\[ e_{i}J = N(1, \alpha) \oplus N(1, \beta) \oplus \cdots \]

and \( C_{1} \supset D_{1} \) (resp. \( C_{2} \supset D_{2} \)) submodules in \( N(1, \alpha) \approx e_{i}R \) (resp. \( N(1, \beta) \approx e_{i}R \)), \( i_{1} \leq i_{2} \) such that \( C_{i}/D_{i} \) is simple and \( f^{-1}: C_{i}/D_{i} \approx C_{2}/D_{2} \). We shall show that \( f \) is extendible to an element in \( \text{Hom}_{R}(N(1, \beta)/D_{2}, N(1, \alpha)/D_{1}) \).

First we note for any \( R \)-module \( E \) in \( e_{k}R \),

\[ E = E(\sum_{j \neq h} e_{j}) = \sum_{j \neq h} E_{e_{j}} \quad \text{and} \quad E_{e_{j}} \subset M_{ik} \tag{13} \]

Since \( C_{1}/D_{1} \approx C_{2}/D_{2} \), \( N(1, \alpha, \cdots, \gamma') \) appear in \( e_{k}R \) for some \( |1, \alpha, \cdots, \gamma'| = h \) from (13). Hence \( N(1, \beta) \approx e_{k}R \) is uniserial by i) and \( C_{1} = M_{i\alpha} \oplus M_{i\alpha} \oplus \cdots \oplus M_{i\alpha} \oplus M_{i\alpha} \) from (13), where \( h < h_{1} < \cdots < h_{t} \). We may identify \( N(1, \alpha) \) with \( e_{i}R \). Let \( M_{i\alpha} = x\Delta_{h} \) and take a representative \( f(x) \) of \( f(x + D_{t}) \) in \( M_{i\alpha} \) from (13); \( f(x) = \sum x_{p}; 0 \neq x_{p} \in N(1, \alpha, \cdots, \gamma') \) from (10) \( (|1, \alpha, \cdots, \gamma'| = h) \). Since \( x_{p} \neq 0, N(1, \alpha, \cdots, \gamma') \subset N(1, \alpha, \cdots, \gamma'; \delta) \) \( (|i, \alpha, \cdots, \gamma'; \delta| = i_{2}) \) from i), and \( N(1, \alpha, \cdots, \delta; \delta') \neq N(1, \alpha, \cdots, \delta'; \delta) \) if \( p \neq p' \), since \( e_{i}R \) is uniserial. Put \( N = \sum_{i \neq h} N(1, \alpha, \cdots, \delta; \delta) \subset N(1, \alpha), C'_{1} = C_{1} \cap N \) and \( D'_{1} = D_{1} \cap N, f(x) \) being in \( C'_{1} \) and \( f(x) \notin D_{1}, C_{1} = C_{1}' + D_{1} \), and so \( C_{1}/D_{1} \approx C_{1}'/(C_{1}' \cap D_{1}) \approx C_{1}'/D_{1}' \). On the other hand, \( x_{p} = x_{p} e_{h} \) for all \( p \). Hence the mapping: \( x_{i} \rightarrow x_{p} \) is extendible to an element \( g_{p} \) in \( \text{Hom}_{R}(N(1, \alpha, \cdots, \delta; \delta), N(1, \alpha, \cdots, \delta; \delta)) \approx \Delta_{i} \) from i) and ii). Then \( N = N(1, \alpha, \cdots, \delta; \delta) \) \( (\sum_{e \neq h} g_{e}) \oplus \sum_{e \neq h} N(1, \alpha, \cdots, \delta; \delta) \) and \( f(x) \notin N(1, \alpha, \cdots, \delta; \delta) \) \( (\sum_{e \neq h} g_{e}) = N(1, \alpha, \cdots, \delta; \delta) \approx N(1, \alpha, \cdots, \delta; \delta) \) as above. Now \( C_{1}' \subset N(1, \alpha, \cdots, \delta; \delta) \subset N \approx e_{i}R \subset N \subset N(1, \alpha) \) and \( D_{1}' \subset N \approx N(1, \alpha) \approx D_{1}' \). Hence we obtain the natural homomorphism

\[ N(1, \beta)/D_{2} \rightarrow N(1, \alpha)/D_{1} \]

where \( u \) is an extension of \( f \) given by i) and ii), which is an extension of \( f \).

4)→1) This is clear from the definition of \( (\ast, 1) \).

3)→4) This is trivial.

1)→3) This is clear from (8) and Proposition 2.
REMARK 2. We shall study the situation of 2)—ii) of Theorem 1. Let $e_i R$ and $e_j R$ be as in i). Assume
\[ e_{i+1} R = (0 \ldots \Delta_j 0 M_{j+1} 0 \ldots M_{j+1} 0 \ldots M_{j+1} 0), \quad (j = j_1 \text{ and } M_{j_1} = 0). \]
Then
\[ e_{i+2} R = (0 \ldots \Delta_j 0 \ldots M_{j+1} 0 \ldots M_{j+1} 0 \ldots M_{j+1} 0). \]
(14)
\[ \ldots \ldots \ldots \]
\[ e_{i+1} R = (0 \ldots \Delta_j 0 \ldots M_{j+1} 0 \ldots M_{j+1} 0 \ldots M_{j+1} 0), \]
since $e_{i+1} R$ is uniserial. Further $M_{i+1} = m_{i+1} \Delta_j$. In order to simplify the notations, we express $j_i$ by $i$. Then $M_{i+1} = 0$ for $i < j$. Every element in
\[ \text{End}_R(M_{i+1} R/M_{i+1+1} R) \]
is extendible to an element in $\text{End}_R(e_i R/M_{i+1+1} R)$ by the proof after (12). Further, since
\[ (0 \ldots M_{i+1} \ldots M_{i+1}) \approx (0 \ldots M_{i+1} \ldots M_{i+1}) \]
for all $I$ and $s$, every element in $\text{End}_R(M_{i+1} R/M_{i+1+1} R) = \Delta_i$ is extendible to an element in $\text{End}_R(e_i R/M_{i+1+1} R) = \Delta_i$. Hence there exists an isomorphism $\varphi_i : \Delta_i \rightarrow \Delta_i$ (since $M_{i+1} = m_{i+1} \Delta_i$, $\varphi_i$ is an epimorphism) such that
\[ m_{i+1} x = \varphi_i(x) m_{i+1}, \quad \text{where } x \in \Delta_i \text{ and } M_{i+1} = m_{i+1} \Delta_i \]
from the proof of Theorem 1. We fix generators $m_{i+1}$ of $M_{i+1+1}$ for all $i$ and
\[ \varphi_{i+1} : \Delta_{i+1} \rightarrow \Delta_i \]
related with the fixed $m_{i+1}$ in (15). Then $m_{i+1} m_{i+1+1} m_{i+1+2} \ldots m_{i+k+1} = m_{i+k+1}$ is a generator of $M_{i+k+1+1}$ and $\varphi_{i+1} = \varphi_{i+1} \ldots \varphi_{i+k+1} : \Delta_{i+k+1+1} \rightarrow \Delta_i$ is an isomorphism and satisfies (15) (cf [1], Lemma 13). Hence we may assume
\[ (e_i + \ldots + e_i) R(e_i + \ldots + e_i) \approx \begin{pmatrix} \Delta_i & \Delta_i & \ldots & \Delta_i \\ \Delta_i & \ldots & \Delta_i \\ 0 & \ldots & \ldots & \Delta_i \\ \end{pmatrix} \]
(16)
Next assume that $e_i R$ is uniserial only as in (14). Then by the similar argument as above, we obtain
\[ (e_i + \ldots + e_i) R(e_i + \ldots + e_i) \approx \begin{pmatrix} \Delta_i & \Delta_i & \ldots & \Delta_i \\ \Delta_i & \ldots & \Delta_i \\ 0 & \ldots & \ldots & \Delta_i \\ \end{pmatrix} \]
(16')
and the $\varphi_{i} : \Delta_i \rightarrow \Delta_j (i < j)$ are monomorphisms (cf [1], Lemma 13). By $T_i(\Delta_i)$ and $T_i(\Delta_i, \Delta_i, \ldots, \Delta_i)$ we denote the above rings (16) and (16'), respectively.

3. Hereditary rings with (*, 2)

We shall give a characterization of hereditary rings with (*, 2).
Theorem 2. Let $R$ be a hereditary (basic) ring. Then $(\ast, 2)$ holds for any two hollow right $R$-modules if and only if, for each $e_i \in \{e_i\}$,

$$e_iJ = \sum_{k=1}^{n_i} A_k,$$

where the $A_k$ are uniserial modules, which satisfy the following conditions:

i) If $A_k \neq A_{k'}$ for $k \neq k'$, any sub-factor modules of $A_k$ are not isomorphic to ones of $A_{k'}$.

ii) If $A_k \cong A_{k'}$, $(\cong e_j R)$ $(k \neq k')$ and $M_{j} = x \Delta_p$ $(j < p)$, there exists an isomorphism $\delta: \Delta_p \rightarrow \Delta_j$ as in 2)-ii) of Theorem 1.

Proof. Assume that $(\ast, 2)$ holds. Then the $A_i$ are uniserial by [8], Proposition 7. As in the proof of Theorem 1, we consider a case $i=1$ from Lemma 1. Let

$$e_1J = N_{11} \oplus N_{12} \oplus \cdots \oplus N_{1l},$$

$$+ N_{21} \oplus N_{22} \oplus \cdots \oplus N_{2s},$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots 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We put \( F_i = \sum_{i=1}^q \oplus A_k \) then \( e_f = \sum_{i=1}^q \oplus F_i \), (cf. (17)). Since these \( F_i \) have the particular property above, \( E_i = \sum_{i=1}^q \oplus C_i \); \( C_i \subset F_i \), \( E_2 = \sum_{i=1}^q \oplus G_i \); \( G_i \subset F_i \) and \( g(C_i) \subset F_i / G_i \), where \( g \) is induced from \( g \). Hence

\[
M \approx (e_R \oplus e_f / E_2) \oplus \sum \oplus C_i(g).
\]

Next we consider \( C_i(g) \). Assume that \( A_i \) has the structure given in ii) of the theorem. Now \( A_i \) has the structure of \( e_i R \) in (16), and so every element in the endomorphism ring of sub-factor module \( T/L \) of \( A_i \) extendible to an element in \( \text{End}(A_i / L) \). Further \( T_i / L_i \cong T'_i / L'_i \) for sub-factor modules \( T_i / L_i \); \( T'_i / L'_i \) if and only if \( T_i = T'_i \) (and \( L_i = L'_i \)). From this remark and the following fact: since \( C_i(g) \subset e_f \oplus F_i / G_i \), for any submodule \( L \) in \( e_f \oplus F_i \), \( (e_f R \oplus F_i) / L \cong e_f R / X'_i \oplus F_i / G'_i \), where \( G'_i \) is a (standard) submodule of \( F'_i \) and \( X'_i \) is a submodule of \( e_f \) (cf. [8], Proposition 8), we can find an isomorphism:

\[
\text{(19)} \quad (e_R \oplus e_f / E_2) / C_i(g) \cong e_f R / X'_i \oplus \sum \oplus F_i / G_i
\]

(see the proof of Theorem 5 below and [8], Proposition 8).

Finally assume \( F_i = A_i \), i.e., \( I \) is a singleton. Then \( C_i / X_i \cong g(C_i) \), where \( X_i = g^{-1}(0) \cap C_i \). Since \( g \) is an isomorphism of \( A_i \) to \( F_i \) and \( A_i \) is uniserial, \( g(X_i) = G_i \). Hence we have the same situation as above (take \( g^{-1} \)). Accordingly we finally obtain from (19)

\[
M \approx e_f R / \sum X'_i \oplus \sum \oplus F_i / G'_i : F_i \cong F_i,
\]

which is a direct sum of hollow modules by Theorem 1.

Let \( R \) be a hereditary ring with \((*, 2)\). We shall assume \( e_i R = (\Delta_i M_{1i} \oplus M_{1i} \oplus M_{2i} \oplus M_{2i} \oplus M_{3i}) \) and \( M_{1i} \neq 0 \) for all \( j \) from Lemma 3. \( e_i J = (0 M_{1i} \oplus \cdots \oplus M_{1i} \oplus M_{2i} \oplus M_{2i} \oplus M_{3i} \oplus M_{3i} \oplus M_{3i} \oplus M_{3i} \oplus M_{3i}) = \sum_{i=1}^q \oplus F_i \) as in the proof of Theorem 2. Following \{\( F_i \}_{i=1}^q \) we divide the index set \{1, 2, 3, \ldots, \( n \)\} into \( q \)-parts \( I = I_1 \cup I_2 \cup \cdots \cup I_q \) such that \( F_i e_j = 0 \leftrightarrow j \in I_i \). Then \( I_i \cap I_j = \emptyset \) if \( i \neq j \) by i) of Theorem 2. Put \( |F_i / F_i J| = p_i \). If \( p_i = 1 \), \( F_i \) is uniserial, and so \( F_i = m_{i} \Delta_i \oplus m_{i} \Delta_i \oplus m_{i} \Delta_i \oplus \cdots \oplus m_{i} \Delta_i \), where the \( m_{i} \Delta_i \) runs over through \( I_i \) and \( \Delta_i \subset \Delta_i \subset \cdots \subset \Delta_i \) are division rings (see (16')). If \( p_i \geq 2 \), \( F_i = (m_{i} \Delta_i) \oplus (m_{i} \Delta_i \oplus m_{i} \Delta_i \oplus m_{i} \Delta_i \oplus \cdots \oplus (m_{i} \Delta_i)^{(p)} \) \( \oplus \cdots \oplus (m_{i} \Delta_i)^{(p)} \), where \( (m_{i} \Delta_i)^{(p)} \) means a direct sum of \( p \) copies of \( m_{i} \Delta_i \). Since \( e_i R e_j = 0 \) and \( R \) is hereditary, \( e_i R \) is monomorphic to \( e_i R / L \) by Lemma 2. On the other hand, \( e_i R e_j \) is a submodule of \( F_j \) for some \( j \) by i) of Theorem 2. Hence \( e_i R \approx m_{i} \Delta_j \oplus m_{i} \Delta_j \oplus m_{i} \Delta_j \oplus \cdots \oplus m_{i} \Delta_j \) or \( m_{i} \Delta_j \oplus m_{i} \Delta_j \oplus m_{i} \Delta_j \oplus \cdots \oplus m_{i} \Delta_j \) (1 < \( i = j \)) from Lemma 2. Therefore \( R \) is determined by \( \{F_i\} \), provided \( e_i R e_i \neq 0 \) for all \( i \). Since \( R \) is hereditary and \( I_i \cap I_j = \emptyset (i \neq j) \), \( M_{1i} = 0 \).
Next let $R_0$ be a hereditary ring as in (1) and assume $R_0 \approx \Sigma \oplus S_i$ as rings. Then after renumbering $\{e_i = e_{ii}\}$, we may assume

$$R_0 = \begin{pmatrix} S_1 & 0 \\ S_2 & \cdots \\ 0 & \cdots \\ S_i & \end{pmatrix}. $$

By $E_i$ we denote the identity element in $S_i$. On the other hand, for any hereditary ring $R$ as in (1)

$$R = e_i R \oplus R'_i$$

as $R$-modules,

where $R'_i = (1-e_i)R(1-e_i)$ and $e_i R$ is a two-sided ideal of $R$ by Lemma 1. If $R_0 \approx \Sigma \oplus S_i$ as above, $e_i R = \Sigma \oplus e_i RE_j$. Put $A_j = e_i RE_j$, and $A_j$ is a right ideal in $e_i R$. We use those notations in the following theorem. Thus we obtain

**Theorem 3.** Let $R$ be a (basic) hereditary ring such that $e_i R e_j \neq 0$ for all $j$. Then the following conditions are equivalent:

1) $(\ast, 2)$ holds for any two hollow modules.

2) $R e_i R$ is a direct sum of right serial rings $S_j$; 1) $S_j = T_r(\Delta_{i_1}, \Delta_{i_2}, \ldots, \Delta_{i_r})$ or 2) $T_r(\Delta_{i_1})$ and $A_j = (\Delta_{i_1}, \Delta_{i_2}, \ldots, \Delta_{i_r})$ in Case 1), $A_j = (\Delta \gamma_1, \ldots, \Delta \gamma_r)$ is a left $\Delta$ (= $e_i R e_j$)- and right $\Delta_j$-modules in Case 2), where $\Delta \subseteq \Delta_i \subseteq \cdots \subseteq \Delta_{i_r}$ are division rings.

3) $R$ is isomorphic to

$$\begin{pmatrix} \Delta & A_1 & \cdots & A_1 \\ S_1 & \cdots & \cdots \\ 0 & S_2 & \cdots \\ \cdots & \cdots & \cdots \\ 0 & \cdots & S_r \end{pmatrix}$$

(20)

where $S_k = T_{r_k}(\Delta_{i_1}, \Delta_{i_2}, \ldots, \Delta_{i_{r_k}})$ or $T_{r_k}(\Delta_k)$.

**Theorem 3'.** Let $R$ be a (basic) hereditary ring. Then $(\ast, 2)$ holds if and only if $R$ is a patched ring of hereditary rings given in (20).

**Lemma 4.** Let $R$ be a hereditary and connected (basic) ring. 1) If $R$ is a left serial ring, then $e_i R e_j \neq 0$ for all $j > 1$. 2) Conversely, if $e_i R e_j \neq 0$ for all $j$, and $[M_{ij}; \Delta_j] \leq 1$, $[M_{ij}; \Delta_i] \leq 1$ for all $i$ and $j$, then $R$ is left serial.

**Proof.** 1) Let $e_i R = e_i \Delta \oplus M_{i_1} \oplus \cdots \oplus M_{i_n}$. We divide the index set $\{2, 3, \ldots, n\}$ into two sets $I, J$ such that $M_{ii} \neq 0$ provided $i \in I$ and $M_{ij} = 0$ provided $j \in J$. Take $M_{ii}$ and consider $M_{ii}$. If $M_{ji} \neq 0$ for $j \in J$, $RM_{ji} \oplus M_{ii}$,
since $M_{ij}=0$. Hence $M_{ii}=0$ for all $i \in J$ by assumption. Hence $R=(e_t R \oplus \sum_{k \in J} \oplus e_k R) \oplus (\sum_{k \in J} \oplus e_k R)$ as rings from (2). Therefore $J=\phi$ by assumption.

2) Assume $0 \neq e_i R_j \in \Delta_{i} m_{ij} = m_{ij} \Delta_i$ for all $j$. Since $R$ is hereditary, $e_j R = \sum \oplus A_i$; the $A_i$ are hollow and no sub-factor modules of $A_i$ are isomorphic to any ones of $A_j (i \neq j)$ from (13) and the assumption $[M_{ij} : \Delta_i] \leq 1$. Similarly $J(A_i) = \sum \oplus A_{ij}$ and so on (cf. [7]). Hence any indecomposable (projective) module in $e_j R$ is equal to some $A_{i(t_i-t_i)}$. Let $M_{ii} = m_{ii} \Delta_i = \Delta_i m_{ii}$ and $M_{ij} = m_{ij} \Delta_i = \Delta_i m_{ij} (i < j)$ for a fixed $t$. Then $m_{ii} e_i R$ and $e_j e_j R$ have a common sub-factor module in $e_i R$. Hence $e_j R$ is monomorphic to $e_i R$ from the initial remark, and so $e_i R e_j = 0$, which implies $R m_{ii} \subset R m_{ji}$. Therefore $R$ is left serial.

**Theorem 4.** Let $R$ be a connected (basic) hereditary ring. Then $R$ is a left serial ring with $(*, 2)$ as right $R$-modules if and only if $R$ is isomorphic to 

$$
\begin{pmatrix}
\Delta & \Delta & \cdots & \Delta & \Delta & \cdots & \Delta \\
T_r(\Delta_i) & 0 & \cdots & 0 & T_r(\Delta) & \cdots & T_r(\Delta_i)
\end{pmatrix}
$$

where $\Delta_i \subset \Delta$ are division rings.

Proof. Assume that $R$ is a left serial ring with $(*, 2)$ as right $R$-modules. Then $R$ is isomorphic to the ring in (20) by Theorem 3 and Lemma 4. Since $R$ is left serial, the $A_i$ in (20) are isomorphic to $\Delta$ as left $\Delta$-modules and $\Delta_{ii} = \Delta_{ik} = \cdots = \Delta_{kr}$ in (20). If we take a generator of $A_i$, we know $\Delta_i \subset \Delta$. The converse is clear from the structure of the diagram.

**4. Hereditary rings with $(*, 3)$**

We have already obtained a characterization of artinian rings with $(*, 3)$ and $|e_i f |e^2| \leq 2$ in [5]. As is seen in [5], Theorem 1, the structure of such artinian rings is a little complicated. However if $R$ is a hereditary ring with $|e_j f |e_t |^2| \leq 2$, we obtain the following theorem.

We quote here a particular property of a vector space (cf. [2] and [7]).

($\#$, $m$) Let $\Delta_1$ and $\Delta_2$ be division rings and $V$ a left $\Delta_1$, right $\Delta_2$-space. For any two right $\Delta_2$-subspaces $V_1$, $V_2$ with $|V_1| = |V_2| = m$, there exists $x$ in $\Delta_1$ such that $xV_1 = V_2$.

**Theorem 5.** Let $R$ be a hereditary (basic) ring with $|e_j f |e^2| \leq 2$ for each $e = e_t$. Then $(*, 3)$ holds for any three hollow modules if and only if $e_j = A_1 \oplus A_2$ such that

1) The $A_i$ are as in Theorem 2, and further if $A_1 \approx A_2$, $2) [\Delta : \Delta(A_i)] = 2$ and
3) $eJ/eJ^2$ satisfies $(\#, 1)$ as a left $\Delta$-module and right $\Delta'$-module, where $A_i \approx e_i R$, $\Delta = eRe$, $\Delta' = eRe_j$, and $\Delta(A_i) = \{x | x \Delta, xA_i \subset A_i\}$.

Proof. Assume $eJ = A_i \oplus B_1$ as in the theorem. If $A_i \not\cong B_j$, $\Delta(C) = \Delta$ for every submodule $C$ in $eJ$ by i) of Theorem 2. Assume $A_i \cong B_j (\approx e_j R)$. Then $A_i$ and $B_j$ have the structure of $eRe_i$ as in (16). For any $C$, there exists submodules $C_1 \supseteq D_1$ in $A_i$ and $C_2 \supseteq D_2$ in $B_j$ such that $f: C_1/D_1 \cong C_2/D_2$ and $C = \{x + D_1 + f(x) + D_2 | x \in C_1\}$. From (16), $f$ is extendible to an element $g: A_i/D_1 \rightarrow B_j/D_2$. Since $(\#, 1)$ is satisfied for $eJ/eJ^2 = u_1 \Delta_J \oplus v_1 \Delta_J$, there exist $u_1$ in $\Delta$ and $z$ in $\Delta_j$ such that $u_1 + g(u_1) = \alpha u_1 z + w$, $w \in eJ^2$. However, since $u_1$, $v_1$ are in $eJ - eJ^2$ and $u_1 e_j = u_1$, $v_1 = v_1 e_j$, $w = 0$. Hence $C = C_1(f) + D_1 \oplus D_2 = \alpha(C_1 \oplus D_2)$, (note that $D_1 \approx D_2$ and $\alpha(D_1 \oplus D_2) = D_1 \oplus D_2$ and that $A_i$ is uniserial). It is clear that $\Delta(A_i) \subseteq \Delta(C_1 \oplus D_2) = \Delta(\alpha^{-1} C) = \alpha^{-1} \Delta(C) \alpha$ and so $[\Delta: \Delta(C)] \leq 2$. Thus the conditions in [5], Theorem 1 are fulfilled, and hence $(\#, 3)$ holds by [5], Theorem 2. Conversely, assume $(\#, 3)$ holds. Then 1) and 2) are clear from Theorem 2 and [5], Theorem 1. We shall show 3). We may assume from Lemma 1 and [2], Lemma 1 that $\Delta_{j+1} = \cdots = \Delta_n = 0$. Then 2) of [2], Theorem 1 is nothing but $(\#, 1)$.

As in Lemma 3, if $e_i R e_j = 0$ for all $j$, $R$ in Theorem 5 is isomorphic to

$$\begin{pmatrix}
\Delta & \Delta_1 & \cdots & \Delta_r & \Delta_{r+1} & \cdots & \Delta_{r+n} \\
T_r(\Delta_1 \Delta_2 \cdots \Delta_r) & 0 \\
0 & 0 & T_r(\Delta_{r+1} \cdots \Delta_{r+n})
\end{pmatrix},$$

where $\Delta \subset \Delta_1 \subset \cdots \subset \Delta_r$ and $\Delta \subset \Delta_{r+1} \subset \cdots \subset \Delta_{r+n}$, or

$$\begin{pmatrix}
\Delta & \Delta^{(2)} & \cdots & \Delta^{(2)} \\
0 & T_r(\Delta_1)
\end{pmatrix},$$

where $\Delta^{(2)}$ is a left $\Delta$ and right $\Delta$ space satisfying $(\#, 1)$ and $[\Delta: \Delta(\Delta_1, \cdots, \Delta_1)] = 2$.

In the former ring, $eJ = A_i \oplus A_j$ and $A_i \not\cong A_j$. Hence $(\#, n)$ holds for all $n$ by [5], Theorem 3. We do not know this fact for the latter ring.

5. Hereditary algebras

In this section we consider particular algebras over a field $K$ such that

(21) $e_i R e_j = e_i K$ \quad ([2], Condition II').

(e.g. an algebraically closed field.)

Under the assumption (21), every $\Delta_i$ in (1) is equal to $K$. In this case, if $eR$ is uniserial, $[eRe': K] \leq 1$ (cf. (14)). Hence

(22) $\text{End}_K(A/A') \approx K \approx \text{End}_R(eR/A')$.
for any submodules $A \supset A'$ in $eR$. Accordingly, from the proof of Theorem 2 (cf. [8], Theorem 2) we obtain

**Theorem 6.** Let $R$ be a hereditary $K$-algebra satisfying (21). Then the following conditions are equivalent:
1) $(\ast, 2)$ holds for any two hollow modules.
2) Every factor module of $eR \oplus eJ^m$ is a direct sum of hollow modules for each primitive idempotent $e$ and any integer $m$. (It is sufficient in case $m=1$.)

If every finitely generated $R$-module is a direct sum of hollow modules, $R$ is called a ring of right local type [10]. It is clear from the definition that $(\ast, n)$ holds for a ring of right local type. By $T_n(\Delta)$ we denoted the ring of upper tri-angular matrices over a division ring $\Delta$ (see (14)).

**Theorem 7.** Let $R$ be a hereditary (basic) $K$-algebra satisfying (21). Then the following are equivalent:
1) $(\ast, 3)$ holds for any three hollow modules, and $e_iRe_j \neq 0$ for all $j$, (and hence $(\ast, n)$ holds for all $n$).
2) $R$ is isomorphic to 
\[ T_{m_1}(K) \begin{pmatrix} K & K & \cdots & K \\ 0 & T_{m_2}(K) \end{pmatrix} \]
3) $R$ is of right local type and connected.

Proof. 1)$\rightarrow$2). Since $|eJ^i| \leq 2$ from [4], Theorem 3, we obtain it from the remark after (21) and the last part in §4.
2)$\rightarrow$3). It is clear that the ring in 2) is connected and of right local type from Lemma 4 and [10] (see [9]).
3)$\rightarrow$1). $(\ast, 3)$ holds for any three hollow modules. Since $R$ is left serial by [10], and connected, $M_{ij} \neq 0$ by Lemma 4.

**Theorem 8.** Let $R$ be a hereditary algebra as above. Then the following conditions are equivalent:
1) $(\ast, 3)$ holds for any three hollow right $R$-modules.
2) $ef=A_1 \oplus A_2$, where the $A_i$ are uniserial, and any non-trivial sub-factor modules of $A_1$ are not isomorphic to ones of $A_2$. In this case $(\ast, n)$ holds for all $n$.
3) Let $\{N_i\}_{i=1}^k$ be any set of submodules in $eR$. Then every factor module of $\sum \oplus N_i^{(n)}$ is a direct sum of hollow modules.
4) Every factor modules of $eR^{(m)} \oplus eJ^{(m)}$ is a direct sum of hollow modules for any integers $n$ and $m$. (It is sufficient in case $n=2$ and $m=1$).

Proof. 1)$\rightarrow$2) This is clear from Theorem 5 and [2], Theorem 2'.
1)$\rightarrow$3). Let $e=e_i$ and let $R_i$ and $X_i$ be as before Lemma 3. Then $R_i$ is of a right local type by Theorem 7. Since $R_iX_i=0$ and $R/X_i=R_i$, every submodule in $eR$ is an $R_i$-module. Hence every factor module of $\sum \oplus N_i^{(n)}$ is also
an \( R_r \)-module. Therefore it is a direct sum of \( R_r \)-hollow (and hence \( R \)-hollow) modules.

3)→4). This is clear. (We can show directly 1)→4) in the similar manner to [8], Theorem 2, cf. the proof of Theorem 2.)

3)→1). Let \( D = \sum_{i=1}^{3} \oplus eR/E_i \) and \( M \) a maximal submodule in \( D \). Then \( D' = eR^{(3)} \) contains the submodule \( M' \) such that \( M' \supset \sum_{i=1}^{3} \oplus E_i \) and \( M'/\sum \oplus E_i = M \). Since \( D' \) has the lifting property of simple modules modulo the radical, \( D' \) has a decomposition \( \sum F_i \) such that \( F_i \approx eR \) and \( M' = F_1 \oplus F_2 \oplus J(F_3) \). Hence \( M \) is a factor module of \( eR^{(3)} \oplus eJ \). Therefore \( M \) is a direct sum of hollow modules from 3).

**Theorem 9.** Let \( R \) be as in Theorem 8. Then \(*, 3) \) holds for any three hollow modules if and only if \( R \) is the patched ring of serial rings \( T_r(K) \) and rings of right local type \( (\mathbb{T}_r, (K)) \).

Proof. This is clear from Proposition 1 and Theorem 7.

6. **US-n algebras**

We have studied special types of hereditary algebras in §5. We shall show, in this section, that they are related with US-\( n \) algebras defined in [4].

As another generalization of right serial ring (cf. \(*, n)\), we considered

\(*, n) \) Every maximal submodule in a direct sum \( D \) of \( n \) hollow modules contains a non-zero direct summand of \( D \) [4].

It is clear that if \( D/J(D) \) is not homogeneous, \( D \) satisfies \(*, n)\). Hence we may restrict ourselves to hollow modules of a form \( eR/E \), where \( e \) is a primitive idempotent and \( E \) is a submodule of \( eR \). If \(*, n) \) holds for any direct sum of \( n \) hollow modules, we call \( R \) a right US-\( n \) ring [4]. We showed in [4] that \( R \) is right US-1 (resp. US-2) if and only if \( R \) is semisimple (resp. right uniserial). On the other hand,

**Proposition 3** ([6], Proposition 8). Let \( R \) be a right artinian ring. Then \( R \) is a right US-\( m \) ring for some \( m \) if and only if the number of isomorphism classes of hollow modules \( eR/A \) is finite and \([\Delta: \Delta(A)] < \infty \).

If \( R \) is an algebra of finite dimension over a field \( K, [\Delta: \Delta(A)] < \infty \). Hence from Proposition 3, we know that an algebra of finite representation type is a US-\( n \) algebra for some \( n \). Further we note that if \( K \) is a finite field, \( R \) is a finite ring. Then, since there are only finite non-isomorphic hollow modules,
$R$ is a US-$n$ algebra. Hence we may assume that $K$ is an infinite field.

From now on we assume that $R$ is a $K$-algebra satisfying (21). Let $e$ be a primitive idempotent in $R$. Let \{$A_i$, $A_{i+1}$, ..., $A_i$\} be a set of submodules in $eR$ such that $A_i \sim A_{i+1}$ for any pair $i$ and $j$, where $A_i \sim A_{i+1}$ means that there exists a unit element $x$ in $eRe$ such that $xA_i \subset A_{i+1}$ or $xA_{i+1} \supset A_i$. Let $m(e)$ be the maximal number $t$ among the above sets.

**Proposition 4.** Let $R$ be an algebra over $K$ satisfying (21). Then $R$ is a right US-$m$ if and only if $m = \max_e \{m(e)\} + 1 < \infty$.

Proof. This is clear from [3], Corollaries 1 and 2 of Theorem 2.

**Theorem 10.** Let $R$ be as above. We assume further $J^2 = 0$. Then $R$ is a right US-$m$ algebra if and only if $ef$ is square-free for each primitive idempotent $e$.

Proof. Assume that $R$ is right US-$m$. Since $J^2 = 0$, $ef = \sum \oplus A_i$ the $A_i$ are simple, i.e. $A_i \approx \bar{e}iK$, ($R$ is basic). If $A_i \approx A_{i+1}$, $(a_i + a_i k)K \approx A_i$ and $(a_i + a_j k)K \approx (a_i + a_j k')K$ for any $k \neq k'$ in $K$, where $A_i = a_iK$ ([6], Lemma 15). Then $R$ is not right US-$m$ for any $m$. Hence $ef$ is square-free. Conversely if $ef$ is square-free, every submodule in $ef$ is a sum of some $A_i$. Hence the number of hollow modules is finite, and so $R$ is right US-$m$ for some $m$ from Proposition 4.

**Corollary.** Let $R$ be as above. If $R$ is right US-$m$, $eJ_i ef^{i+1}$ is square-free for all $i$.

Proof. It is clear that if $R$ is right US-$m$, so is $R/J_i$ for any $i$ (cf. [4], Lemma 1). If $J_i^{i+1} = 0$, $ef^n$ is semisimple and hence we can employ the same argument given above. Therefore we obtain the corollary by induction on $n$ and the initial remark.

It is clear that the converse is not true provided $J^2 \neq 0$.

Finally we study the ring of generalized tri-angular matrices over division rings $\Delta_j$ as (1). If $R$ is a (basic) hereditary ring (more generally if $\text{gl dim } R/J^2 < \infty$), $R$ has the structure of (1) [1].

**Theorem 11.** Let $R$ be a (basic) algebra satisfying (21). Assume $\text{gl dim } R/J^2 < \infty$. Then $R$ is a US-$m$ algebra for some $m$ if and only if $[e_iRe_j: K] \leq 1$ for all $i, j$.

Proof. Assume that $R$ is a US-$m$ algebra for some $m$. We may assume that $\Delta_{i+1} = \cdots = \Delta_s = 0$ in (1) by [4], Lemma 1. Let $M_{i_k} = x_iK \oplus x_{i+1}K \oplus \cdots$. Then $[M_{i_k}: K] \leq 1$ as the proof of Theorem 10. Conversely, if $[M_{i_k}: K] \leq 1$, $e_iR$ contains only finitely many right ideals. Hence $R$ is a US-$m$ algebra for
some $m$.

7. Examples

We shall give several examples of hereditary algebras with $(\ast, n)$.

Let $K \subset L$ be fields.

1. \[
\begin{pmatrix}
K & L \\
0 & K
\end{pmatrix}
\]
is a hereditary ring with $(\ast, 2)$ and hence $(\ast, 1)$. (If $L \neq K$, $(\ast, 3)$ does not hold from Theorem 8.)

2. \[
\begin{pmatrix}
K & (K) & (K) & (K) \\
0 & K & 0 & 0 \\
0 & 0 & K & K \\
0 & 0 & 0 & K
\end{pmatrix}
\]
is a hereditary ring with $(\ast, 1)$ but not $(\ast, 2)$. In this ring, $ef$ is a direct sum of uniserial modules (cf. [8], Theorem 3).

3. \[
\begin{pmatrix}
K & L & L \\
0 & L & 0 \\
0 & 0 & L
\end{pmatrix}
\]
is a hereditary ring satisfying $(\ast, n)$ for all $n$ by Theorem 8.

4. \[
\begin{pmatrix}
K & K & (K) & (K) & (K) \\
0 & K & (K) & (K) & (K) \\
0 & 0 & K & K & K \\
0 & 0 & 0 & K & K \\
0 & 0 & 0 & 0 & K
\end{pmatrix}
\]
satisfies all conditions in Theorem 1 except the last one of i).

5. Let $R$ be an algebra satisfying (21), and $\text{gl \, dim } R/J^2 < \infty$. Then if $R$ is right $US-n$, $R$ is left $US-m$ from Theorem 10 for some $m$. However $n \neq m$ in general. For example $R = \begin{pmatrix} K & 0 & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$. Then $R$ is right $US-2$ and left $US-3$.

If $R$ does not satisfy (21), then the above fact is not true. Let $L \supseteq K$ be fields with $[L: K] = 5$ (not small) and $R = \begin{pmatrix} K & L \\ 0 & L \end{pmatrix}$. Then $R$ is right $US-2$ but not left $US-n$ for any $n$.

6. Let $K$ be a field. We can give the complete list of connected algebras given in Theorem 11, provided that $R$ is hereditary and $|R/J|$ is enough small. For instance, let $|R/J| = 6$. We shall give some samples of them.
US-11 (and (*, 2))

\[
\begin{pmatrix}
K & K & K & K & K \\
K & 0 & 0 & 0 & 0 \\
K & 0 & 0 & 0 & 0 \\
0 & K & 0 & 0 & K \\
0 & K & 0 & K & K
\end{pmatrix}
\]

US-8 (and (*, 2))

\[
\begin{pmatrix}
K & K & K & K & K \\
K & 0 & 0 & 0 & 0 \\
K & 0 & 0 & 0 & 0 \\
0 & K & 0 & 0 & K \\
0 & K & 0 & K & K
\end{pmatrix}
\]

US-7 (and (*, 1))

\[
\begin{pmatrix}
K & K & K & K & K \\
K & 0 & 0 & 0 & 0 \\
K & 0 & 0 & 0 & 0 \\
0 & K & K & K & K \\
0 & K & K & K & K
\end{pmatrix}
\]

US-6 (and (*, 2))

\[
\begin{pmatrix}
K & K & K & K & K \\
K & 0 & 0 & 0 & 0 \\
K & 0 & 0 & 0 & 0 \\
0 & K & K & K & K \\
0 & K & K & K & K
\end{pmatrix}
\]

US-5 (and (*, 2))

\[
\begin{pmatrix}
K & K & K & K & K \\
K & 0 & 0 & 0 & 0 \\
K & 0 & 0 & 0 & 0 \\
0 & K & K & K & K \\
0 & K & K & K & K
\end{pmatrix}
\]

US-4 (and (*, 3))

\[
\begin{pmatrix}
K & K & K & K & K \\
K & 0 & 0 & 0 & 0 \\
K & 0 & 0 & 0 & 0 \\
0 & K & K & K & K \\
0 & K & K & K & K
\end{pmatrix}
\]

US-3 (and (*, 3))

\[
\begin{pmatrix}
K & K & K & K & K \\
K & 0 & 0 & 0 & 0 \\
K & K & K & K & K \\
0 & K & K & K & K \\
0 & K & K & K & K
\end{pmatrix}
\]

US-2 (and (*, 3))

\[
\begin{pmatrix}
K & K & K & K & K \\
K & K & K & K & K \\
K & K & K & K & K \\
0 & K & K & K & K \\
0 & K & K & K & K
\end{pmatrix}
\]

US-1 (and (*, 3))

\[
\begin{pmatrix}
K & K & 0 \\
K & K & 0 \\
0 & K & K
\end{pmatrix}
\]
where $e = e_1$.

We do not have US-9 and US-10 algebras under the assumption $|R/J| = 6$.

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References


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