Nakagiri, S.
Osaka J. Math.
25 (1988), 353–398

STRUCTURAL PROPERTIES OF FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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(Received November 13, 1986)

Contents

1. Introduction and summary of the results

In the present work we study the structural properties of linear autonomous functional differential equations in Banach spaces within the framework of linear operator theory. We shall explain our motivation of this study.

In a series of papers Bernier and Manitius [4], Manitius [29] and Delfour and Manitius [15] they have developed an excellent state space theory for linear retarded functional differential equations (FDE's) in the product space $\boldsymbol{R}^n \times L_p([-h, 0]; \boldsymbol{R}^n)$, $h>0$. The theory is based on certain relations between semigroups associated with the FDE's and the socalled structural operators F and G. The structural operators have enriched the qualitative theory of linear FDE's and have provided various new and efficient techniques for the study of control theory involving retarded FDE's. The power of the theory has been shown nad increased by a number of contributions (refer to Delfour [12], Delfour, Lee and Manitius [14], Manitius [30,31], Salamon [39] and Vinter and Kwong [47]). Recently Salamon has been extended the state space theory to the controlled neutral FDE's and has used the theory to expand a system theory for neutral systems in his book [40].

We now pose a question. Is it possible to construct an analogous theory for partial FDE's? For the question we shall give an affirmative answer for certain class of partial FDE's. We study partial FDE's in the class consists of abstract delay evolutional equations in Banach spaces similarly as in Travis and Webb [44,45], Webb [48], Datko [9, 10] and Kunisch and Schappacher [26, 27]. Partial FDE's in this class are very general and are appropriate for system theoretical study as shown in Curtain and Pritchard [8] and Fuhrmann [16], so we take this class. Let X be a reflexive Banach space and consider the evolution equation with delay

$$(E) \qquad \frac{dx(t)}{dt} = A_0 x(t) + \int_{-h}^{0} d\eta(s) x(t+s)\,, \qquad t>0$$

on X, where A_0 generates a C_0-semigroup and η is a bounded Stieltjes measure on $I_h=[-h, 0]$. We study the equation (E) on the state space $M_p = X \times L_p(I_h; X)$. The structural operator F is concerned with the retarded part of (E) and is defined through the measure η quite similarly as in [4]. In [33, 34] the author has constructed the fundamental solution of (E) under the natural condition on η and has shown its prominent role in the optimal control theory involving (E). The introduction of fundamental solution permits us to define the structural operator G, and these G and F have made it possible to develop the state space for (E).

The objective of this paper is to extend and give certain new contributions to the state space theory for the equation (E) on a reflexive Banach Space. Many results obtained here, which are useful in applications, are considered to be possible generalizations of the results in [4, 15, 29] to infinite dimensions. However it is also the objective of this paper to propose an approach for simplifying the state space theory. Due to our approach heavily depending on functional analysis method, many of the proofs can be improved. The author believes that the results presented here will provide a useful tool in studying the control theory for partial FDE's.

We enumerate the contents of this paper. Section 2 gives some preliminary results on the equation (E). The notations and terminology to be used for (E) are given in Subsection 2.1. In Subsection 2.2 various fundamental concepts relating to (E) are introduced; e.g., the fundamental solution $W(t)$, the retarded resolvent $R(\lambda; A_0, \eta)$ which is a bounded inverse of $\Delta(\lambda) = \lambda I - A_0 - \int_{-h}^{0} e^{\lambda s} d\eta(s)$, the three kinds of retarded, point spectrum $\sigma_P(A_0, \eta)$, continuous spectrum $\sigma_C(A_0, \eta)$, and the residual spectrum $\sigma_R(A_0, \eta)$, the mild solution; and the basic fact that $R(\lambda; A_0, \eta)$ is given by the Laplace transform of $W(t)$ for $\mathrm{Re}\,\lambda$ large is stated. Also, a variation of constants formula for the mild solution in terms of $W(t)$, which is essential in our treatment, is given. In the remainder part of this subsection we introduce the transposed equation (E^T) on the adjoint space M_p^* of M_p and give an elementary adjoint theory. In Section 3 we define semigroups $S(t)$ and $S_T(t)$ associated with (E) and (E^T) respectively, by the translation segments of mild solutions. The basic properties of the semigroups like infinitesimal generator or compactness for $t>h$ are investigated as well as those for their adjoint semigroups $S^*(t)$ and $S_T^*(t)$. Section 4 is devoted to study the properties of structural operators. As in Bernier and Manitius [4] we define the structural operators F and G_t, $t>0$, then a key relation $S(t)=G_tF$, $t \geq h$ in our theory follows from the variation of constants formula. The representations of the adjoints F^* and G_t^* are shown to be of same type as F and G_t, so analogous decomposition for $S_T(t)$ holds. As a consequence of such decompositions we can show that the adjoint semigroup $S^*(t)$ is realized via a modified transposed equation with non-zero forcing term by regarding the forcing term as the initial state of the transposed equation. Other fundamental properties of F and $G=G_h$ shown in [4, 15, 29] are the intertwined property $S(t)G=GS_T^*(t)$ and $FS(t)=S_T^*(t)F$. These relations are extended to our Banach space case and their simple proofs based on the new formula $S(t)G= G_{t+h}$ are presented. It is also proved in Section 4 that the null space $\mathrm{Ker}\,G$ of G is $\{0\}$ and the image $\mathrm{Im}\,G$ of G is dense in M_p. We know that similar conditions for F are hopeful in establishing good qualitative properties of (E), but these are not true in general. Thus, in Section 5 we examine conditions for F such that $\mathrm{Ker}\,F=\{0\}$, $\mathrm{Im}\,F=M_p$ or $\mathrm{Cl}\,(\mathrm{Im}\,F)=M_p$, where Cl denotes the closure operation. A number of necessary and/or sufficient conditions for these criterion expressed by η are established by solving a Volterra integral equation with delays induced by the operator F. Among those it is shown that for differential difference equations with the retarded term $\int_{-h}^{0} d\eta(s)x(t+s) = \sum_{r=1}^{m} A_r x(t-h_r)$, $0<h_1<\cdots<h_k=h$, an equivalent condition to $\mathrm{Ker}\,F=\{0\}$ is $\mathrm{Ker}\,A_m=\{0\}$. Section 6 is devoted to studying the resolvent operators of infinitesimal generators which generates the semigroups given above. According to [7, 15, 40] various spectral operators containing exponential function terms are introduced and the relations each other and connections between F and/or $\Delta(\lambda)$ are investigated. Using such relations we show, via the characterizations of generators given in Section 3, that each resolvent is described as a composition of F (or F^*),

retarded resolvent and other spectral operators. Such representations for the resolvents play an important role in the spectral analysis for (E). With the help of such forms a detailed and somewhat complicated spectral theory than [15], [18] is developed in Section 7 and Section 8. Section 7 studies the spectral decomposition theory for (E). In Subsection 7.1 the spectrum of the generator A of $S(t)$ is determined. The spectrum of A coincides with the retarded spectrum completely. Strictly speaking, it is shown that $\sigma_P(A)=\sigma_P(A_0, \eta)$, $\sigma_C(A)=\sigma_C(A_0, \eta)$ and $\sigma_R(A)=\sigma_R(A_0, \eta)$. In Subsection 7.2 a rather sophistcated spectral decomposition is presented. A characterization of the null space $\operatorname{Ker}(\lambda I-A)^l$, $l=1, 2, \cdots$ in terms of $\Delta(\lambda)$ and its derivatives is established for $\lambda\in\sigma_P(A)$. If λ is a pole of $R(\mu; A_0, \eta)$ of order k_λ, then M_p can be decomposed as the direct sum of the generalized eigenspace $\mathcal{M}_\lambda=\operatorname{Ker}(\lambda I-A)^{k_\lambda}$ and its complementary space $\mathcal{M}_\lambda^\perp=\operatorname{Im}(\lambda I-A)^{k_\lambda}$. In view of the representation of the resolvent $R(\lambda; A)$ of A given in Section 6, the cannonical spectral projection P_λ on $\mathcal{M}_\lambda$ is expressed as a composition of F and other operators containing the retarded resolvent. Finally in this section we restrict a set $\Lambda\subset\sigma(A)$ to a subset of discrete spectrum and establish the group property of $S(t)$ on the decomposed space $\mathcal{M}_\Lambda=\bigoplus_{\lambda\in\Lambda}\mathcal{M}_\lambda$ (direct sum) with a clear picture of the asymptotic behaviour of the mild solution of (E). In Section 8 we develop the adjoint spectral decomposition theory by emphasizing the role of structural operators F and G. The main concern in Subsection 8.1 is to clarify the relation between the spectrums of the adjoint A^* of A and the generator A_T of $S_T(t)$. Thus it is shown that three kinds of spectrums of A^* and A_T coincide entirely and the generalized eigenspace $\mathcal{M}_\lambda^*$ of A^*, $\lambda\in\sigma_P(A^*)$ is given by $\mathcal{M}_\lambda^*=F^*\mathcal{M}_\lambda^T$, where $\mathcal{M}_\lambda^T$ denotes the generalized eigenspace of A_T corresponding to λ. We now denote by $\sigma_d(A^*)$ the discrete spectrum of A^*, i.e., $\dim\mathcal{M}_\lambda^*<\infty$ if $\lambda\in\sigma_d(A^*)\subset\sigma_P(A^*)$. Then it is also established that $\sigma_d(A^*)=\sigma_d(A_T)$ and $G^*\mathcal{M}_\lambda^*=\mathcal{M}_\lambda^T$. This implies, by the property of G^*, that $\dim\mathcal{M}_\lambda^*=\dim\mathcal{M}_\lambda^T$. The last result in Subsection 8.1 gives the M_p-adjoint result for A, in which a fact that $\dim\mathcal{M}_\lambda=\dim\mathcal{M}_\lambda^*$ is shown for a pole λ of $R(\mu; A_0, \eta)$. In Subsection 8.2 we are concerned with the representations of spectral projections. From the results in Subsection 8.1 we know that $\dim\mathcal{M}_\lambda^*=\dim\mathcal{M}_\lambda^T<\infty$ for $\lambda\in\sigma_d(A)$. Using this fact the spectral projection P_λ for $\lambda\in\sigma_d(A)$ is expressed in terms of the bases of $\mathcal{M}_\lambda$, $\mathcal{M}_\lambda^T$ and the operator F. In Section 9 we study the problem of completeness of generalized eigenfunctions, which means $\mathrm{Cl}(\bigcup_{\lambda\in\sigma_P(A)}\mathcal{M}_\lambda)=M_p$. First a characterization of the null space $\operatorname{Ker}P_\lambda$ for a pole λ of $R(\mu; A_0, \eta)$ is given. Then a number of necessary and sufficient conditions for the completeness are established by the use of the representation of $\operatorname{Ker}P_\lambda$. In the final Section 10 we give some examples of practical partial FDE's which illustrate the contents of this paper.

2. Linear functional differential equations in Banach spaces

2.1. Notation

The sets of real and complex numbers are denoted by $\boldsymbol{R}^1$ and $\boldsymbol{C}^1$, respectively. $\boldsymbol{R}^+$ denotes the set of non-negative numbers and $\boldsymbol{R}^n$ denotes the n-dimensional Euclidean space. Let X and Y be complex (separable) Banach spaces with norms $|\cdot|$ and $\|\cdot\|_Y$, respectively. For $E\subset Y$ the closure of E is denoted by $\mathrm{Cl}(E)$. The adjoint spaces of X, Y are denoted by X^*, Y^* and their norms are denoted by $|\cdot|_*$, $\|\cdot\|_{Y^*}$, respectively. For a closed linear operator A on a dense domain $D(A)\subset X$ into Y, its adjoint operator is denoted by A^*. The symbols Im A and Ker A will denote the image and the null space of A, respectively. The duality pairing between X and X^* is denoted by $\langle\ ,\ \rangle$ and the pairing between Y and Y^* by $\langle\ ,\ \rangle_Y$. For $E\subset Y$ the orthogonal complement $\{y^*\in Y^*: \langle y, y^*\rangle_Y=0$ for all $y\in E\}$ of E is denoted by $E^\perp$. $B(Y, X)$ denotes the Banach space of bounded linear operators from Y into X. When $X=Y$, $B(Y, X)$ is denoted by $B(X)$. Every operator norm simply is denoted by $\|\cdot\|$.

Given an interval $I\subset R^1$, $L_p(I; X)$ and $C(I; X)$ will denote the usual Banach space of X-valued measurable functions which are p-Bochner integrable $(1\leq p<\infty)$ or essentially bounded $(p=\infty)$ on I and the Banach space of strongly continuous functions on I, respectively. The norm of $L_p(I; X)$ is denoted by $\|\cdot\|_{p,I}$. $W_p^{(1)}(I; X)$ denotes the Sobolev space of X-valued functions $x(s)$ on I such that $x(s)$ and its distributional derivative $\dot{x}(s)=\dfrac{dx(s)}{ds}$ belong to $L_p(I; X)$. For each integer $k\geq 1$, $C^k(I; X)$ denotes the Banach space of all k-times continuously differentiable functions from I into X. $C(\boldsymbol{R}^+; X)$ (resp. $L_p^{\mathrm{loc}}(\boldsymbol{R}^+; X)$) will denote the Frèchet space of functions which belong to $C([0, t]; X)$ (resp. $L_p([0, t]; X)$) for any $t>0$. Let $M_p(I; X)$ denote the product space $X\times L_p(I; X)$. Given an element $g\in M_p(I; X)$, $g^0\in X$, $g^1(\cdot)\in L_p(I; X)$ will denote the two coordinates of g, i.e., $g=(g^0, g^1)$. $M_p(I; X)$ is the Banach space with norm

$$\|g\|_{M_p(I\,;\,X)}=\begin{cases}(|g^0|^p+\|g^1\|_{p,I}^p)^{1/p} & \text{if}\quad 1\leq p<\infty\\ |g^0|+\|g^1\|_{\infty,I} & \text{if}\quad p=\infty\,.\end{cases}$$

The symbol χ_E denotes the characteristic function of the set E.

2.2. Fundamental solution, mild solution and retarded resolvent

We shall review some basic results on linear functional differential equations (FDE's) in Banach spaces. Let $h>0$ be fixed and $I_h=[-h, 0]$. Consider the following autonomous retarded FDE (E) on a Banach space X:

$$\frac{dx(t)}{dt}=A_0x(t)+\int_{-h}^{0}d\eta(s)x(t+s)+u(t)\qquad \text{a.e.}\quad t\geq 0 \tag{2.1}$$

$$(2.2)\qquad x(0)=g^0\,,\quad x(s)=g^1(s)\qquad \text{a.e.}\quad s\in[-h,0)\,,$$

where $g=(g^0,g^1)\in M_p\equiv M_p(I_h;X)$, $u\in L_p^{\mathrm{loc}}(\boldsymbol{R}^+;X)$, $p\in[1,\infty]$, and A_0 generates a C_0-semigroup $T(t)$ on X. The Stieltjes measure η in (2.1) is given by

$$(2.3)\qquad \eta(s)=-\sum_{r=1}^{m}\chi_{(-\infty,-h_r]}(s)A_r-\int_s^0 A_I(\xi)d\xi\,,\qquad s\in I_h\,,$$

where $0<h_1<\cdots<h_m=h$, $A_r\in B(X)$ $(r=1,\cdots,m)$ and $A_I\in L_1(I_h;B(X))$.

Let $W(t)$ be the fundamental solution of (E), which is a unique solution of

$$W(t)=\begin{cases} T(t)+\int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)W(\xi+s)ds\,, & t\geq 0\\ O\,, \quad t<0\,. \end{cases}$$

Then $W(t)$ is strongly continuous on $\boldsymbol{R}^+$ and satisfies, for some M, $\gamma_0>0$,

$$(2.4)\qquad \|W(t)\|\leq M\exp(\gamma_0 t)\,,\qquad t\geq 0\,.$$

If the condition

$$(2.5)\qquad A_I(\cdot)\in L_{p'}(I_h;B(X))\,,\qquad 1/p+1/p'=1$$

is satisfied, then for each $t\in\boldsymbol{R}^+$ the operator valued function $U_t(\cdot)$ given by

$$(2.6)\qquad U_t(s)=\sum_{r=1}^{m}W(t-s-h_r)A_r\chi_{[-h_r,0]}(s)+\int_{-h}^{s}W(t-s+\xi)A_I(\xi)d\xi$$
$$=\int_{-h}^{s}W(t-s+\xi)d\eta(\xi)\,,\qquad \text{a.e.}\quad s\in I_h$$

belongs to $L_p(I_{h'};B(X))$. This follows from the Hausdorff-Young inequality. Hence the function

$$(2.7)\qquad x(t;g,u)=\begin{cases} W(t)g^0+\int_{-h}^0 U_t(s)g^1(s)\,ds+\int_0^t W(t-s)u(s)ds\,, & t\geq 0\\ g^1(t) \qquad \text{a.e.}\quad t\in[-h,0)\,. \end{cases}$$

is well defined and is an element of $C(\boldsymbol{R}^+;X)\cap L_p(I_h;X)$. From (2.4)–(2.7) we can derive the following estimate

$$(2.8)\qquad |x(t;g,u)|\leq(M_0\|g\|_{M_p}+M_1\|u(\cdot)\|_{p,[0,t]})\exp(\gamma_0 t)\,,\qquad t\geq 0\,,$$

where M_0 and M_1 are constants depending only on p, η and A_0.

Theorem 2.1. *Let* (2.5) *be satisfied. Then the function* $x(t)=x(t;g,u)$ *in* (2.7) *is the unique solution of the following functional integral equation*:

$$(2.9)\qquad x(t)=\begin{cases} T(t)g^0+\int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)x(s+\xi)ds+\int_0^t T(t-s)u(s)ds\,, & t\geq 0\\ g^1(t) \qquad \text{a.e.}\quad t\in[-h,0)\,. \end{cases}$$

In the sense of Theorem 2.1 we shall call this $x(t)$ the mild solution of (E). The formula (2.7) is well known as a variation of constants formula for retarded FDE's in $\boldsymbol{R}^n$ (cf. Hale [18, Chap. 6]). Since we use the class of mild solutions (2.7) throughout this paper, the condition (2.5) is always assumed. A sufficient condition for the existence of differentiable solution of (E) is given by the next corollary (for the proof see [34]).

Corollary 2.1. *Let X be reflexive. If $g=(g^0, g^1)$ and u satisfy*

$$g^1 \in W_p^{(1)}(I_h; X), \quad g^1(0) = g^0 \in D(A_0),$$
$$u \in W_p^{(1)}([0, t]; X) \qquad \text{for each} \quad t>0,$$

then the function $x(t)=x(t; g, u)$ given in (2.7) *is a strong solution of* (E), *i.e., $x(t)$ satisfies* (*i*) $x \in C(\boldsymbol{R}^+; X) \cap W_p^{(1)}([0, t]; X)$ *for all* $t>0$; (*ii*) $x(t) \in D(A_0)$ *for* a.e. $t \geq 0$, *$x(t)$ is strongly differentiable and satisfies the equation* (2.1); (*iii*) $x(0)=g^0$, $x(s)=g^1(s)$ a.e. $s \in [-h, 0)$.

For each $\lambda \in \boldsymbol{C}^1$ we define the densely defined closed linear operator $\Delta(\lambda)= \Delta(\lambda; A_0, \eta)$ by

$$\Delta(\lambda) = \lambda I - A_0 - \int_{-h}^{0} e^{\lambda s} d\eta(s), \tag{2.10}$$

where I denotes the identity operator on X. The retarded resolvent set $\rho(A_0, \eta)$ we understand the set of all values λ in $\boldsymbol{C}^1$ for which the operator $\Delta(\lambda)$ has a bounded inverse with dense domain in X. In this case $\Delta(\lambda)^{-1}$ is denoted by $R(\lambda; A_0, \eta)$ and is called the retarded resolvent. The complement of $\rho(A_0, \eta)$ in the complex plane is called the retarded spectrum and is denoted by $\sigma(A_0, \eta)$. The three different types of retarded spectrum can be defined as in the following manner. The continuous retarded spectrum $\sigma_C(A_0, \eta)$ is the set of values λ for which $\Delta(\lambda)$ has an unbounded inverse with dense domain in X. The residual retarded spectrum $\sigma_R(A_0,\eta)$ is the set of values λ for which $\Delta(\lambda)$ has an inverse whose domain is not dense in X. The point retarded spectrum $\sigma_P(A_0, \eta)$ is the set of values λ for which no inverse of $\Delta(\lambda)$ exists (cf. Hille and Phillip [31, p. 54], Tanabe [43, Chap. 8]).

We know that the retarded resolvent set $\rho(A_0, \eta)$ is open in $\boldsymbol{C}^1$ and contains right half plane and the retarded resolvent $R(\lambda; A_0, \eta)$ is holomorphic on $\rho(A_0, \eta)$. In fact, we have the following

Theorem 2.2. *Let*

$$\omega_0 = \inf \{\alpha : \|W(t)\| \leq M e^{\alpha t}, \quad t>0 \quad \text{for some} \quad M>0\}. \tag{2.11}$$

If $\mathrm{Re}\lambda>\omega_0$, *then* $\lambda \in \rho(A_0, \eta)$ *and the retarded resolvent* $R(\lambda; A_0, \eta)$ *is given by the Laplace transform of $W(t)$, i.e.,*

$$(2.12)\qquad R(\lambda;\, A_0,\, \eta) = \int_0^\infty e^{-\lambda t} W(t)dt\,.$$

Next we give an elementary adjoint theory for (E) under the assumption that X is reflexive and $p \neq \infty$. Then the adjoint space M_p^* of M_p is identified with the product space $X^* \times L_{p'}(I_h; X^*)$, where $1/p+1/p'=1$. Let $f=(f^0, f^1) \in M_p^*$ and $(v \in L_{p'}^{\text{loc}}(\boldsymbol{R}^+; X^*)$. The transposed equation (E^T) on X^* is defined by

$$(2.13)\qquad \frac{dz(t)}{dt} = A_0^* z(t) + \int_{-h}^0 d\eta^*(s) z(t+s) + v(t) \qquad \text{a.e.} \quad t \geq 0$$

$$(2.14)\qquad z(0) = f^0, \quad z(s) = f^1(s) \qquad \text{a.e.} \quad s \in [-h,\, 0)\,.$$

Since X is reflexive, the adjoint operator A_0^* generates a C_0-semigroup $T^*(t)$ on X^* which is given by the adjoint of $T(t)$ (see [37]). Hencc we can construct the fundamental solution $W^T(t)$ of (E^T) as the unique solution of the equation

$$W^T(t) = \begin{cases} T^*(t) + \int_0^t T^*(t-s) \int_{-h}^0 d\eta^*(\xi) W^T(\xi+s) ds\,, & t \geq 0 \\ O\,, \quad t < 0\,. \end{cases}$$

We denote by $W^*(t)$ the adjoint of $W(t)$. Then we can show that $W^T(t) = W^*(t)$, $t \in \boldsymbol{R}^1$. This implies that $W^*(t)$ is strongly continuous on $\boldsymbol{R}^+$. Throughout this paper the condition

$$A_1^*(\cdot) \in L_p(I_h;\, B(X^*))\,, \quad 1/p + 1/p' = 1$$

is assumed whenever the transposed equation (E^T) is in consideration. Thus, the (unique) mild solution $z(t)$ of (E^T) exists and is represented by

$$(2.15)\quad z(t) = z(t; f,\, v) = W^*(t) f^0 + \int_{-h}^0 V_t(s) f^1(s) ds + \int_0^t W^*(t-s) v(s) ds\,, \qquad t \geq 0\,,$$

where

$$(2.16)\quad V_t(s) = \sum_{r=1}^m W^*(t-s-h_r) A_r^* \chi_{[-h_r, 0]}(s) + \int_{-h}^s W^*(t-s+\xi) A_1^*(\xi) d\xi\,, \qquad \text{a.e.} \quad s \in I_h\,,$$

For $\lambda \in \boldsymbol{C}^1$ define the operator

$$\Delta_T(\lambda) = \Delta(\lambda;\, A_0^*,\, \eta^*) = \lambda I - A_0^* - \int_{-h}^0 e^{\lambda s} d\eta^*(s)\,.$$

The retarded resolvent and three kinds of spectrum corresponding to $\Delta_T(\lambda)$ are defined similarly as for $\Delta(\lambda)$.

Theorem 2.3. (i) $\lambda \in \rho(A_0, \eta)$ *if and only if* $\bar{\lambda}$*(complex conjugate)* $\in \rho(A_0^*, \eta^*)$ *and*

$$(2.17)\qquad R(\lambda;\, A_0,\, \eta)^* = R(\bar{\lambda};\, A_0^*,\, \eta^*)\,.$$

(ii) *Both retarded resolvent sets* $\rho(A_0, \eta)$ *and* $\rho(A_0^*, \eta^*)$ *contain the half plane*

$\{\lambda \in C^1: \mathrm{Re}\,\lambda > \omega_0\}$, *where* ω_0 *is given in* (2.11).

(iii) $$R(\lambda; A_0^*, \eta^*) = \int_0^\infty e^{-\lambda t} W^*(t)dt \qquad \text{for} \quad \mathrm{Re}\,\lambda > \omega_0 .$$

Here in Theorem 2.3 (i) we remark that the duality pairing $\langle\, ,\rangle$ between X and X^* satisfies

(2.18) $$\langle x, \alpha x^* \rangle = \langle \bar{\alpha} x, x^* \rangle \qquad \text{for} \quad \alpha \in C^1, \quad (x, x^*) \in X \times X^* .$$

Complete proofs of these results in this section can be found in [34, 35].

3. Semigroups associated with functional differential equations

This section is devoted to studying basic properties of semigroups associated with the equations (E) and (E^T). In what follows we assume that X is reflexive and $1 < p < \infty$.

Let $x(t; g)$ be the mild solution of (E) with $u \equiv 0$ and $g \in M_p$. The solution operator $S(t): M_p \to M_p$, $t \geq 0$ is defined by

(3.1) $$S(t)g = (x(t; g), x_t(\cdot\,; g)) \qquad \text{for} \quad g \in M_p ,$$

where $x_t(s; g) = x(t+s; g)$ a.e. $s \in I_h$. The operator $S(t)$ is bounded and linear on M_p by (2.7) and has the following properties (for similar results, see [2,4,5, 6,44,46,48]).

Proposition 3.1. (i) *The family of operators* $\{S(t): t \geq 0\}$ *is a* C_0*-semigroup on* M_p.
(ii) *If* $T(t)$ *is compact for all* $t > 0$, *then* $S(t)$ *is compact for* $t > h$.
(iii) *The infinitesimal generator* A *of* $S(t)$ *is given by*

(3.2) $$D(A) = \{g = (g^0, g^1) \in M_p: g^1 \in W_p^{(1)}(I_h; X), g^1(0) = g^0 \in D(A_0)\} ,$$

(3.3) $$Ag = (A_0 g^0 + \int_{-h}^0 d\eta(s) g^1(s), \frac{dg^1}{ds}(\cdot)) \qquad \text{for} \quad g = (g^0, g^1) \in D(A) ,$$

and for $g \in D(A)$,

(3.4) $$\frac{dS(t)g}{dt} = AS(t)g = S(t)Ag , \qquad t > 0 .$$

Proof. (i) The semigroup property $S(t+s) = S(t)S(s)$, $S(0) = I$ is obvious from the definition (3.1). Strong continuity of $S(t)$ on M_p follows from that $x(t; g) \to g^0$ in X as $t \to 0+$ by (2.9) and that $x_t(\cdot\,; g) \to g^1$ in $L_p(I_h; X)$ as $t \to 0+$ by the absolute continuity of Bochner integrable functions (cf. Ahmed and Teo [1, p. 16]).
(ii) First we introduce the operator $Q^t: M_p \to X$, $t \geq 0$ defined by

$$Q^t g = \int_0^t T(t-s)k(s;g)ds\,, \qquad g\in M_p\,, \tag{3.5}$$

where

$$k(s;g) = \int_{-h}^0 d\eta(\xi)x(s+\xi;g)\,, \qquad s\geq 0\,. \tag{3.6}$$

Using Hölder inequality and the estimate (2.8), we have

$$\|k(\cdot\,;g)\|_{p,[0,t]} \leq M_2(t)\|g\|_{M_p}\,, \tag{3.7}$$

where

$$M_2(t) = (\sum_{r=1}^m \|A_r\| + \|A_I(\cdot)\|_{p',I_h}\cdot t^{1/p})\,(1+M_0 t^{1/p}\exp(\gamma_0 t))\,.$$

In order to prove the compactness of Q^t for $t>0$ under the compactness of $T(t)$, $t>0$, we define the ε-approximation $Q_\varepsilon^t\colon M_p\to X$ of Q^t for $\varepsilon\in(0,t]$ by

$$Q_\varepsilon^t g = T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)k(s;g)ds\,, \qquad g\in M_p \tag{3.8}$$

Since $T(\varepsilon)$ is compact, Q_ε^t is also compact. The compactness of Q^t follows from

$$|(Q_\varepsilon^t - Q^t)g| = |\int_{t-\varepsilon}^t T(t-s)k(s;g)ds| \leq M_3(t)\cdot\varepsilon^{1/p'}\cdot\|g\|_{M_p}\,, \tag{3.9}$$

where $M_3(t)=(\sup_{s\in[0,t]}\|T(s)\|)M_2(t)$.

Now let $t>h$ be fixed and let the operator $R^t\colon M_p\to C([t-h,t];X)$ be defined by

$$(R^t g)(s) = x(s;g)\,, \qquad s\in[t-h,t]\,.$$

Let E be a bounded set in M_p. Since $T(s)$ and Q^s are compact for $s>0$, from the equation (2.9) it follows that for each $s\in[t-h,t]$, the set $\{(R^t g)(s)\in X\colon g\in E\}$ is precompact in X. Next we shall prove that $\{R^t g; g\in E\}$ is an equicontinuous family of $C([t-h,t];X)$. Let $0<a<t-h$, $g\in E$ and $t-h\leq s'<s\leq t$. Then we obtain from (3.5) and (3.9) that

$$\begin{aligned} &|(R^t g)(s)-(R^t g)(s')| \\ &\leq \|T(s)-T(s')\|\cdot|g^0| + \int_{s'}^s \|-T(s-\cdot)\|\cdot|k(\tau;g)|d\tau \\ &+\int_0^{s'-a}\|T(s-\tau)-T(s'-\tau)\|\cdot|k(\tau;g)|d\tau \\ &+\int_{s'-a}^{s'}\|T(s-\tau)-T(s'-\tau)\|\cdot|k(\tau;g)|d\tau \end{aligned} \tag{3.10}$$

$$\leq \|T(s)-T(s')\|\cdot|g^0|+M_3(t)\|g\|_{M_p}(s-s')^{1/p'}$$
$$+(\sup\{\|T(\tau)-T(\tau')\|:\tau,\tau'\in[a,t],\ |\tau-\tau'|=|s-s'|\})$$
$$\times t^{1/p'}M_3(t)\|g\|_{M_p}+2M_3(t)\|g\|_{M_p}\cdot a^{1/p'}.$$

For each fixed $a>0$, it is verified via Hille and Phillips [21, p. 304] that $T(s)$ is uniformly continuous on $[a, t]$ in the operator norm topology of $B(X)$. Taking $a>0$ sufficiently small and applying the uniform continuity to (3.10), we have the desired equi-continuity. Therefore by Royden [38, p. 155], R^t is compact. Now we introduce the immersion I^t: $C([t-h, t]; X)\to M_p$ by $I^t x(\cdot)=(x(t), x_t(\cdot))$ for $x\in C([t-h, t]; X)$. Clearly I^t is bounded. Since $S(t)$ can be decomposed as $S(t)=I^tR^t$ for $t>h$, $S(t)$ is compact for $t>h$.
(iii) We denote by $\tilde{A}$ and $D(\tilde{A})$ the infinitesimal generator of $S(t)$ and its domain, respectively. Let $g\in D(\tilde{A})$ and

$$\tilde{A}g=(y^0, y^1)\,. \tag{3.11}$$

Since the second coordinate of $S(t)g$ is the t-shift $x(t+\cdot\,; g)$, it follows immediately that

$$x(\cdot\,; g)=g^1\in W_p^{(1)}(I_h; X) \quad \text{and} \quad \frac{d^+}{ds}x(\cdot\,; g)=\dot{g}^1=y^1 \quad \text{in} \quad L_p(I_h; X)\,, \tag{3.12}$$

where $\frac{d^+}{ds}$ denotes the right hand derivative. By redefining on the set of measure 0 we can suppose that $x(s; g)=g^1(s)$ is absolutely continuous from I_h to X (cf. Barbu [3, p. 19, Theorem 2.2]). Since $x(0; g)=g^0$, this implies $g^1(0)=g^0$ and $x(\cdot\,; g)\in C([-h, \infty); X)$. Then the function $k(s; g)$ in (3.6) is continuous in $s\geq 0$ and satisfies $\lim_{t\to 0+} k(t; g)=\int_{-h}^0 d\eta(s)g^1(s)$. So that

$$\lim_{t\to 0+}\frac{1}{t}\int_0^t T(t-s)k(s; g)ds=\int_{-h}^0 d\eta(s)g^1(s)\,. \tag{3.13}$$

Applying (2.9) and (3.13) to the first coordinate of (3.11), we obtain that

$$\begin{aligned} y^0 &= \lim_{t\to 0+}\frac{1}{t}(x(t; g)-g^0) \\ &= \lim_{t\to 0+}\frac{1}{t}\left(T(t)g^0+\int_0^t T(t-s)k(s; g)ds-g_0\right) \\ &= \lim_{t\to 0+}\frac{1}{t}(T(t)g^0-g)+\int_{-h}^0 d\eta(s)g^1(s) \qquad \text{exists in } X. \end{aligned} \tag{3.14}$$

Hence $\lim_{t\to 0+} t^{-1}(T(t)g^0-g^0)$ exists in X, i.e., $g^0\in D(A_0)$ and $y^0=A_0g^0+\int_{-h}^0 d\eta(s)g^1(s)$. This shows

$$D(\tilde{A})\subset D(A) \quad \text{and} \quad Ag=\tilde{A}g \qquad \text{for} \quad g\in D(\tilde{A})\,.$$

Next we show the reverse inclusion. Let $g \in D(A)$. According to Corollary 2.1 we have $x(\cdot\,; g) \in W_p^{(1)}([-h, a]; X)$ for any $a>0$, from which (3.13) follows. Combining this with $g^0 \in D(A_0)$ we see that

$$\lim_{t\to+0} \frac{1}{t}(x(t; g)-g^0) = A_0 g^0 + \int_{-h}^0 d\eta(s) g^1(s)\,.$$

Noting

$$(3.15)\quad \frac{1}{t}(x_t(\xi; g)-g^1(\xi))-\dot{g}^1(\xi)$$
$$= \frac{1}{t}(x(t+\xi; g)-x(\xi; g))-\dot{x}(\xi; g) = \frac{1}{t}\int_0^t (\dot{x}(s+\xi; g)-\dot{x}(\xi; g))ds\,,$$

for $\xi \in [-h, 0]$ we obtain with the aid of Hölder inequality that

$$(3.16)\quad \|\frac{1}{t}(x_t(\cdot\,; g)-g^1)-\dot{g}^1\|_{p,I_h}^p \leq \frac{1}{t}\int_0^t [\int_{-h}^0 |\dot{x}(s+\xi; g)-\dot{x}(\xi; g)|^p d\xi]ds\,.$$

This implies that $\lim_{t\to 0+} t^{-1}(x_t(\cdot\,; g)-g^1)$ exists in $L_p(I_h; X)$ and equals $\dot{g}^1$. Thus, we prove $D(A) \subset D(\tilde{A})$ and $Ag=\tilde{A}g$ for $g \in D(A)$, and hence (3.2), (3.3) are shown. The remaining equality (3.4) is obvious.

Concerning the transposed equation (E^T) we define the semigroup $S_T(t)$ on M_p^* in an analogous manner. Thus we have:

Proposition 3.2. (i) *The family of operators* $\{S_T(t): t \geq 0\}$ *is a* C_0*-semigroup on* M_p^*.
(ii) *If* $T(t)$ *is compact for* $t>0$, *then* $S_T(t)$ *is compact for* $t>h$.
(iii) *The infinitesimal generator* A_T *of* $S_T(t)$ *is given by*

$$D(A_T) = \{f=(f^0, f^1) \in M_p^*: f^1 \in W_{p'}^{(1)}(I_h; X^*), f^1(0) = f^0 \in D(A_0^*)\}\,,$$
$$A_T f = (A_0^* f^0 + \int_{-h}^0 d\eta^*(s) f^1(s), \frac{df^1}{ds}(\cdot)) \qquad \text{for} \quad f=(f^0, f^1) \in D(A_T)\,,$$

where $1/p+1/p'=1$.

Since M_p is reflexive, we know that the adjoint $S^*(t)$ of $S(t)$ generates a C_0-semigroup on M_p^*. Probably it was Vinter [46] who first characterized the infinitesimal generator of the semigroup $S^*(t)$ in the case $X=\mathbf{R}^n$ and $p=2$. His article seems hardly to available, however, we shall give a complete proof of the result in our Banach space case.

Proposition 3.3. *The infinitesimal generator* A^* *of* $S^*(t)$ *is given by*

$$(3.17)\quad D(A^*) = \{f=(f^0, f^1) \in M_p^*: w(f) \in W_{p'}^{(1)}(I_h; X^*), w(f)(-h) = 0,$$
$$f^0 \in D(A_0^*)\}$$

(3.18) $A^*f = (A_0^*f^0 + f^1(0), \dot{w}(f))$ *for* $f = (f^0, f^1) \in D(A^*)$,

where

(3.19) $$w(f)(s) = \int_{-h}^{s} d\eta^*(s) f^0 - f^1(s), \quad s \in I_h.$$

Proof. Note first that the infinitesimal generator of $S^*(t)$ is given by the adjoint A^* of A. Let $(g^0, g^1) \in D(A)$ and $(f^0, f^1) \in M_p^*$. Assume that there exists a $(k^0, k^1) \in M_p^*$ such that for all $(g^0, g^1) \in D(A)$,

$$\langle A(g^0, g^1), (f^0, f^1) \rangle_{M_p} = \langle (g^0, g^1), (k^0, k^1) \rangle_{M_p}$$

or, equivalently by Proposition 3.1,

(3.20) $$\langle A_0 g^0 + \int_{-h}^{0} d\eta(s) g^1(s), f^0 \rangle + \int_{-h}^{0} \langle \dot{g}^1(s), f^1(s) \rangle ds = \langle g^0, k^0 \rangle + \int_{-h}^{0} \langle g^1(s), k^1(s) \rangle ds$$

Set $M(s) = \int_{-h}^{s} k^1(\xi) d\xi$, $s \in I_h$. $M(-h) = 0$ is evident. It is easy to see that, by using integration by parts,

(3.21) $$\int_{-h}^{0} \langle g^1(s), k^1(s) \rangle ds = \langle g^1(0), M(0) \rangle - \int_{-h}^{0} \langle \dot{g}^1(s), M(s) \rangle ds.$$

Next we set $h_0 = 0$ and

$$N(s) = \int_{-h}^{s} d\eta^*(s) f^0 = \sum_{r=1}^{m} A_r^* \chi_{[-h_r, 0]}(s) f^0 + \int_{-h}^{s} A_1^*(\xi) f^0 d\xi.$$

Again, using integration by parts on each $[-h_r, -h_{r-1}]$, $r = 1, \cdots, m$, it is not difficult to show that

(3.22) $$\int_{-1}^{0} \langle \dot{g}^1(s), N(s) \rangle ds = \langle g^0, N(0) \rangle - \langle \int_{-h}^{0} d\eta(s) g^1(s), f^0 \rangle$$

Then by (3.20)–(3.22), we see for $(g^0, g^1) \in D(A)$,

(3.23) $$\int_{-h}^{0} \langle \dot{g}^1(s), f^1(s) - N(s) + M(s) \rangle ds + \langle A_0 g^0, f^0 \rangle = \langle g^0, k^0 - N(0) + M(0) \rangle.$$

For $g^0 \in D(A_0)$ and $g^1(s) \equiv g^0$, it is obvious that $(g^0, g^1) \in D(A)$ and $\dot{g} = 0$. Hence applying such (g^0, g^1) to (3.23), we have

(3.24) $$\langle A_0 g^0, f^0 \rangle = \langle g^0, k^0 - N(0) + M(0) \rangle \qquad \text{for all} \quad g^0 \in D(A_0).$$

This proves that $f^0 \in D(A_0^*)$ and

(3.25) $$k^0 = A_0^* f^0 + \int_{-h}^{0} d\eta^*(s) f^0 - \int_{-h}^{0} k^1(s) ds.$$

Since $\{\dot{g}^1 : (g^0, g^1) \in D(A)\}$ is dense in $L_p(I_h; X)$, from (3.23) and (3.24) it follows that

$$f^1(s) - N(s) + M(s) = 0 \qquad \text{a.e.} \quad s \in I_h . \tag{3.26}$$

If we put $w(f)(s) = N(s) - f^1(s)$, $s \in I_h$, then by (3.26) $w(f)$ satisfies

$$w(f) \in W_{p'}^{(1)}(I_h; X^*) , \quad \dot{w}(f) = \dot{M} = k^1 \qquad \text{in} \quad L_{p'}(I_h; X^*) \tag{3.27}$$

and

$$w(f)(-h) = 0 , \quad w(f)(0) = \int_{-h}^{0} k^1(s) ds = \int_{-h}^{0} d\eta^*(s) f^0 - f^1(0) . \tag{3.28}$$

Therefore, by (3.25), (3.27) and (3.28) we conclude that $D(A^*)$ is given by (3.17) and $A^*f, f \in D(A^*)$ is represented by $A^*f = (k^0, k^1) = (A_0^* f^0 + f^1(0), \dot{w}(f))$, which is (3.18). Conversely it is not difficult ot show that any element of the right hand side of (3.17) belongs to $D(A^*)$. Thus the proof is complete.

4. Structural operators F, G and their adjoint operators

In this section we extend the structural operator F and G_t introduced in [4] for the case $X = \boldsymbol{R}^n$ to our Banach space case and study their basic properties including the decomposition formula as well as their adjoint operators.

Define the operator $F_1 : L_p(I_h; X) \to L_p(I_h; X)$ by

$$\begin{aligned} [F_1 g^1](s) &= \int_{-h}^{s} d\eta(\xi) g^1(\xi - s) \\ &= \sum_{r=1}^{m} A_r \chi_{[-h_r, 0]}(s) g^1(-h_r - s) + \int_{-h}^{s} A_I(\xi) g^1(\xi - s) d\xi \qquad \text{a.e.} \quad s \in I_h . \end{aligned} \tag{4.1}$$

By direct calculations using Hölder inequality it is verified that F_1 is into, linear and bounded.

First we give an equivalent representation formula of the mild solution $x(t; g)$ to (2.7) in terms of $W(t)$ and F_1, which is given by another complicated form in [4, p. 902]. The following one is explicit.

Lemma 4.1. *The mild solution $x(t; g)$ is represented by*

$$x(t; g) = W(t) g^0 + \int_{-h}^{0} W(t+s) [F_1 g^1](s) ds , \qquad t \geq 0 . \tag{4.2}$$

Proof. In view of (2.7) we are left to prove the equality

$$\int_{-h}^{0} U_t(s) g^1(s) ds = \int_{-h}^{0} W(t+s) [F_1 g^1](s) ds . \tag{4.3}$$

With the aid of suitable changes of variables and Fubini's theorem we obtain

$$\begin{aligned}\int_{-h}^{0} U_t(s)g^1(s)ds &= \sum_{r=1}^{m}\int_{-h_r}^{0} W(t-s-h_r)A_r g^1(s)ds \\ &\quad +\int_{-h}^{0}\left(\int_{-h}^{s} W(t-s+\xi)A_I(\xi)d\xi\right)g^1(s)ds \\ &= \int_{-h}^{0} W(t+s)\left\{\sum_{r=1}^{m} A_r \chi_{[-h_r,0]}(s)g^1(-h_r-s)\right\}ds \\ &\quad +\int_{-h}^{0} W(t+s)\left\{\int_{-h}^{s} A_I(\xi)g^1(\xi-s)d\xi\right\}ds \\ &= \int_{-h}^{0} W(t+s)[F_1 g^1](s)ds\,.\end{aligned}$$

The first structural operator F: $M_p \to M_p$ is defined by $F=\begin{bmatrix} I & O \\ O & F_1 \end{bmatrix}$, i.e.,

$$[Fg]^0 = g^0\,,\quad [Fg]^1 = F_1 g^1 \qquad \text{for}\quad g=(g^0, g^1)\in M_p\,. \tag{4.4}$$

By (2.7), (4.4) and Lemma 4.1, we have

$$x(t+s;g) = \begin{cases} W(t+s)g^0 + \int_{-h}^{0} W(t+s+\xi)[Fg]^1(\xi)d\xi\,, & t+s\geq 0 \\ g^1(t+s)\,, & t+s<0\,. \end{cases} \tag{4.5}$$

The equality (4.5) suggests us to introduce the operator G_t: $M_p \to M_p$, $t\geq 0$ defined by

$$[G_t g]^1(s) = W(t+s)g^0 + \int_{-h}^{0} W(t+s+\xi)g^1(\xi)d\xi\,, \qquad s\in I_h\,, \tag{4.6}$$

$$[G_t g]^0 = [G_t g^1](0)\,, \qquad g=(g^0, g^1)\in M_p\,. \tag{4.7}$$

Clearly G_t is linear and bounded. Notice that the right hand side of (4.6) vanishes if $t+s<0$. Especially we define the second structural operator G: $M_p \to M_p$ by

$$G = G_h\,. \tag{4.8}$$

We remark here that $G_t g \in C(I_h; X)$ for $t\geq h$ and $g\in M_p$.

The following proposition is obvious from (4.5) and the definitions of F, G_t, G and $\kappa(t)$.

Proposition 4.1. *The semigroup $S(t)$ is represented by*

$$S(t) = G_t F + \kappa(t)\,, \qquad t\geq 0\,, \tag{4.9}$$

where $\kappa(t)$: $M_p \to M_p$ is given by

$$[\kappa(t)g]^0 = 0\,,\quad [\kappa(t)g]^1(s) = g^1(t+s)\chi_{[-h,-t)}(s) \qquad \text{a.e.}\quad s\in I_h\,. \tag{4.10}$$

In particular, $S(h)$ is decomposed as

(4.11) $$S(h) = GF.$$

To obtain a similar representation formula for the transposed semigroup $S_T(t)$, we have to compute the adjoints of G_t and F (cf. (2.15), (2.16)).

The following proposition can be established by a direct calculation.

Proposition 4.2. *The adjoint* $F^*: M_p^* \to M_p^*$ *of* F *is given by*

$$F^* = \begin{bmatrix} I & O \\ O & F_1^* \end{bmatrix},$$

where $F_1^*: L_{p'}(I_h: X^*) \to L_{p'}(I_h; X^*)$ *denotes the adjoint of* F_1 *and is represented by*

(4.12) $$[F_1^* f^1](s) = \int_{-h}^{s} d\eta^*(\xi) f^1(\xi - s)$$
$$= \sum_{r=1}^{m} A_r^* \chi_{[-h_r, 0]}(s) f^1(-h_r - s) + \int_{-h}^{s} A_1^*(\xi) f^1(\xi - s) d\xi \quad \text{a.e. } s \in I_h.$$

The following proposition is also easily proved.

Proposition 4.3. *The adjoint* $G_t^*: M_p^* \to M_p^*$ *of* G_t, $t \geq 0$ *is represented by*

$$\begin{cases} [G_t^* f]^1(s) = W^*(t+s) f^0 + \int_{-h}^{0} W^*(t+s+\xi) f^1(\xi) d\xi, & s \in I_h, \\ [G_t^* f]^0 = [G_t^* f]^1(0), & f = (f^0, f^1) \in M_p^*. \end{cases}$$

Consider the transposed equation (E^T). By (2.15) and Proposition 4.2, we see that the mild solution $z(t; f) = z(t; f, 0)$ of (E^T) is written as

$$z(t; f) = W^*(t) f^0 + \int_{-h}^{0} W^*(t+s) [F^* f]^1(s) ds, \qquad t \geq 0.$$

Hence by Proposition 4.3, we obtain the following

Proposition 4.4. *The semigroup* $S_T(t)$ *is represented by*

$$S_T(t) = G_t^* F^* + \kappa(t), \qquad t \geq 0,$$

where $\kappa(t): M_p^* \to M_p^*$ *is same as given in* (4.10). *In particular,*

(4.13) $$S_T(h) = G^* F^*.$$

We can verify by standard manupulation involving the pairing $\langle \, , \rangle_{M_p}$ that the adjoint $\kappa^*(t): M_p^* \to M_p^*$ of $\kappa(t)$ in (4.10) is given by

(4.14) $$[\kappa^*(t) f]^0 = 0, \quad [\kappa^*(t) f]^1(s) = \chi_{[0, s+h]}(t) f^1(s-t), \qquad f \in M_p^*.$$

Since the same operator as in (4.14) can be defined on M_p, we denote this operator by the same symbol $\kappa^*(t)$. Then taking adjoints of $S(t)$ and $S_T(t)$, we have the following result.

Corollary 4.1. *The adjoint semigroups $S^*(t)$ and $S_T^*(t)$ are represented by*

$$S^*(t) = F^*G_t^* + \kappa^*(t)\,, \quad S_T^*(t) = FG_t + \kappa^*(t)\,, \qquad t \geq 0\,, \tag{4.15}$$

respectively. In particular,

$$S^*(h) = F^*G^*\,, \quad S_T^*(h) = FG\,. \tag{4.16}$$

It is well known that the adjoint semigroup $S^*(t)$ plays an important role in the study of linear quadratic optimal control problem associated with FDE's including their numerical computations (cf. [11, 12, 14, 17, 47]). The structure of $S^*(t)$ is not straightforward compared with $S_T(t)$, since a functional differential equation which realizes $S^*(t)$ has not been unknown. The advantage of the use of transposed semigroup $S_T(t)$ depends on this fact and that $S^*(t)$ and $S_T(t)$ are connected by the operators F^* and G^* in an appropriate way (see Theorems 4.1, 4.2 below).

A somewhat simple property of G and G^* is the following

Proposition 4.5. (i) $\mathrm{Cl}(\mathrm{Im}\, G) = M_p$, $\mathrm{Ker}\, G = \{0\}$;
(ii) $\mathrm{Cl}(\mathrm{Im}\, G^*) = M_p^*$, $\mathrm{Ker}\, G^* = \{0\}$.

Proof. First we shall show $\mathrm{Ker}\, G = \{0\}$. Assume $Gg = 0$ in M_p. Then by (4.6) and (4.8), $0 = [Gg]^1(-h) = W(0)g^0 = g^0$. Using this and changing variables $\xi \to -\xi$ and $h+s \to s$ in (4.6), we have

$$[Gg]^1(s-h) = \int_0^s W(s-\xi)g^1(-\xi)d\xi = 0 \qquad \text{for all} \quad s \in [0, h]\,. \tag{4.17}$$

Now we can use a convolution type result on the fundamental solution in Nakagiri [34, Lemma 5.1] to obtain from (4.17) that $g^1(-\xi) = 0$ a.e. $\xi \in [0, h]$, i.e., $g^1 = 0$ in $L_p(I_h; X)$. Hence $g = (g^0, g^1) = 0$ in M_p, which proves $\mathrm{Ker}\, G = \{0\}$. Similarly, by Proposition 4.3 $\mathrm{Ker}\, G^* = \{0\}$ holds. Since M_p is reflexive, it follows from the duality theorem (cf. Kato [25, p. 243], Tanabe [43, Chapter III]) that $\mathrm{Ker}\, G = \{0\}$ (resp. $\mathrm{Ker}\, G^* = \{0\}$) is equivalent to $\mathrm{Cl}(\mathrm{Im}\, G^*) = M_p^*$ (resp. $\mathrm{Cl}(\mathrm{Im}\, G) = M_p$). This proves (i) and (ii).

In the special case where A_0 is bounded, we have the following sharper result for G than Proposition 4.5.

Proposition 4.6. *Let A_0 be bounded. Then*
(i) $\mathrm{Im}\, G = D(A)$ *and* $G\colon M_p \to D(A)$ *is bijective*;
(ii) $G^{-1}\colon D(A) \to M_p$ *is given by*

$$(4.18)\quad \begin{cases} [G^{-1}g]^1(s) = \dot{g}^1(-s-h) - A_0 g^1(-s-h) - \int_s^0 d\eta(\xi) g^1(\xi-s-h) & \text{a.e. } s \in I_h \\ [G^{-1}g]^0 = g^1(-h)\,, \qquad g \in D(A); \end{cases}$$

(iii)

$$(4.19)\qquad \operatorname{Im} S(t) \subset D(A) \qquad \textit{for} \quad t \geq h\,.$$

Proof. Since A_0 is bounded, we see from Delfour [13, Theorems 1.1, 1.2] that the function

$$(4.20)\qquad W(t)k^0 + \int_0^t W(t-s)k^1(s)ds\,, \qquad t \in [0, h]\,,$$

where $k^0 \in X$, $k^1 \in L_p([0, h]; X)$, gives a unique strong solution of (E) with $g^0 = k^0$, $g^1 = 0$ and $u = k^1$. For $g = (g^0, g^1) \in M_p$ the function $y(t) = [Gg]^1(t-h)$, $t \in [0, h]$ is given by (4.20) with $k^0 = g^0$ and $k^1(s) = g^1(-s)$. Then $[Gg]^1(\cdot) = y(\cdot + h) \in W_p^{(1)}(I_h; X)$, and hence by $[Gg]^1(0) = [Gg]^0$, $Gg \in D(A)$ i.e., $\operatorname{Im} G \subset D(A)$. To prove the reverse inclusion let $g = (g^1(0), g^1) \in D(A)$ and define $\psi = (\psi^0, \psi^1)$ by the right hand side of (4.18). It is clear that $\psi \in M_p$. Put $y(t) = g^1(t-h)$, $t \in [0, h]$ and $y(0) = g^1(-h)$, $y(s) = 0$ a.e. $s \in I_h$. Then from (4.18) it follows immediately that $y(t)$ is a strong solution of (E) with $k^0 = \psi^0$ and $k^1(s) = \psi^1(-s)$. Hence by uniqueness, $y(t) = [G\psi]^1(t-h) = g^1(t-h)$, $t \in [0, h]$ and especially $y(h) = [G\psi]^1(0) = [G\psi]^0 = g^1(0)$. So that $G\psi = g$, or $\psi = G^{-1}g$. This shows (i) and (ii) simultaneously. Since $S(t) = S(h)S(t-h) = GFS(t-h)$ by (4.11) and $\operatorname{Im} G = D(A)$, we have (iii). In other words, $S(t)$ is differentiable for $t \geq h$.

We note that $S^*(t)$ also satisfies $\operatorname{Im} S^*(t) \subset D(A^*)$ for $t \geq h$. Next we establish a simple and fundamental relation between $S(t)$ and G_t which can not be found in any literatures studying state space theory for (E).

Proposition 4.7. (i) $S(t)G = G_{t+h}$, $S_T(t)G^* = G^*_{t+h}$, $t \geq 0$;
(ii) $G^*S^*(t) = G^*_{t+h}$, $GS^*_T(t) = G_{t+h}$, $t \geq 0$.

Proof. (i) With the aid of (2.7), (4.6), (4.8), we have for $g = (g^0, g^1) \in M_p$,

$$(4.21)\qquad \begin{aligned}[S(t)Gg]^1(s) = {} & \Big(W(t+s)W(h) + \int_{-h}^0 U_{t+s}(\xi)W(h+\xi)d\xi\Big)g^0 \\ & + \int_{-h}^0 W(t+s)W(h+\xi)g^1(\xi)d\xi \\ & + \int_{-h}^0 \int_{-h}^0 U_{t+s}(\xi)W(h+\xi+\alpha)g^1(\alpha)d\alpha d\xi = I_1 + I_2 + I_3\,. \end{aligned}$$

As is easily seen

$$(4.22)\qquad I_2 + I_3 = \int_{-h}^0 \Big(W(t+s)W(h+\xi) + \int_{-h}^0 U_{t+s}(\alpha)W(h+\xi+\alpha)d\alpha\Big)g^1(\xi)d\xi\,.$$

We now recall the following quasi-semigroup property of $W(t)$ given in [35, Eq. (4.9)];

$$W(t_1+t_2) = W(t_1)W(t_2)+\int_{-h}^{0} U_{t_1}(\xi)W(t_2+\xi)d\xi\,, \qquad t_1,\, t_2\geq 0\,. \tag{4.23}$$

Applying (4.23) to I_1 in (4.21) and the integrand in (4.22), we obtain that

$$[S(t)Gg]^1(s) = W(t+s+h)g^0+\int_{-h}^{0} W(t+s+h+\xi)g^1(\xi)d\xi = [G_{t+h}g]^1(s)\,. \tag{4.24}$$

Substituting $s=0$ in (4.24), we have $[S(t)Gg]^0=[G_{t+h}g]^0$. Therefore, $S(t)G=G_{t+h}$ is proved. Similarly $S_T(t)G^*=G^*_{t+h}$ is true.

(ii) Take adjoints of the equalities in (i).

We are now ready to give the main theorem which is one of the key results in the state space theory. A similar result for $X=\boldsymbol{R}^n$ is already proved by Manitius [29, Theorem 3.3], however his proof is much complicated and can not be carried to our Banach space case ($W(t)$ is not differentiable!). Here we shall give a very simple proof based on Proposition 4.7.

Theorem 4.1. (i)

$$S(t)G = GS^*_T(t)\,, \quad G^*S^*(t) = S_T(t)G^*\,, \qquad t\geq 0; \tag{4.25}$$

(ii)

$$GD(A^*_T)\subset D(A) \quad \textit{and} \quad AG = GA^*_T \quad \textit{on} \quad D(A^*_T); \tag{4.26}$$

$$G^*D(A^*)\subset D(A_T) \quad \textit{and} \quad G^*A^* = A_TG^* \quad \textit{on} \quad D(A^*)\,. \tag{4.27}$$

Proof. The part (i) is a direct consequence from Proposition 4.7 and the part (ii) follows from (i) and the definition of infinitesimal generator.

The next is the second key result related to F, which is first proved by Bernier and Manitius [4, Theorem 5.4] and later by Delfour and Manitius [15, Theorem 3.1] for more general measure η. Compare their proofs and our simple proof.

Theorem 4.2. (i)

$$FS(t) = S^*_T(t)F\,, \quad S^*(t)F^* = F^*S_T(t)\,, \qquad t\geq 0\,; \tag{4.28}$$

(ii)

$$FD(A)\subset D(A^*_T) \quad \textit{and} \quad FA = A^*_TF \quad \textit{on} \quad D(A); \tag{4.29}$$

$$F^*D(A_T)\subset D(A^*) \quad \textit{and} \quad A^*F^* = F^*A_T \quad \text{on} \quad D(A_T)\,. \tag{4.30}$$

Proof. Since (ii) follows from (i), we prove only (i). By (4.25) and (4.11),

$$\begin{aligned} G(S^*_T(t)F) &= (GS^*_T(t))F = S(t)GF = S(t)S(h) = S(h)S(t) \\ &= GFS(t) = G(FS(t))\,, \qquad t\geq 0\,. \end{aligned} \tag{4.31}$$

Since Ker $G=\{0\}$, it follows from (4.31) that $S_T^*(t)F=FS(t)$, $t\geq 0$. The second equality in (i) is proved analogously.

Corollary 4.2. (i)

$$\text{Ker}\, F = \{g\in M_p\colon x(t; g) = 0 \quad for \quad t\in[0, h]\}\,; \tag{4.32}$$

(ii)

$$\text{Ker}\, F^* = \{f\in M_p^*\colon z(t; f) = 0 \quad for \quad t\in[0, h]\}\,. \tag{4.33}$$

Proof. Since $S(t)$ is a semigroup defined by (3.1) and (E) is autonomous, we see easily that

$$\begin{aligned} S(h)g = 0 \quad &\text{if and only if} \quad S(t+h)g = 0 \quad \text{for all} \quad t\geq 0\,, \\ &\text{if and only if} \quad x(t; g) = 0 \quad \text{for all} \quad t\geq 0\,, \\ &\text{if and only if} \quad x(t; g) = 0 \quad \text{for all} \quad t\in[0, h]\,. \end{aligned} \tag{4.34}$$

From (4.11) and Ker $G=\{0\}$, we have Ker $F=$Ker $S(h)$. Hence (4.34) implies (4.32). Similarly (4.33) is proved by (4.13) and Proposition 4.5 (ii).

Lastly in this section we introduce a bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle$ between M_p and M_p^* defined by

$$\langle\!\langle g, f\rangle\!\rangle = \langle Fg, f\rangle_{M_p} = \langle g, F^*f\rangle_{M_p}\,. \tag{4.35}$$

The form $\langle\!\langle\ ,\ \rangle\!\rangle$ is considered a time reversing one of the Hale's bilinear form (see [18, p. 173]) and appears in the representation of basis for generalized eigenspaces associated with (E) (which will be given in Section 8 below). The following corollary is obvious from Theorem 4.2 and the definition (4.35).

Corollary 4.3. (i) $\langle\!\langle Ag, f\rangle\!\rangle=\langle\!\langle g, A_T f\rangle\!\rangle$, $(g, f)\in D(A)\times D(A_T)$.
(ii) $\langle\!\langle S(t)g, f\rangle\!\rangle=\langle\!\langle g, S_T(t)f\rangle\!\rangle$, $t\geq 0$, $(g, f)\in M_p\times M_p^*$.

5. Characterizations of Ker F and Im F

In this section we shall give a number of necessary and/or sufficient conditions for Ker $F=\{0\}$, Im $F=M_p$ and Cl(Im F)$=M_p$ in terms of the coefficient operators appearing in the measure η.

By definition it is clear that

$$\text{Ker}\, F = \{0\}\times\text{Ker}\, F_1\,, \quad \text{Im}\, F = X\times\text{Im}\, F_1\,. \tag{5.1}$$

Further, we know by the duality theorem that

$$(\text{Im}\, F_1)^\perp = (\text{Cl}(\text{Im}\, F_1))^\perp = \text{Ker}\, F_1^*\,.$$

Since F_1^* is an operator of the same type as F_1 (Proposition 4.2), we mainly investigate the structure of $\operatorname{Ker} F_1$. From the definition (4.1) of F_1, we see that the condition $g^1 \in \operatorname{Ker} F_1$ is equivalent to that g^1 satisfies

$$\sum_{r=1}^{m} A_r \chi_{[-h_r,0]}(s) g^1(-h_r - s) + \int_{-h}^{s} A_I(\xi) g^1(\xi - s) d\xi = 0 \qquad \text{a.e.} \quad s \in I_h . \tag{5.2}$$

The equation (5.2) can be written by the following homogeneous Volterra integral equation with delays:

$$A_m \psi(t) + \sum_{r=1}^{m-1} A_r \chi_{[\tau_r, h]}(t) \psi(t - \tau_r) + \int_0^t A_I(\xi - h) \psi(t - \xi) d\xi = 0 \qquad \text{a.e.} \quad t \in [0, h] \tag{5.3}$$

where $\psi(t) = g^1(-t)$, $t \in [0, h]$ and $\tau_r = h - h_r > 0$, $r = 1, \cdots, m-1$.

Hence by the first equality in (5.1), $\operatorname{Ker} F = \{0\}$ is equivalent to that the equation (5.3) admits a unique trivial solution $\psi(t) = 0$ a.e. $t \in [0, h]$. In order to give conditions for $\operatorname{Ker} F = \{0\}$ we introduce the following null space $N(A_I; \alpha)$, $\alpha \in I_h$ associated with the kernel $A_I(\xi)$:

$$N(A_I; \alpha) = \{x \in X \colon A_I(\xi) x = 0 \quad \text{for a.e. } \xi \in [-h, \alpha]\} .$$

Proposition 5.1. *A necessary condition for* $\operatorname{Ker} F = \{0\}$ *is*

$$\operatorname{Ker} A_m \cap N(A_I; \alpha) = \{0\} \qquad \textit{for each} \quad \alpha \in (-h, -h_{m-1}] . \tag{5.4}$$

Proof. Suppose (5.4) does not hold. Then there exist $\alpha_0 \in (-h, -h_{m-1}]$ and $x_0 \neq 0$ such that $x_0 \in \operatorname{Ker} A_m \cap N(A_I; \alpha_0)$, i.e.,

$$A_m x_0 = 0 \quad \text{and} \quad A_I(\xi) x_0 = 0 \qquad \text{a.e.} \quad \xi \in [-h, \alpha_0] . \tag{5.5}$$

Define

$$\psi(t) = \begin{cases} 0, & t \in [0, -\alpha_0] \\ x_0, & t \in (-\alpha, h_0] , \end{cases} \tag{5.6}$$

where $\psi(\cdot) \neq 0$ in $L_p([0, h]; X)$. Making use of (5.5) we can verify straightforwardly that $\psi(t)$ in (5.6) satisfies (5.3). Hence $\psi(-\cdot) \in \operatorname{Ker} F_1$, so that $\operatorname{Ker} F \neq \{0\}$. This proves the proposition.

Theorem 5.1. *Assume that* $A_I(s) = 0$ *in a neighbourhood of* $-h$. *A necessary and sufficient condition for* $\operatorname{Ker} F = \{0\}$, *or equivajntly* $\operatorname{Cl}(\operatorname{Im} F^*) = M_p^*$, *is*

$$\operatorname{Ker} A_m = \{0\} . \tag{5.7}$$

Proof. Since the condition (5.7) is necessary by Proposition 5.1 and assumption, it sufficies to prove that $\operatorname{Ker} A_m = \{0\}$ implies $\operatorname{Ker} F_1 = \{0\}$. Let $g^1 \in \operatorname{Ker} F_1$ and ψ be given in (5.3). Suppose $A_I(s) = 0$ a.e. $s \in [-h, -h + \tau]$ for some $\tau \in (0, h - h_{m-1}]$ by assumption. Then by (5.3), (5.8) $$A_m\psi(t)+\sum_{r=1}^{m-1}A_r\chi_{[\tau_r,h]}(t)\psi(t-\tau_r)=0$$

for a.e. $t\in[0,\tau]$; in particular $A_m\psi(t)=0$ a.e. $t\in[0,\min(\tau,\tau_{m-1})]$. So that $\psi(t)=0$ a.e. $t\in[0,\min(\tau,\tau_{m-1})]$ by (5.7). Using this, via step by step argument, we obtain from (5.8) that $\psi(t)=0$ a.e. $t\in[0,\tau]$. Let $k\geq 1$ and suppose $\psi(t)=0$ a.e. $t\in[0,k\tau]$. Then for $t\in[k\tau,(k+1)\tau]$, we have

$$\int_0^t A_I(\xi-h)\psi(t-\xi)d\xi=\int_0^{t-k\tau}A_I(\xi-h)\psi(t-\xi)d\xi=0\,.$$

Thus, $\psi(t)$ satisfies (5.8) for a.e. $t\in[0,(k+1)\tau]$. Consequently we have $\psi(t)=0$ a.e. $t\in[0,(k+1)\tau]$ similarly as above. Then by mathematical induction $\psi=0$, or $g^1=0$ in $L_p(I_h;X)$ follows. This shows $\operatorname{Ker}F=\{0\}$.

Proposition 5.2. *If* $0\in\rho(A_m)$, *then*

(5.9) $$\operatorname{Im}F=M_p\,,\quad \operatorname{Im}F^*=M_p^*\,.$$

Proof. Since $0\in\rho(A_m)$, the inverse A_m^{-1} exists and is bounded. Let $\phi\in L_p([0,h];X)$ be given. Consider the following inhomogeneous Volterra integral equation with delays

(5.10) $$\psi(t)+\sum_{r=1}^{m-1}C_r\chi_{[\tau_r,h]}(t)\psi(t-\tau_r)+\int_0^t C_I(\xi-h)\psi(t-\xi)d\xi=A_m^{-1}\phi(t)\,,$$

where $C_r=A_m^{-1}A_r\in B(X)$, $r=1,\cdots,m-1$ and $C_I(\cdot)=A_m^{-1}A_I(\cdot)\in L_{p'}(I_h;B(X))$. For $t\in[0,\tau_{m-1}]$ the equation (5.10) becomes a Volterra integral equation

(5.11) $$\psi(t)+\int_0^t C_I(t-h-\xi)\psi(\xi)d\xi=A_m^{-1}\phi(t)\qquad \text{a.e.}\quad t\in[0,\tau_{m-1}]\,.$$

Since the term $A_m^{-1}\phi(t)$ belongs to $L_p([0,\tau_{m-1}];X)$, the equation (5.11) admits a unique solution $\psi\in L_p([0,\tau_{m-1}];X)$. This can be proved in the usual manner using the contraction mapping principle in L_p-space (see e.g. Miller [32] or Hönig [22]). Then (5.10) is solvable on $[0,\tau_{m-1}]$. Suppose that ψ solves (5.1) a.e. in $[0,k\tau_{m-1}]$, $k\geq 1$. Then for a.e. $t\in[k\tau_{m-1},(k+1)\tau_{m-1}]$, the equation (5.10) is written by the equivalent form as

(5.12) $$\psi(t)+\int_{k\tau_{m-1}}^t C_I(t-h-\xi)\psi(\xi)d\xi$$
$$=A_m^{-1}\phi(t)-\int_0^{k\tau_{m-1}}C_I(t-h-\xi)\psi(\xi)d\xi-\sum_{r=1}^{m-1}C_r\chi_{[\tau_r,h]}(t)\psi(t-\tau_r)\,.$$

Because $t-\tau_r\leq t-\tau_{m-1}\leq k\tau_{m-1}$, $r=1,\cdots,m-1$, the last term in the righth and side of (5.12) is a known function, and hence the right hand side denotes a known function in $L_p([k\tau_{m-1},(k+1)\tau_{m-1}];X)$. We then have that the equation (5.12) is a Volterra integral equation which can be solved in the space $L_p([k\tau_{m-1}, (k+1)\tau_{m-1}]; X)$. This concludes that (5.10) is solvable on $[0, (k+1)\tau_{m-1}]$. Hence by induction, (5.10) is solvable on whole $[0, h]$. By a change of variables $t \to -s$ and an application of A_m to (5.10), we derive $\operatorname{Im} F_1 = L_p(I_h; X)$, and this implies $\operatorname{Im} F = M_p$. It is well known (cf. Kato [25, p. 184]) that $0 \in \rho(A_m)$ is equivalent to $0 \in \rho(A_m^*)$. Thus we have the second equality in (5.9) similarly as above.

The following corollaries are obvious from Theorem 5.1 and Proposition 5.2.

Corollary 5.1. *For the differential difference equation*

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{r=1}^{m} A_r x(t-h_r),$$

a necessary and sufficient condition for $\operatorname{Ker} F = \{0\}$ *(resp.* $\operatorname{Ker} F^* = \{0\}$*) is* $\operatorname{Ker} A_m = \{0\}$ *(resp.* $\operatorname{Ker} A_m^* = \{0\}$*).*

Corollary 5.2. *Assume that* $A_l(s) = 0$ *in a neighbourhood of* $-h$. *If* $0 \in \rho(A_m)$, *then* $0 \in \rho(F)$ *and* $0 \in \rho(F^*)$, *in other words, F and* F^* *are boundedly invertible.*

The above results are infinite dimensional analogue of those given in Delfour and Manitius [15, Section 2.1], in which the proofs are more complicated than those given here, because they have intended to include a very general Stieltjes measure η on $\boldsymbol{R}^n$ of bounded variation. Our proofs are simple and easy because of the restricted form of η given in (2.3).

6. Representations of resolvent operators

This section is devoted to give convenient forms of the resolvents of A, A_T, A^* and A_T^*. In order to give such forms we require some definitions. According to Delfour and Manitius [15], Burns and Herdman [7] and Salamon [40] we introduce the following linear operators E_λ, T_λ, K_λ and H_λ. Let $\lambda \in \boldsymbol{C}^1$ and an ordered pair of spaces (Y, Z_p) be the pair (X, M_p) or (X^*, M_p^*). Define $E_\lambda\colon Y \to Z_p$, $T_\lambda\colon Z_p \to Z_p$, $K_\lambda\colon Z_p \to Z_p$ and $H_\lambda\colon Z_p \to Y$ by

$$(6.1) \qquad \begin{cases} [E_\lambda z]^0 = z, & \\ [E_\lambda z]^1(s) = e^{\lambda s} z, & s \in I_h \end{cases} \qquad \text{for } z \in Y,$$

$$(6.2) \qquad \begin{cases} [T_\lambda y]^0 = 0, & \\ [T_\lambda y]^1(s) = \int_s^0 e^{\lambda(s-\xi)} y^1(\xi) d\xi, & s \in I_h \end{cases} \qquad \text{for } y = (y^0, y^1) \in Z_p,$$

$$(6.3)\qquad \begin{cases} [K_\lambda y]^0 = 0\,, \\ [K_\lambda y]^1(s) = \int_{-h}^{s} e^{\lambda(\xi-s)} y^1(\xi)d\xi\,, \qquad s\in I_h \end{cases} \qquad \text{for}\quad y=(y^0, y^1)\in Z_p\,,$$

$$(6.4)\qquad H_\lambda y = y^0 + \int_{-h}^{0} e^{\lambda s} y^1(s)\,ds \qquad\qquad \text{for}\quad y=(y^0, y^1)\in Z_p\,,$$

respectively. The operator E_λ is often called the exponential map. All above are operator valued entire functions in λ. In what follows we denote the k-th derivative $\frac{d^k}{d\lambda^k}f(\lambda)$ of $f(\lambda)$ by $f^{(k)}(\lambda)$, or simply $f^{(k)}$ for $k=0, 1, 2, \cdots$.

Proposition 6.1. *For each* $\lambda\in \boldsymbol{C}^1$ *and integer* $k\geq 0$,

(i)

$$(6.5)\qquad FT_\lambda^k = K_\lambda^k F\,, \quad F^*T_\lambda^k = K_\lambda^k F^*;$$

(ii)

$$(6.6)\qquad T_\lambda^k E_\lambda = \frac{(-1)^k}{k!} E_\lambda^{(k)};$$

(iii)

$$(6.7)\qquad (FE_\lambda)^{(k)} = FE_\lambda^{(k)}, \quad (F^*E_\lambda)^{(k)} = F^*E_\lambda^{(k)};$$

(iv)

$$(6.8)\qquad H_\lambda FT_\lambda^k E_\lambda = \frac{(-1)^k}{(k+1)!}\Delta^{(k+1)}(\lambda), \quad H_\lambda F^*T_\lambda^k E_\lambda = \frac{(-1)^k}{(k+1)!}\Delta_T^{(k+1)}(\lambda).$$

Proof. (i). First we shall show $FT_\lambda^k=K_\lambda^k F$ for $k=1$. Let $g=(g^0, g^1)\in M_p$. Since $[FT_\lambda g]^0=[K_\lambda Fg]^0=0$, in order to prove $FT_\lambda=K_\lambda F$ we have to prove, by (6.2) and (6.3), that

$$(6.9)\qquad \int_{-h}^{s} e^{\lambda(\xi-s)}\int_{-h}^{\xi} d\eta(\tau)g^1(\tau-\xi)d\xi = \int_{-h}^{s} d\eta(\tau)\int_{\tau-s}^{0} e^{\lambda(\tau-s-\beta)}g^1(\beta)d\beta\,, \qquad s\in I_h\,.$$

Since the relation can be shown with the aid of the Fubini theorem, the detailed proof is omitted. The equality $FT_\lambda^k=K_\lambda^k F$ for $k\geq 2$ follows easily by induction. Since F^* has the same form as F (Proposition 4.2), we can verify $F^*T_\lambda^k= K_\lambda^k F^*$ similarly as above.

(ii), (iii). These parts are proved easily by straightforward calculations using the definitions (4.1), (6.1), (6.2) of F, E_λ, T_λ.

(iv). We prove only the first equality in (6.8). By virtue of (6.4) and (6.6), the element $H_\lambda FT_\lambda^k E_\lambda x$, $x\in X$ is written by

$$(6.10)\qquad \begin{aligned} H_\lambda FT_\lambda^k E_\lambda x &= \frac{(-1)^k}{k!} H_\lambda FE_\lambda^{(k)} x \\ &= \delta_{k,0}x + (-1)^k \int_{-h}^{0} e^{\lambda\xi}\int_{-h}^{\xi} d\eta(s)\,\frac{(s-\xi)^k}{k!}e^{\lambda(s-\xi)}x ds\,. \end{aligned}$$

Thus, the equations (6.10) and (6.9) with $g^1(s)=\frac{1}{k!}s^k e^{\lambda s}x$ imply that

$$\begin{aligned}
H_\lambda F T_\lambda^k E_\lambda x &= \delta_{k,0}x+(-1)^k\int_{-h}^0 d\eta(s)\int_s^0 e^{\lambda s}\cdot\frac{\xi^k}{k!}xd\xi \\
&= \delta_{k,0}x+(-1)^{k+1}\int_{-h}^0 d\eta(s)e^{\lambda s}\cdot\frac{s^{k+1}}{(k+1)!}x \\
&= \delta_{k,0}x-\frac{(-1)^k}{(k+1)!}\cdot\frac{d^{k+1}}{d\lambda^{k+1}}\int_{-h}^0 e^{\lambda s}d\eta(s)x \\
&= \frac{(-1)^k}{(k+1)!}\Delta^{(k+1)}(\lambda)x\,.
\end{aligned}$$

This completes the proof.

Now we can give explicit representations of the resolvents of A and A_T in terms of the retarded resolvent, structural operator F and other operators introduced in this section.

Theorem 6.1. (i) $\rho(A)=\rho(A_0,\eta)$ *and the resolvent* $R(\lambda;A)$ *of* A *is given by*

(6.11) $$R(\lambda;A)=E_\lambda R(\lambda;A_0,\eta)H_\lambda F+T_\lambda\,,\qquad \lambda\in\rho(A)\,.$$

(ii) $\rho(A_T)=\rho(A_0^*,\eta^*)$ *and the resolvent* $R(\lambda;A_T)$ *of* A_T *is given by*

(6.12) $$R(\lambda;A_T)=E_\lambda R(\lambda;A_0^*,\eta^*)H_\lambda F^*+T_\lambda\,,\qquad \lambda\in\rho(A_T)\,.$$

Proof. For a given $\phi=(\phi^0,\phi^1)\in M_p$, we construct a $g=(g(0),g(\cdot))\in D(A)$ such that $(\lambda I-A)g=\phi$. This is equivalent, in view of Proposition 3.1 (iii), to that

(6.13) $$\lambda g(0)-A_0g(0)-\int_{-h}^0 d\eta(s)g(s)=\phi^0,\qquad g(0)\in D(A_0)$$

(6.14) $$\lambda g(s)-\frac{d}{ds}g(s)=\phi^1(s)\,,\qquad s\in I_h\,.$$

We solve the differential equation (6.14) to obtain

(6.15) $$g(s)=e^{\lambda s}g(0)+\int_s^0 e^{\lambda(s-\xi)}\phi^1(\xi)d\xi\,,$$

i.e.,

(6.16) $$(g(0),g)=E_\lambda g(0)+T_\lambda\phi\,.$$

Substituting (6.15) in (6.13) and using (6.9), we have

(6.17) $$\begin{aligned}
\Delta(\lambda)g(0)&=\int_{-h}^0 d\eta(s)\int_s^0 e^{\lambda(s-\xi)}\phi^1(\xi)d\xi+\phi^0 \\
&=\int_{-h}^0 e^{\lambda s}\int_{-h}^s d\eta(\xi)\phi^1(\xi-s)ds+\phi^0=H_\lambda F\phi\,.
\end{aligned}$$

Assume that $\lambda\in\rho(A_0, \eta)$. Then by definition, $\Delta(\lambda)$ has a bounded inverse $\Delta^{-1}(\lambda)=R(\lambda; A_0, \eta)$. So that by (6.16) and (6.17), we derive

$$(g(0), g) = E_\lambda R(\lambda; A_0, \eta)H_\lambda F\phi+T_\lambda\phi . \tag{6.18}$$

Since all operators appearing in (6.18) are bounded, $\lambda\in\rho(A)$ and the resolvent $R(\lambda; A)$ is given by (6.11). Next we show the inclusion $\rho(A)\subset\rho(A_0, \eta)$. Let $\lambda\in\rho(A)$. Then for any $\phi=(\phi^0, \phi^1)\in M_p$ there exists a unique $g=(g(0), g)\in D(A)$ such that $(\lambda I-A)g=\phi$, or equivalently, (6.16) and (6.17) hold. We note that $\Delta(\lambda)$ is one to one. Because if not, there exists a $g^0\in D(A_0)$, $g^0\neq 0$ such that $\Delta(\lambda)g^0=0$. The element $g=E_\lambda g^0\in D(A)$ satisfies $(\lambda I-A)g=0$, $g\neq 0$, which contradicts to $\lambda\in\rho(A)$. For special $\phi=(\phi^0, 0)$, $\phi^0\in X$, the equality (6.17) means that there exists a $g=(g(0), g)\in D(A)$ such that $\Delta(\lambda)g(0)=\phi^0$. This concludes that the densely defined closed linear operator $\Delta(\lambda)$: $D(A_0)\subset X\to X$ is onto and one to one. Hence by open mapping theorem, $\Delta(\lambda)^{-1}$ exists and is bounded, i.e., $\lambda\in\rho(A_0, \eta)$. Therefore (i) is proved. The part (ii) is proved in quite analogous manner as in (i).

Next we characterize the resolvents of the adjoint operators A^* and A_T^*.

Lemma 6.1. *The relation*

$$(\lambda I-A^*)f = \psi , \qquad f\in D(A^*) , \ \psi\in M_p^* \tag{6.19}$$

is equivalent to

$$\Delta_T(\lambda)f^0 = H_\lambda\psi , \quad f^0\in D(A_0^*) \quad \textit{and} \quad f = K_\lambda\psi+F^*E_\lambda f^0 . \tag{6.20}$$

Proof. In view of Proposition 3.3, (6.19) is written by the following equivalent condition

$$\lambda f^0-A_0^* f^0-f^1(0) = \psi^0 , \qquad f^0\in D(A_0^*) , \tag{6.21}$$

$$\lambda f^1(s)-\frac{d}{ds}w(f)(s) = \psi^1(s) , \qquad s\in I_h , \tag{6.22}$$

where $w(f)$ is given in (3.19). Put $\tilde{f}^0=(f^0, f^0)\in M_p^*$ i.e., $[\tilde{f}^0]^0=f^0$, $[\tilde{f}^0]^1(s)\equiv f^0$, and $\tilde{w}=-w(f)$. Since $f^1=[F^*\tilde{f}^0]^1+\tilde{w}$ by (3.19), we can solve the differential equation (6.22) with the initial condition $\tilde{w}(-h)=0$ to obtain

$$\begin{aligned}\tilde{w}(s) &= e^{-\lambda s}\int_{-h}^{s} e^{\lambda\xi}\{\psi^1(\xi)-\lambda[F^*\tilde{f}^0]^1(\xi)\}\,d\xi \\ &= \int_{-h}^{s} e^{-\lambda(s-\xi)}\psi^1(\xi)\,d\xi-\int_{-h}^{s} e^{\lambda(\xi-s)}\lambda\int_{-h}^{\xi} d\eta^*(\beta)f^0\,d\xi , \qquad s\in I_h .\end{aligned}$$

By (6.3) and applying the Fubini theorem to the last term of the above equality, we obtain without difficulty that

(6.23) $$\tilde{w}(s) = [K_\lambda \psi]^1(s) - [F^* \tilde{f}^0]^1(s) + [F^* E_\lambda f^0]^1(s), \qquad s \in I_h.$$

Thus,

$$f^1 = [F^* \tilde{f}^0]^1 + \tilde{w} = [K_\lambda \psi]^1 + [F^* E_\lambda f^0]^1,$$

which shows the second equality in (6.20). Substituting $s=0$ in (3.19) and (6.23), we have

$$f^1(0) = \int_{-h}^0 d\eta^*(s) f^0 + \tilde{w}(0) = \int_{-h}^0 e^{\lambda s} \psi^1(s) ds + \int_{-h}^0 e^{\lambda s} d\eta^*(s) f^0.$$

Hence the equality (6.21) is rewritten as

$$(\lambda I - A_0^* - \int_{-h}^0 e^{\lambda s} d\eta^*(s)) f^0 = \psi^0 + \int_{-h}^0 e^{\lambda s} \psi^1(s) ds, \qquad f^0 \in D(A_0^*),$$

which is the first equality in (6.20).

Theorem 6.2. (i) $\rho(A^*) = \rho(A_0^*, \eta^*)$ *and the resolvent* $R(\lambda; A^*)$ *of* A^* *is given by*

(6.24) $$R(\lambda; A^*) = F^* E_\lambda R(\lambda; A_0^*, \eta^*) H_\lambda + K_\lambda, \qquad \lambda \in \rho(A^*).$$

(ii) $\rho(A_T^*) = \rho(A_0, \eta)$ *and the resolvent* $R(\lambda; A_T^*)$ *of* A_T^* *is given by*

(6.25) $$R(\lambda; A_T^*) = F E_\lambda R(\lambda; A_0, \eta) H_\lambda + K_\lambda, \qquad \lambda \in \rho(A_T^*).$$

Proof. Using Lemma 6.1 we can prove (i) by analogous argument as in the proof of Theorem 6.1. The proof of the remaining part (ii) is similar.

Here we give important relations between the operators E_λ, H_λ, T_λ and K_λ. Taking into account of the relation (2.18), we can verify by direct computations involving the pairing $\langle\ ,\ \rangle_{M_p}$ that for each $\lambda \in \boldsymbol{C}^1$,

(6.26) $$E_\lambda^* = H_{\bar\lambda}, \quad H_\lambda^* = E_{\bar\lambda}, \quad T_\lambda^* = K_{\bar\lambda}, \quad K_\lambda^* = T_{\bar\lambda}.$$

Consequently, by using the equality (2.17) and (6.26) Theorem 6.2 can be derived as the adjoint version of Theorem 6.1. This may be a simple proof of Theorem 6.2.

Corollary 6.1. (i)

(6.27) $$FR(\lambda; A) = R(\lambda; A_T^*) F, \quad R(\lambda; A) G = GR(\lambda; A_T^*)$$
$$\text{for} \quad \lambda \in \rho(A) = \rho(A_T^*) = \rho(A_0, \eta).$$

(ii)

(6.28) $$R(\lambda; A^*) F^* = F^* R(\lambda; A_T), \quad G^* R(\lambda; A^*) = R(\lambda; A_T) G^*$$
$$\text{for} \quad \lambda \in \rho(A^*) = \rho(A_T) = \rho(A_0^*, \eta^*).$$

Proof. (i) follows from Theorems 4.1, 4.2, 6.1 and 6.2. (ii) follows from Proposition 6.1 (i), Theorems 6.1 and 6.2.

7. Spectral decomposition

In this and following sections we study the spectral decomposition theory for the FDE's in Banach spaces. The spectral theory for various types of FDE's in $\boldsymbol{R}^n$ is further developed by many authors (see [6, 15, 18, 19, 20, 24, 36, 40] for examples). An attempt to extend the spectral theory to retarded FDE's in infinite dimensional spaces was first made by Travis and Webb [44] whose main concern is the stability of mild solutions. Their analysis and investigations have been carried in the space $C(I_h; X)$, but seems incomplete compared with those for $X=\boldsymbol{R}^n$. The purpose here is to construct a rather complete spectral decomposition theory for the equation (E) on the space M_p, which extends the work of [15,29] to general Banach space case. Our analysis, however, is more delicate than those in [15, 29] because of permitting X being infinite dimensional.

7.1. Classification of spectrum

The retarded spectrum introduced in Section 2 is efficiently used to determine the spectrum of the infinitesimal generators associated with (E) and (E^T).

Proposition 7.1. *Three kinds of spectrum of A and A_T are given by*

$$(7.1)\qquad \sigma_P(A)=\sigma_P(A_0,\eta)\,,\quad \sigma_C(A)=\sigma_C(A_0,\eta)\,,\quad \sigma_R(A)=\sigma_R(A_0,\eta)\,,$$

$$(7.2)\qquad \sigma_P(A_T)=\sigma_P(A_0^*,\eta^*)\,,\quad \sigma_C(A_T)=\sigma_C(A_0^*,\eta^*)\,,\quad \sigma_R(A_T)=\sigma_R(A_0^*,\eta^*)\,,$$

respectively.

Proof. First we recall the following fact which is already shown in the proof of Theorem 6.1. That is, the relation $(\lambda I-A)g=\phi$, $g\in D(A)$, $\phi\in M_p$ is equivalent to that $\Delta(\lambda)g^0=H_\lambda F\phi$, $g^0\in D(A_0)$, $g=E_\lambda g^0+T_\lambda\phi$. If we substitute $\phi=0$ in the above equivalence, then we have that $\operatorname{Ker}(\lambda I-A)=\{0\}$ is equivalent to that $\operatorname{Ker}\Delta(\lambda)=\{0\}$, and hence $\operatorname{Ker}(\lambda I-A)\neq\{0\}$ if and only if $\operatorname{Ker}\Delta(\lambda)\neq\{0\}$. This concludes, by definition, $\sigma_P(A)=\sigma_P(A_0,\eta)$. By the same reason, from Lemma 6.1 it follows that $\operatorname{Ker}(\lambda I-A^*)=\{0\}$ if and only if $\operatorname{Ker}\Delta_T(\lambda)=\{0\}$. Then by putting $\lambda=\bar{\lambda}$ and using the duality theorem, we have that $\operatorname{Cl}(\operatorname{Im}(\bar{\lambda}I-A^*)^*)=\operatorname{Cl}(\operatorname{Im}(\lambda I-A))=M_p$ if and only if $\operatorname{Cl}(\operatorname{Im}\Delta_T(\bar{\lambda})^*)=\operatorname{Cl}(\operatorname{Im}\Delta(\lambda))=X$. This implies, by contradiction, that $\operatorname{Im}\Delta(\lambda)$ is not dense in X if and only if $\operatorname{Im}(\lambda I-A)$ is not dense in M_p. Now we are ready to prove $\sigma_R(A_0,\eta)=\sigma_R(A)$. From the definition of residual spectrum, $\lambda\in\sigma_P(A_0,\eta)$ if and only if

$$(7.3)\qquad \Delta(\lambda)^{-1} \text{ exists (i.e., } \operatorname{Ker}\Delta(\lambda)=\{0\}\text{) but } \operatorname{Im}\Delta(\lambda) \text{ is not dense in } X\,.$$

It then follows that (7.3) is equivalent to

(7.4) $(\lambda I-A)^{-1}$ exists but $\operatorname{Im}(\lambda I-A)$ is not dense in M_p.

The statement (7.4) is exactly the definition of $\lambda\in\sigma_R(A)$. Hence $\sigma_R(A_0, \eta)=\sigma_R(A)$ is proved. The rest equality $\sigma_C(A_0, \eta)=\sigma_C(A)$ is now evident. The part for A_T is proved in a same manner as above.

REMARK 7.1. In the case where $X=\boldsymbol{R}^n$, it is well known that $\sigma(A)=\sigma_P(A)=\{\lambda\colon \det\Delta(\lambda)=0\}$ is countable and isolated. However there exists an operator A defined by (3.2) and (3.3) such that $\sigma_C(A)\neq\phi$ or $\sigma_R(A)\neq\phi$ in our infinite dimensional case.

7.2. Generalized eigenspaces and spectral decomposition

Let $\lambda\in\sigma_P(A)$. The generalized eigenspace $\mathcal{M}_\lambda$ of A corresponding to λ is defined by

$$\mathcal{M}_\lambda=\bigcup_{l=0}^{\infty}\operatorname{Ker}(\lambda I-A)^l. \tag{7.5}$$

To characterize the structure of Ker $(\lambda I-A)^l$, $l=1, 2, \cdots$, we introduce operator valued matrices $\mathcal{A}_l=\mathcal{A}_l(\lambda)$ defined by

$$\mathcal{A}_l=\begin{pmatrix} D_1 & D_2 & \cdots\cdots & D_l \\ O & D_1 & \cdots\cdots & D_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots\cdots & D_1 \end{pmatrix}, \qquad l=1, 2, \cdots. \tag{7.6}$$

where

$$D_{j+1}=D_{j+1}(\lambda)=\frac{1}{j!}\Delta^{(j)}(\lambda), \qquad j=0, 1, 2, \cdots. \tag{7.7}$$

The following result extends the results of [15, Proposition 4.3] and [18, Lemma 3.3, p. 177] to reflexive Banach space case.

Proposition 7.2. *Let* $\lambda\in\sigma_P(A)$. *Then* $\operatorname{Ker}(\lambda I-A)^l$ *coincides with the space of functions* $\phi\in M_p$ *of the form*

$$\phi=\sum_{j=0}^{l-1}\frac{1}{j!}(E_\lambda y_{j+1})^{(j)}, \tag{7.8}$$

where $\mathcal{Y}=\operatorname{col}(y_1, \cdots, y_l)$, $y_{j+1}\in D(A_0)$, $j=0, 1, \cdots, l-1$ *satisfies* $\mathcal{A}_l\mathcal{Y}=0$ *in* X^l.

Proof. Let $\phi\in\operatorname{Ker}(\lambda I-A)^l$. Set $\phi_0=\phi$ and $\phi_j=(\lambda I-A)\phi_{j-1}$, $j=1, \cdots, l$; then ϕ is characterized by $\phi_l=0$. The relation $\phi_j=(\lambda I-A)\phi_{j-1}$ is written as

$$\phi_{j-1}=E_\lambda\phi_{j-1}^0+T_\lambda\phi_j \quad\text{and}\quad \Delta(\lambda)\phi_{j-1}^0=H_\lambda F\phi_j, \quad \phi_{j-1}^0\in D(A_0), \quad j=1, \cdots, l. \tag{7.9}$$

Since $\phi_l=0$, we have

$$(7.10)\quad \begin{cases} \phi_{l-1}=E\phi_{l-1}, & \phi_{l-2}=E_\lambda\phi^0_{l-2}+T_\lambda\phi_{l-1}=E_\lambda\phi^0_{l-2}+T_\lambda E_\lambda\phi^0_{l-1}, \cdots, \\ \phi_j=\sum_{r=j}^{l-1}T_\lambda^{r-j}E_\lambda\phi^0_r, & \cdots, \quad \phi=\phi_0=\sum_{r=0}^{l-1}T_\lambda^r E_\lambda\phi^0_r, \end{cases}$$

so that by (6.6) and a change $r\to j$,

$$\phi=\sum_{j=0}^{l-1}\frac{1}{j!}(E_\lambda(-1)^j\phi^0_j)^{(j)}.$$

If we put $y_{i+1}=(-1)^j\phi^0_j\in D(A_0)$, $j=0, 1, \cdots, l-1$, we obtain (7.8). From the second equality in (7.9) it follows by (7.10) that

$$(7.11)\quad \Delta(\lambda)y_j=(-1)^{j-1}H_\lambda F\phi_j=(-1)^{j-1}(\sum_{r=j}^{l-1}H_\lambda FT_\lambda^{r-j}E_\lambda(-1)^{-r}y_{r+1})$$
$$=(-1)\sum_{r=j}^{l-1}(-1)^{j-r}H_\lambda FT_\lambda^{r-j}E_\lambda y_{r+1}.$$

By virtue of (6.8), the equation (7.11) is written as

$$\Delta(\lambda)y_j=(-1)\sum_{r=j}^{l-1}\frac{1}{(r-j+1)!}\Delta^{(r-j+1)}(\lambda)y_{r+1},$$

and hence (by changing $r\to r+1$),

$$(7.12)\quad \sum_{r=j}^{l}\frac{1}{(r-j)!}\Delta^{(r-j)}(\lambda)y_r=0, \qquad j=1, \cdots, l.$$

The system of equations (7.12) is rewritten simply by $\mathcal{A}_l\mathcal{Y}=0$ in X^l, where $\mathcal{Y}=\mathrm{col}(y_1, \cdots, y_l)$. This completes the proof.

In order to go into a further spectral decomposition theory we have to restrict λ to the isolated spectrum. We now require the following definitions:

$$\sigma_I(A_0, \eta)=\{\lambda\in\boldsymbol{C}^1\colon \lambda \textit{ is an isolated singular point of } R(\mu; A_0, \eta)\}.$$
$$\sigma_0(A_0, \eta)=\{\lambda\in\boldsymbol{C}^1\colon \lambda \textit{ is a pole of } R(\mu; A_0, \eta)\}.$$

Similarly we define the spectral sets $\sigma_I(A)$ and $\sigma_0(A)$. It is obvious that $\sigma_0(A_0, \eta)\subset\sigma_I(A_0, \eta)$, $\sigma_0(A)\subset\sigma_I(A)$. Since all E_λ, H_λ and T_λ are entire functions, Theorem 6.1 (i) implies that λ is a pole of $R(\mu; A)$ of order k_λ if and only if λ is a pole of $R(\mu; A_0, \eta)$ of same order k_λ. Then, $\sigma_0(A)=\sigma_0(A_0, \eta)$.

Let $\lambda\in\sigma_I(A)$ and let P_λ be the projection operator

$$(7.13)\quad P_\lambda=\frac{1}{2\pi i}\int_{\Gamma_\lambda}R(\mu; A)d\mu,$$

where Γ_λ is a closed rectifiable curve containing λ inside and all other singular points of $R(\mu; A)$ outside. By Yosida [49, p. 228–231] (see also Taylor and Lay [41], Tanabe [43] and Kato [25]), we obtain the following decomposition of the space M_p.

Theorem 7.1. *Let* λ *be a pole of* $R(\mu; A_0, \eta)$ *of order* k_λ. *Then* $\lambda \in \sigma_P(A)$ *and the direct sum decomposition*

$$M_p = \operatorname{Ker}(\lambda I - A)^{k_\lambda} \oplus \operatorname{Im}(\lambda I - A)^{k_\lambda}, \quad \mathcal{M}_\lambda = P_\lambda M_p = \operatorname{Ker}(\lambda I - A)^{k_\lambda} \tag{7.14}$$

holds. Both $\mathcal{M}_\lambda$ *and* $\operatorname{Im}(\lambda I - A)^{k_\lambda} = \operatorname{Ker} P_\lambda$ *are closed and invariant under* $S(t)$. *Moreover the resolvent* $R(\mu; A)$ *has the Laurent series expansion*

$$R(\mu; A) = \sum_{n=-k_\lambda}^{\infty} (\mu - \lambda)^n Q_n \tag{7.15}$$

in a neighbourhood of λ, *where* Q_n *is given by*

$$Q_n = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - \lambda)^{-n-1} R(\mu; A) d\mu . \tag{7.16}$$

Clearly $P_\lambda = Q_{-1}$. Put $Q_\lambda = Q_{-2}$, then from the expression (7.16) (cf. Kato [25, p. 180] it follows that

$$Q_{-n} = Q_\lambda^{n-1}, \quad n = 2, \cdots, k_\lambda, \quad Q_\lambda^{k_\lambda} = O \text{ (nilpotent)}, \tag{7.17}$$

$$P_\lambda Q_\lambda = Q_\lambda P_\lambda = Q_\lambda, \quad A P_\lambda = \lambda P_\lambda + Q_\lambda . \tag{7.18}$$

The decomposition of M_p in Theorem 7.1 is slightly generalized as

$$M_p = (\bigoplus_{\lambda \in \Lambda} \mathcal{M}_\lambda) \oplus \mathcal{R}_\lambda, \quad \mathcal{R}_\lambda = \bigcap_{\lambda \in \Lambda} \operatorname{Im}(\lambda I - A)^{k_\lambda}, \tag{7.19}$$

where $\Lambda \subset \sigma_0(A)$ is a finite set (see e.g., Kato [25, p. 181]).

REMARK 7.2. Proposition 7.2 tells us $\dim \operatorname{Ker}(\lambda I - A)^l = \dim \operatorname{Ker} \mathcal{A}_l$, but the dimension may be infinity even if λ is a pole of $R(\mu; A)$.

From (7.13), (7.16), (7.18) and (6.11) the following corollary follows at once. Notice that T_λ is entire.

Corollary 7.1. *For* $\lambda \in \sigma_0(A_0, \eta)$,

(i)

$$P_\lambda = \mathcal{E}_\lambda F, \quad \mathcal{E}_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} E_\mu R(\mu; A_0, \eta) H_\mu d\mu; \tag{7.20}$$

$$Q_\lambda = \mathcal{F}_\lambda F, \quad \mathcal{F}_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - \lambda) E_\mu R(\mu; A_0, \eta) H_\mu d\mu; \tag{7.21}$$

(ii)

$$\operatorname{Ker} F \subset \operatorname{Ker} P_\lambda \subset \operatorname{Ker} Q_\lambda . \tag{7.22}$$

Next we consider the case for the transposed operator A_T. Let $\mathcal{M}_\lambda^T$ denote the generalized eigenspace of A_T corresponding to $\lambda\in\sigma_P(A_T)$; let the matrices $\mathcal{A}_l^T=\mathcal{A}_l^T(\lambda)$, $l=1, 2, \cdots$ be defined by (7.6) in which D_{j+1} is replaced by $\frac{1}{j!}\Delta_T^{(j)}(\lambda)$, $j=0, 1, 2, \cdots$; and let P_λ^T denote the spectral projection corresponding to $\lambda\in\sigma_I(A_T)$. Then we have:

Theorem 7.2. (i) *If* λ *is a pole of* $R(\mu; A_0^*, \eta^*)$ *of order* m_λ, *then* $\lambda\in\sigma_P(A_T)$ *and the decomposition*

$$M_p^* = \mathrm{Ker}\,(\lambda I-A_T)^{m_\lambda}\oplus\mathrm{Im}\,(\lambda I-A_T)^{m_\lambda}\,, \quad \mathcal{M}_\lambda^T = P_\lambda^T M_p^* = \mathrm{Ker}\,(\lambda I-A_T)^{m_\lambda}$$

holds. Moreover the resolvent $R(\mu; A_T)$ *has the Laurent series expansion*

$$R(\mu; A_T) = \sum_{n=-m_\lambda}^{\infty} (\mu-\lambda)^n Q_n^T$$

in a neighbourhood of λ, *where* Q_n^T *is given by* (7.16) *in which* $R(\mu; A)$ *is replaced by* $R(\mu; A_T)$.
(ii) *For* $\lambda\in\sigma_P(A_T)$, *the space* $\mathrm{Ker}\,(\lambda I-A_T)^l$ *coincides with the set of functions* $\psi\in M_p^*$ *of the form*

$$\psi = \sum_{j=0}^{l-1}\frac{1}{j!}(E_\lambda y_{j+1}^T)^{(j)}\,,$$

where $\mathcal{Y}^T=\mathrm{col}\,(y_1^T, \cdots, y_l^T)$, $y_{j+1}^T\in D(A_0^*)$, $j=0, 1, \cdots, l-1$ *satisfies* $\mathcal{A}_l^T\mathcal{Y}^T=0$ *in* X^{*l}.
(iii) *For* $\lambda\in\sigma_0(A_0^*, \eta^*)$,

$$P_\lambda^T = \mathcal{E}_\lambda^T F^*; \quad \mathcal{E}_\lambda^T = \frac{1}{2\pi i}\int_{\Gamma_\lambda} E_\mu R(\mu; A_0^*, \eta^*) H_\mu\, d\mu\,,$$

$$Q_\lambda^T = \mathcal{F}_\lambda^T F^*; \quad \mathcal{F}_\lambda^T = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (\mu-\lambda) E_\mu R(\mu; A_0^*, \eta^*) H_\mu\, d\mu\,,$$

where $Q_\lambda^T=Q_{-2}^T$ *and*

$$\mathrm{Ker}\,F^*\subset\mathrm{Ker}\,P_\lambda^T\subset\mathrm{Ker}\,Q_\lambda^T\,.$$

We shall describe a group property of $S(t)$. For this the following discrete spectrum $\sigma_d(A)$ of A is needed to be defined by

$$\sigma_d(A) = \{\lambda\in\sigma_I(A)\colon \dim\,(\mathrm{Im}\,P_\lambda)<+\infty\}\,.$$

It is well known (Kato [25, p. 181]) that $\sigma_d(A)\subset\sigma_0(A)\subset\sigma_P(A)$ and $\mathrm{Im}\,P_\lambda=\mathcal{M}_\lambda=\mathrm{Ker}\,(\lambda I-A)^{k_\lambda}$ for $\lambda\in\sigma_0(A)$. Let $\lambda\in\sigma_d(A)$ and let $d_\lambda=\dim\mathcal{M}_\lambda$. We shall write by $\Phi_\lambda=\{\phi_{\lambda,1}, \cdots, \phi_{\lambda,d_\lambda}\}$ a basis of $\mathcal{M}_\lambda$ of the form (7.8). Since $A\mathcal{M}_\lambda\subset\mathcal{M}_\lambda$, there exists a $d_\lambda\times d_\lambda$ matrix B_λ such that

$$A\Phi_\lambda = \Phi_\lambda B_\lambda \quad\text{and}\quad S(t)\Phi_\lambda=\Phi_\lambda e^{B_\lambda t} \qquad \text{for}\quad t\geq 0\,. \tag{7.23}$$

Hence the only eigenvalue of B_λ is λ and $S(t)$ can be extended to a holomorphic group on $\mathcal{M}_\lambda$. Now we can prove the following result in a similar fashion as in Hale [18, Chapter 7, Theorem 2.2] with suitable modifications to the space M_p.

Proposition 7.3. *Assume that* $\Lambda \subset \sigma_d(A)$ *is a finite set* $\{\lambda_1, \cdots, \lambda_n\}$. *Let* $\Phi_\Lambda = \{\Phi_{\lambda_1}, \cdots, \Phi_{\lambda_n}\}$ *and* $B_\Lambda = \mathrm{diag}\,(B_{\lambda_1}, \cdots, B_{\lambda_n})$. *Then for any column vector* a *of the same dimension as* $\Phi_\Lambda (= d_{\lambda_1} + \cdots + d_{\lambda_n})$, *we have*
(i) $S(t)\Phi_\Lambda a$ *can be defined on* $(-\infty, \infty)$ *by the relation*

$$S(t)\Phi_\Lambda a = \Phi_\Lambda \exp(tB_\Lambda) a$$

where

$$\begin{cases} [\Phi_\Lambda]^1(s) = [\Phi_\Lambda]^0 \exp(sB_\Lambda)\,, & s \in I_h\,, \\ [\Phi_\Lambda]^0 = \{\Phi_{\lambda_1}(0), \cdots, \Phi_{\lambda_n}(0)\}\,; \end{cases}$$

(ii) $$\begin{cases} [S(t)\Phi_\Lambda a]^1(s) = [\Phi_\Lambda \exp((t+s)B_\Lambda)a]^0\,, & s \in I_h\,, \\ [S(t)\Phi_\Lambda a]^0 = [\Phi_\Lambda]^0 \exp(tB_\Lambda) a\,, \end{cases} \quad \text{for} \quad t \geq 0;$$

(iii) $x(t) = [S(t)\Phi_\Lambda a]^0$ *is a mild solution (in fact, a strong solution) of (E) on* $(-\infty, \infty)$ *with the initial condition* $g = \Phi_\Lambda a$ *and* $u = 0$;
(iv) M_p *is decomposed into the direct sum*

$$M_p = \mathcal{M}_\Lambda \oplus \mathcal{R}_\Lambda\,, \quad \mathcal{M}_\Lambda = \bigoplus_{\lambda \in \Lambda} \mathcal{M}_\lambda$$

as in (7.19), *where* $\mathcal{M}_\Lambda$ *is given by*

$$\mathcal{M}_\Lambda = \{\phi \in M_p \colon \phi = \Phi_\Lambda a \text{ for some } a \in \boldsymbol{C}^{\dim \Phi_\Lambda}\}\,.$$

Moreover,

$$\begin{cases} S(t)\mathcal{M}_\Lambda \subset \mathcal{M}_\Lambda & \text{for all } t \in (-\infty, \infty) \\ S(t)\mathcal{R}_\Lambda \subset \mathcal{R}_\Lambda & \text{for all } t \geq 0\,. \end{cases}$$

The above proposition gives a precise information on the asymptotic behaviour of the mild solution of (E) on generalized eigenspaces for discrete spectrum. Analogous result to Proposition 7.3 for A_T holds true.

REMARK 7.3. If A_0 has compact resolvent, then the retarded resolvent $R(\lambda; A_0, \eta)$ is compact. From this and the representation (7.20) the compactness of P_λ follows, so that $\sigma_I(A) = \sigma_d(A)$ is true in this case.

Proposition 7.4. (i) *For* $\lambda \in \sigma_0(A)$, F *is one to one on* $\mathcal{M}_\lambda$.
(ii) *For* $\lambda \in \sigma_0(A_T)$, F^* *is one to one on* $\mathcal{M}_\lambda^T$.

Proof. We prove only (i). Assume first that $\phi \in \mathrm{Ker}\,(\lambda I - A)$ and $F\phi = 0$. Then by (4.11), $0 = GF\phi = S(h)\phi = e^{\lambda h}\phi$ and hence $\phi = 0$. This shows F is one to one on $\mathrm{Ker}\,(\lambda I - A)$. Next assume $\phi \in \mathcal{M}_\lambda = \mathrm{Ker}\,(\lambda I - A)^{k_\lambda}$ and $F\phi = 0$. If we set $\phi_1 = (\lambda I - A)^{k_\lambda - 1}\phi$, then $\phi_1 \in \mathrm{Ker}\,(\lambda I - A)$ and

$$S(h)\phi_1 = (\lambda I - A)^{k_\lambda - 1} S(h)\phi = (\lambda I - A)^{k_\lambda - 1} GF\phi = 0 .$$

So that $\phi_1 = 0$. Continuing this procedure k_λ times we have $\phi = 0$, i.e., F is one to one on $\mathcal{M}_\lambda$.

8. Adjoint spectral decomposition

In this section we study the spectral decomposition theory for the adjoint operator A^* of A in the space M_p^*, with an emphasis of the relations between A^* and the transposed operator A_T. The structural operators F^* and G^* will appear to key connections between the generalized eigenspaces of A^* and A_T.

8.1. Generalized eigenspaces and structural operators

Let $\mathcal{M}_\lambda^*$ (not the adjoint space of $\mathcal{M}_\lambda$!) denote the generalized eigenspace of A^* corresponding to $\lambda \in \sigma_P(A^*)$. Similarly we denote by $\mathcal{M}_\lambda^{T*}$ the generalized eigenspace of A_T^* corresponding to $\lambda \in \sigma_P(A_T^*)$.

Theorem 8.1. (i) *Three kinds of spectrum of A^* and A_T are identical and are given by*

$$(8.1) \quad \sigma_P(A^*) = \sigma_P(A_T) = \sigma_P(A_0^*, \eta^*) , \quad \sigma_R(A^*) = \sigma_R(A_T) = \sigma_R(A_0^*, \eta^*) ,$$
$$\sigma_C(A^*) = \sigma_C(A_T) = \sigma_C(A_0^*, \eta^*) ,$$

respectively.
(ii) *For each $\lambda \in \sigma_P(A^*) = \sigma_P(A_T)$,*

$$(8.2) \qquad \mathrm{Ker}(\lambda I - A^*)^l = F^* \,\mathrm{Ker}(\lambda I - A_T)^l, \qquad l = 1, 2, \cdots .$$

In particular

$$(8.3) \qquad \mathcal{M}_\lambda^* = F^* \mathcal{M}_\lambda^T .$$

Proof. (i). Using Lemma 6.1 we can prove this part by similar arguments as in the proof of Proposition 7.1.
(ii). By (4.30), $(\lambda I - A^*)F^* = F^*(\lambda I - A_T)$ on $D(A_T)$. Hence, by induction, we have

$$(\lambda I - A^*)^l F^* = F^*(\lambda I - A_T)^l \qquad \text{on} \quad D(A_T^l) .$$

Thus

$$(8.4) \qquad F^* \,\mathrm{Ker}(\lambda I - A_T)^l \subset \mathrm{Ker}(\lambda I - A^*)^l , \qquad l = 1, 2, \cdots .$$

The reverse inclusions in (8.4) were proved by Delfour and Manitius [15] by using mathematical induction. Here we give a direct proof based on Lemma 6.1. Let $\psi \in \mathrm{Ker}(\lambda I - A^*)^l$ and put

$$(8.5) \qquad \psi_0 = \psi \quad \text{and} \quad \psi_j = (\lambda I - A^*)\psi_{j-1} , \qquad j = 1, \cdots, l .$$

Then $\psi \in \mathrm{Ker}(\lambda I - A^*)^l$ is equivalent to $\psi_l = 0$. In view of Lemma 6.1 we see that $\psi_j = (\lambda I - A^*)\psi_{j-1}$ is equivalent to

$$(8.6) \quad \Delta_T(\lambda)\psi_{j-1}^0 = H_\lambda \psi_j, \quad \psi_{j-1}^0 \in D(A_0^*) \quad \text{and} \quad \psi_{j-1} = K_\lambda \psi_j + F^* E_\lambda \psi_{j-1}^0 .$$

Since $\psi_l = 0$, it follows from the last equality in (8.6) that

$$\begin{aligned} \psi_{l-1} &= F^* E_\lambda \psi_{l-1}^0 \\ \psi_{l-2} &= F^* E_\lambda \psi_{l-2}^0 + K_\lambda \psi_{l-1} = F^* E_\lambda \psi_{l-2}^0 + K_\lambda F^* E_\lambda \psi_{l-1}^0 \\ &= F^* E_\lambda \psi_{l-2}^0 + F^* T_\lambda E_\lambda \psi_{l-1}^0 = F^*(E_\lambda \psi_{l-2}^0 + T_\lambda E_\lambda \psi_{l-1}^0), \qquad \text{(by (6.5))} \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ \psi &= \psi_0 = F^*\Big(\sum_{j=0}^{l-1} T_\lambda^j E_\lambda \psi_j^0\Big). \end{aligned}$$

If we set $y_{j+1}^T = (-1)^j \psi_j^0$, $j = 0, 1, \cdots, l-1$, then by (6.6) ψ can be written as $\psi = F^*\Big(\sum_{j=0}^{l-1} \frac{1}{j!}(E_\lambda y_{j+1}^T)^{(j)}\Big)$, where $\mathcal{Y}^T = \mathrm{col}(y_1^T, \cdots, y_l^T)$ satisfies $\mathcal{A}_l^T \mathcal{Y}^T = 0$ in X^{*l}. Hence by Theorem 7.2 (ii), $\psi \in F^* \mathrm{Ker}(\lambda I - A_T)^l$, i.e., the reverse inclusions of (8.4) are proved. Therefore (8.2) is shown. The rest equality (8.3) is clear from (8.2) and the definition of generalized eigenspaces.

The statement (ii) of Proposition 8.1 has concluded that the null space $\mathrm{Ker}(\lambda I - A^*)^l$ is obtained by the application of F^* to $\mathrm{Ker}(\lambda I - A_T)^l$, whose elements are straightforwardly computed as given in Theorem 7.2 (ii).

If we notice $(A_T)_T = A$, we have the following result.

Corollary 8.1. (i) $\sigma_P(A_T^*) = \sigma_P(A) = \sigma_P(A_0, \eta)$,

$$\sigma_R(A_T^*) = \sigma_R(A) = \sigma_R(A_0, \eta), \quad \sigma_C(A_T^*) = \sigma_C(A) = \sigma_C(A_0, \eta).$$

(ii) *For each* $\lambda \in \sigma_P(A_T^*) = \sigma_P(A)$,

$$\mathrm{Ker}(\lambda I - A_T^*)^l = F\, \mathrm{Ker}(\lambda I - A)^l, \qquad l = 1, 2, \cdots .$$

In particular

$$\mathcal{M}_\lambda^{T*} = F \mathcal{M}_\lambda .$$

Analogous inclusions to (8.4) involving the operator G are:

Lemma 8.1. *For each* $\lambda \in \sigma_P(A^*) = \sigma_P(A_T)$ (*resp.* $\lambda \in \sigma_P(A) = \sigma_P(A_T^*)$),

$$\begin{aligned} &G^* \mathrm{Ker}(\lambda I - A^*)^l \subset \mathrm{Ker}(\lambda I - A_T)^l, \qquad l = 1, 2, \cdots \\ &(\textit{resp. } G\, \mathrm{Ker}(\lambda I - A_T^*)^l \subset \mathrm{Ker}(\lambda I - A)^l, \qquad l = 1, 2, \cdots). \end{aligned}$$

Proof. This lemma can be proved similarly as in the proof of (8.4) by using (4.26) and (4.27) instead of (4.30).

Theorem 8.2. (i)

$$\sigma_d(A^*) = \sigma_d(A_T)\,. \tag{8.7}$$

(ii)

$$G^* \operatorname{Ker}(\lambda I - A^*)^l = \operatorname{Ker}(\lambda I - A_T)^l\,, \tag{8.8}$$

$$\dim \operatorname{Ker}(\lambda I - A^*)^l = \dim \operatorname{Ker}(\lambda I - A_T)^l = \dim \operatorname{Ker} \mathcal{A}_l^T(\lambda) < \infty\,, \quad l = 1, 2, \cdots,\ \lambda \in \sigma_d(A_T)\,. \tag{8.9}$$

In particular

$$\dim \mathcal{M}_\lambda^* = \dim \mathcal{M}_\lambda^T \qquad \textit{for} \quad \lambda \in \sigma_d(A^*) = \sigma_d(A_T)\,. \tag{8.10}$$

Proof. (i). Let $\lambda \in \sigma_d(A^*)$. Then $\lambda \in \sigma_0(A^*)$ and from Kato [25, p. 184] and Theorem 7.1, we have

$$\bar{\lambda} \in \sigma_d(A) \text{ is a pole of } R(\mu; A) \text{ of order } k_{\bar{\lambda}} \text{ at } \mu = \bar{\lambda}\,,$$

$$\mathcal{M}_{\bar{\lambda}} = \operatorname{Ker}(\bar{\lambda} I - A)^{k_{\bar{\lambda}}}\,, \quad \dim \mathcal{M}_\lambda^* = \dim \mathcal{M}_{\bar{\lambda}} < \infty\,, \tag{8.11}$$

$$\sup_{l \geqq 1} \dim \operatorname{Ker}(\bar{\lambda} I - A)^l = \dim \mathcal{M}_{\bar{\lambda}} < \infty\,. \tag{8.12}$$

Since G is one to one and

$$G \operatorname{Ker}(\bar{\lambda} I - A_T^*)^l \subset \operatorname{Ker}(\bar{\lambda} I - A)^l\,, \qquad l = 1, 2, \cdots \tag{8.13}$$

for $\bar{\lambda} \in \sigma_d(A) \subset \sigma_P(A) = \sigma_P(A_T^*)$ (by Lemma 8.1), it follows from (8.11)–(8.13) that $\sup_{l \geqq 1} \dim \operatorname{Ker}(\bar{\lambda} I - A_T^*)^l < \infty$. This implies $\dim \mathcal{M}_{\bar{\lambda}}^{T*} < \infty$, and hence $\bar{\lambda} \in \sigma_d(A_T^*)$. Again by using Kato [25, p. 184] we have $\lambda = \bar{\bar{\lambda}} \in \sigma_d(A_T)$, which proves $\sigma_d(A^*) \subset \sigma_d(A_T)$. The reverse inclusion $\sigma_d(A_T) \subset \sigma_d(A^*)$ is obtained similarly as above. Thus, (8.7) is shown.

(ii). For $\lambda \in \sigma_d(A_T)$, the space $\operatorname{Ker}(\lambda I - A_T)^l$ is finite dimensional and is invariant under the semigroup $S_T(t)$. Then the operator $S_T(h) = G^* F^*$ in (4.13) is bijective on $\operatorname{Ker}(\lambda I - A_T)^l$ (cf. Proposition 7.4). Consequently by (8.2),

$$\operatorname{Ker}(\lambda I - A_T)^l = G^* F^* \operatorname{Ker}(\lambda I - A_T)^l = G^* \operatorname{Ker}(\lambda I - A^*)^l\,.$$

Since G^* is one to one and $\dim \operatorname{Ker}(\lambda I - A_T)^l < \infty$, we have (8.9) by (8.8) and Theorem 7.2 (ii).

Corollary 8.2. (i) $\sigma_d(A_T^*) = \sigma_d(A)$.

(ii)

$$G \operatorname{Ker}(\lambda I - A_T^*)^l = \operatorname{Ker}(\lambda I - A)^l\,,$$

$$\dim \operatorname{Ker}(\lambda I - A_T^*)^l = \dim \operatorname{Ker}(\lambda I - A)^l = \dim \operatorname{Ker} \mathcal{A}_l(\lambda) < \infty\,, \quad l = 1, 2, \cdots,\ \lambda \in \sigma_d(A)\,.$$

Lastly in this subsection we give an M_p-adjoint result for A which is an immediate consequence from Kato [25, p. 184].

Theorem 8.3. *Let* λ *be a pole of* $R(\mu;\ A_0,\ \eta)$ *of order* k_λ *at* $\mu=\lambda$. *Then* $\bar{\lambda}$ *is a pole of* $R(\mu;\ A_0^*,\ \eta^*)$ *of same order* k_λ *at* $\mu=\bar{\lambda}$. *Furthermore*

$$\mathcal{M}_{\bar{\lambda}}^* = \operatorname{Ker}(\bar{\lambda}I-A^*)^{k_\lambda} = (P_\lambda)^* M_p^*\ , \quad \dim \mathcal{M}_\lambda = \dim \mathcal{M}_{\bar{\lambda}}^* \ (\textit{may be infinity})\ ,$$

where the adjoint $(P_\lambda)^*$ *of* P_λ *in* (7.13) *is given explicitly by*

$$(P_\lambda)^* = \frac{1}{2\pi i}\int_{\bar{\Gamma}_\lambda} R(\mu;\ A^*)d\mu = P_{\bar{\lambda}}^*$$

with $\bar{\Gamma}_\lambda$ *the miller image of* Γ_λ.

The same result for A_T holds, but we omit to give such a representation.

8.2. Representations of spectral projections

It was shown in the proof of Theorem 8.2 that if $\lambda\in\sigma_d(A)$, then $\bar{\lambda}\in \sigma_d(A^*)=\sigma_d(A_T)$ and

$$\dim \mathcal{M}_\lambda = \dim \mathcal{M}_{\bar{\lambda}}^* = \dim \mathcal{M}_{\bar{\lambda}}^T = d_\lambda < \infty\ . \tag{8.14}$$

Let $\Phi=\{\phi_1,\cdots,\phi_{d_\lambda}\}$ and $\Psi=\{\psi_1,\cdots,\psi_{d_\lambda}\}$ be the bases of $\mathcal{M}_\lambda$ and $\mathcal{M}_{\bar{\lambda}}^T$ respectively. Let M be a $d_\lambda\times d_\lambda$ matrix of element $m_{ij}=\langle\phi_i,\ F^*\psi_j\rangle_{M_p}$. Then by (8.14) and (8.3), M is nonsingular. Hence we can suppose

$$\langle\phi_j,\ F^*\psi_j\rangle_{M_p} = \langle\!\langle\phi_i,\ \psi_j\rangle\!\rangle = \delta_{ij}\ , \qquad i, j = 1, \cdots, d_\lambda\ , \tag{8.15}$$

where $\langle\!\langle\ ,\ \rangle\!\rangle$ denotes the hereditary pairing in (4.35). Now we introduce the continuous projection operator

$$\tilde{P}_\lambda g = \sum_{i=1}^{d_\lambda} \langle g,\ F^*\psi_i\rangle_{M_p}\phi_i\ , \qquad g\in M_p\ .$$

It is easily verified that $\operatorname{Im}\tilde{P}_\lambda=\mathcal{M}_\lambda$ and $\operatorname{Ker}\tilde{P}_\lambda=\operatorname{Im}(\lambda I-A)^{k_\lambda}$, so that $\tilde{P}_\lambda=P_\lambda$. Thus, we obtain the following desired result.

Theorem 8.4. *Let* $\lambda\in\sigma_d(A)$. *Then the spectral projection* P_λ *in* (7.13) *has the following equivalent representation*

$$\tilde{P}_\lambda g = \sum_{i=1}^{d_\lambda} \langle\!\langle g,\ \psi_i\rangle\!\rangle\phi_i\ , \qquad g\in M_p\ , \tag{8.16}$$

where $\{\phi_1,\cdots,\phi_{d_\lambda}\}$ *is a basis of* $\mathcal{M}_\lambda$ *and* $\{\psi_1,\cdots,\psi_{d_\lambda}\}$ *is a basis of* $\mathcal{M}_{\bar{\lambda}}^T$ *satisfying* (8.15).

Corollary 8.4. *Let* $\lambda\in\sigma_d(A_T)$. *Then the spectral projection* P_λ^T *associated with* A_T *has the following equivalent representation*

$$\tilde{P}_\lambda^T f = \sum_{i=1}^{d'_\lambda} \langle\langle \phi_i, f \rangle\rangle \psi_i, \qquad f \in M_p^*,$$

where $d'_\lambda = \dim \mathcal{M}_\lambda^T = \dim \mathcal{M}_{\bar\lambda}$, $\{\psi_1, \cdots, \psi_{d'_\lambda}\}$ *is a basis of* $\mathcal{M}_\lambda^T$ *and* $\{\phi_1, \cdots, \phi_{d'_\lambda}\}$ *is a basis of* $\mathcal{M}_{\bar\lambda}$ *satisfying the same condition in* (8.15).

9. Completeness of generalized eigenfunctions

The problem of completeness of generalized eigenfunctions of retarded FDE's has been studied by Delfour and Manitius [15] and Manitius [29] for n-dimensional equations. The purpose of this section is to extend some of their results to infinite dimensional case.

First we give characterizations of the kernels of spectral projections P_λ and others in terms of F, H_λ and the retarded resolvent. Let $\lambda \in \sigma_0(A)$ and P_λ be the projection in (7.13). Then by (7.15), (7.16) and (7.18), we have that

$$\begin{aligned} g \in \operatorname{Ker} P_\lambda \quad & \text{if and only if} \quad Q_\lambda^n g = ((\lambda I - A)P_\lambda)^n g = 0, \quad n = 1, \cdots, k_\lambda \\ & \text{if and only if} \quad R(\mu; A)g \text{ is holomorphic (h.l.) at } \mu = \lambda . \end{aligned}$$

Since E_λ, H_λ and T_λ are operator valued entire functions, the equality

$$\begin{aligned} (9.1) \qquad & \{g \in M_p \colon R(\mu; A)g \textit{ is } \text{h.l. } \textit{at } \mu = \lambda\} \\ & = \{g \in M_p \colon R(\mu; A_0, \eta) H_\mu F g \textit{ is } \text{h.l. } \textit{at } \mu = \lambda\} \end{aligned}$$

follows from the representation (6.11) of $R(\mu; A)$. Hence $\operatorname{Ker} P_\lambda$ is given by the right hand side of (9.1). Let P_λ^* and P_λ^{T*} denote the spectral projections associated with A^* and A_T^*, respectively. Then the next proposition follows from the representations of $R(\mu; A^*)$ and other resolvents given in Theorems 6.1 and 6.2 as above. Note that $[R(\mu; A^*)f]^0 = R(\mu; A_0^*, \eta^*)H_\mu f$.

Proposition 9.1.

(i) $\operatorname{Ker} P_\lambda = \{g \in M_p \colon R(\mu; A_0, \eta) H_\mu F g \textit{ is } \text{h.l. } \textit{at } \mu = \lambda\}$, $\lambda \in \sigma_0(A)$.

(ii) $\operatorname{Ker} P_\lambda^T = \{f \in M_p^* \colon R(\mu; A_0^*, \eta^*) H_\mu F^* f \textit{ is } \text{h.l. } \textit{at } \mu = \lambda\}$, $\lambda \in \sigma_0(A_T)$.

(iii) $\operatorname{Ker} P_\lambda^* = \{f \in M_p^* \colon R(\mu; A_0^*, \eta^*) H_\mu f \textit{ is } \text{h.l. } \textit{at } \mu = \lambda\}$, $\lambda \in \sigma_0(A^*)$.

(iv) $\operatorname{Ker} P_\lambda^{T*} = \{g \in M_p \colon R(\mu; A_0, \eta) H_\mu g \textit{ is } \text{h.l. } \textit{at } \mu = \lambda\}$, $\lambda \in \sigma_0(A_T^*)$.

For notational brevity we set

$$\mathcal{M} = \bigcup_{\lambda \in \sigma_P(A)} \mathcal{M}_\lambda, \quad \mathcal{M}^T = \bigcup_{\lambda \in \sigma_P(A_T)} \mathcal{M}_\lambda^T, \quad \mathcal{M}^* = \bigcup_{\lambda \in \sigma_P(A^*)} \mathcal{M}_\lambda^*, \quad \mathcal{M}^{T*} = \bigcup_{\lambda \in \sigma_P(A_T^*)} \mathcal{M}_\lambda^{T*}.$$

DEFINITION 9.1. The systems of generalized eignefunctions of A, A^*, A_T and A_T^* are said to be complete if

$$\operatorname{Cl}(\mathcal{M}) = M_p, \quad \operatorname{Cl}(\mathcal{M}^*) = M_p^*, \quad \operatorname{Cl}(\mathcal{M}^T) = M_p^*. \quad \operatorname{Cl}(\mathcal{M}^{T*}) = M_p,$$

respectively.

For a set $E \subset \boldsymbol{C}^1$, $\bar{E}$ denotes the miller image of E. Following the consideration in preceding sections, we know that

$$\overline{\sigma_0(A)} = \sigma_0(A^*) = \sigma_0(A_T) = \sigma_0(A_0^*, \eta^*), \quad (9.2)$$
$$\overline{\sigma_0(A_T)} = \sigma_0(A_T^*) = \sigma_0(A) = \sigma_0(A_0, \eta).$$

Proposition 9.2. (i) *If* $\sigma_P(A)=\sigma_0(A_0, \eta)$, *then*

(9.3) $\mathcal{M}^{\perp} = \{f \in M_p^*: R(\lambda; A_0^*, \eta^*)H_\lambda f$ *is* h.l. *on* $\rho(A_0^*, \eta^*) \cup \sigma_0(A_0^*, \eta^*)\}$,

(9.4) $(\mathcal{M}^{T^*})^{\perp} = \{f \in M_p^*: R(\lambda; A_0^*, \eta^*)H_\lambda F^* f$ *is* h.l. *on* $\rho(A_0^*, \eta^*) \cup \sigma_0(A_0^*, \eta^*)\}$,

(ii) *If* $\sigma_P(A_T)=\sigma_0(A_0^*, \eta^*)$, *then*

(9.5) $(\mathcal{M}^T)^{\perp} = \{g \in M_p: R(\lambda; A_0, \eta)H_\lambda g$ *is* h.l. *on* $\rho(A_0, \eta) \cup \sigma_0(A_0, \eta)\}$,

(9.6) $(\mathcal{M}^*)^{\perp} = \{g \in M_p: R(\lambda; A_0, \eta)H_\lambda Fg$ *is* h.l. *on* $\rho(A_0, \eta) \cup \sigma_0(A_0, \eta)\}$,

Proof. We shall prove only (9.3). Other equalities are proved quite analogously. Using the duality theorem and assumption, we have by the first relation in (9.2) that

$$\mathcal{M}^{\perp} = \bigcap_{\lambda \in \sigma_P(A)} \mathcal{M}_\lambda^{\perp} = \bigcap_{\lambda \in \sigma_0(A)} (\operatorname{Im} P_\lambda)^{\perp} = \bigcap_{\lambda \in \sigma_0(A)} \operatorname{Ker}(P_\lambda)^* \quad (9.7)$$
$$= \bigcap_{\lambda \in \sigma_0(A)} \operatorname{Ker} P_{\bar{\lambda}}^* = \bigcap_{\lambda \in \overline{\sigma_0(A)}} \operatorname{Ker} P_\lambda^* = \bigcap_{\lambda \in \sigma_0(A^*)} \operatorname{Ker} P_\lambda^*.$$

Hence the equality (9.3) follows immediately from Proposition 9.1 (iii). See Theorem 8.3 for completeness in deriving (9.7). Hence the equality (9.3) follows immediately from Proposition 9.1 (iii).

Since $\mathrm{Cl}(\mathcal{M})=M_p$ if and only if $\mathcal{M}^{\perp}=\{0\}$, from Proposition 9.2 we have the following criteria for the completeness of generalized eigenfunctions.

Theorem 9.1. (i) *Assume that* $\sigma(A)=\sigma_0(A_0, \eta)$. *Then the system of generalized eigenfunctions of* A (*resp.* A_T^*) *is complete if and only if*

$$\{f \in M_p^*: R(\lambda; A_0^*, \eta^*)H_\lambda f \text{ is entire}\} = \{0\} \quad (9.8)$$

$$(\text{resp. } \{f \in M_p^*: R(\lambda; A_0^*, \eta^*)H_\lambda F^* f \text{ is entire}\} = \{0\}). \quad (9.9)$$

(ii) *Assume that* $\sigma(A_T)=\sigma_0(A_0^*, \eta^*)$. *Then the system of generalized eigenfunctions of* A_T (*resp.* A^*) *is complete if and only if*

$$\{g \in M_p: R(\lambda; A_0, \eta)H_\lambda g \text{ is entire}\} = \{0\} \quad (9.10)$$

$$(\text{resp. } \{g \in M_p: R(\lambda; A_0, \eta)H_\lambda Fg \text{ is entire}\} = \{0\}). \quad (9.11)$$

We now recall the definition of H_λ appearing in the conditions (9.8) and (9.10):

(9.12) $$H_\lambda \phi = \phi^0 + \int_{-h}^0 e^{\lambda s}\phi^1(s)ds = c + q(\lambda)\,,$$

for $\phi \in M_p^*$ in (9.8) or for $\phi \in M_p$ in (9.10), where $c=\phi^0$. The last term $q(\lambda)$ in (9.12) is a finite Laplace transform of $\phi^1 \in L_p(I_h; X^*)$ in the case (9.8) or of $\phi^1 \in L_p(I_h; X)$ in the case (9.10). We denote these sets of all such functions by FLT_p^* and by FLT_p, respectively.

An additional property of H_λ is given by

Lemma 9.1. $$\bigcap_{\lambda \in C^1} \operatorname{Ker} H_\lambda = \{0\}\,.$$

Proof. We shall show the case $H_\lambda: M_p \to X$. Let $\phi \in \bigcap_{\lambda \in C^1} \operatorname{Ker} H_\lambda$. Then by (9.12),

(9.13) $$\phi^0 + \int_{-h}^0 e^{\lambda s}\phi^1(s)ds = 0 \qquad \text{for all} \quad \lambda \in C^1\,.$$

Tending $\operatorname{Re}\lambda \to \infty$ in (9.13) we have $\phi^0=0$. Then $\phi^1=0$ follows from (9.13) and the bijectivity of Laplace transform.

Corollary 9.1. *Assume that $\sigma(A)=\sigma_0(A_0, \eta)$ (resp. $\sigma(A_T)=\sigma_0(A_0^*, \eta^*)$). The system of generalized eigenfunctions of A (resp. A_T) is complete if and only if for $c \in X^*$ and $q(\lambda) \in FLT_p^*$,*

$$R(\lambda; A_0^*, \eta^*)(c+q(\lambda)) \textit{ is entire } \Rightarrow c+q(\lambda) \equiv 0$$

(resp. for $c \in X$ and $q(\lambda) \in FLT_p$,

$$R(\lambda; A_0, \eta)(c+q(\lambda)) \textit{ is entire } \Rightarrow c+q(\lambda) \equiv 0)\,.$$

Proof. Obvious from Theorem 9.1 and Lemma 9.1.

Corollary 9.1 is interpreted as that the completeness for A is equivalent to the nonexistence of nontrivial entire function in the class $c+q(\lambda)$, $c \in X^*$, $q(\lambda) \in FLT_p^*$, which completely cancellate all poles of $R(\lambda; A_0^*, \eta^*)$ in the form $R(\lambda; A_0^*, \eta^*)(c+q(\lambda))$, provided that $R(\lambda; A_0^*, \eta^*)$ has poles only.

10. Illustrative examples

We shall give some applications of the abstract results of preceding sections to practical partial FDE's in the following examples.

EXAMPLE 10.1. Consider the parabolic partial FDE

(10.1) $$\frac{\partial x(t,\xi)}{\partial t} = \frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial x(t,\xi)}{\partial \xi}\right) + b(\xi)x(t,\xi) + \sum_{r=1}^m a_r(\xi)x(t-h_r,\xi)$$
$$+ \int_{-h}^0 a_I(s,\xi)x(t+s,\xi)ds\,, \qquad t>0,\ \xi \in (0,1)$$

with boundary and initial conditions

(10.2) $$x(t, 0) = x(t, 1) = 0, \qquad t>0,$$

(10.3) $$x(0, \xi) = g^0(\xi), \quad x(s, \xi) = g^1(s, \xi) \quad \text{a.e.} \quad (s, \xi) \in I_h \times [0, 1].$$

For the system (11.1)–(11.3) we assume

(i)

(10.4) $$a(\xi)>0 \quad \text{for} \quad \xi \in [0, 1], \quad a(\cdot) \in C^1[0, 1], \quad b(\cdot) \in C[0, 1];$$

(ii)

(10.5) $$a_r(\cdot) \in L_2[0, 1], \quad r = 1, \cdots, m, \quad 0 \le h_1 < \cdots < h_m = h, \quad a_I(\cdot) \in L_2(I_h \times [0, 1]);$$

(iii)

(10.6) $$g = (g^0(\cdot), g^1(\cdot)) \in L_2[0, 1] \times L_2(I_h \times [0, 1]).$$

The product space $L_2[0, 1] \times L_2(I_h \times [0, 1])$ in (10.6) can be identified with $L_2[0, 1] \times L_2(I_h; L_2[0, 1])$, so we denote this space simply by M_2. Let A_0 be the realization in $L_2[0, 1]$ of the Sturm-Liouville operator $\partial/\partial\xi(a(\xi)\partial/\partial\xi)+b(\xi)$ with Dirichlet boundary condition (10.2). In what follows we shall write L_2 instead of $L_2[0, 1]$ for brevity. Since L_2 is a Hilbert space, we identify L_2 and L_2^* as usual. Now we define the operator $A_r \in B(L_2)$, $r=1, \cdots, m$ and $A_I \in L_2(I_h; B(L_2))$ by

$$(A_r z)(\xi) = a_r(\xi) z(\xi) \quad \text{a.e.} \quad \xi \in [0, 1], \quad r=1, \cdots, m$$

and

$$(A_I(s) z)(\xi) = a_I(s, \xi) z(\xi) \quad \text{a.e.} \quad \xi \in [0, 1] \qquad \text{for a.e.} \quad s \in I_h,$$

respectively. By the condition (10.5) and the use of Schwartz inequality the above operators are well defined. Then the system (10.1)–(10.3) can be written in the same form as of (E) on the space $X=L_2$. The (weak) solution $x(t, \xi; g)$ of (10.1)–(10.3) is interpreted as the mild solution $x(t; g)(\xi)$ of (E) at the point $\xi \in [0, 1]$. So, for each $t>0$, $x(t, \xi; g)$ has sense for a.e. $\xi \in [0, 1]$. Since A_0 is selfadjoint with compact resolvent in L_2 by (10.4), there exists a set of eigenvalues and eigenfunctions $\{\mu_n, \Psi_n : n=1, 2, \cdots\}$ of A_0 such that

(iv) $\{\Psi_n\}$ is a complete orthonormal system in L_2;

(v) $\sqrt{-\mu_n} = Cn + O\left(\frac{1}{n}\right)$ as $n \to \infty$, where C is a constant depending only on the coefficient $a(\xi)$ (cf. Kato [25, p. 277], Ince [23, p. 270–273]).

Consequently, A_0 generates an analytic semigroup $T(t)$ given by

$$T(t)z = \sum_{n=1}^{\infty} e^{\mu_n t} \langle z, \Psi_n \rangle_{L_2} \Psi_n, \qquad z \in L_2,$$

where $\langle z, \Psi_n \rangle_{L_2} = \int_0^1 z(\xi)\Psi_n(\xi)d\xi$. Using the asymptotics of $\{\mu_n\}$ in (v), we can verify that $T(t)$ is compact for all $t>0$. Then by Proposition 3.1 (ii), $S(t)$ is compact for $t>h$. This implies by spectral mapping theorem (cf. Yosida [49, p. 277]) that $\sigma(A)=\sigma_P(A)=\sigma_d(A)$ is countable, bounded from below and $\sigma(A)\cap\{z: \alpha\leq \mathrm{Re}\, z\}$ is a finite set for each $\alpha\in \boldsymbol{R}^1$. Now following the line of Hale's proof in [18, Chapter 7, Section 4] with some obvious modifications, we have the following result on the asymptotic stability.

The zero solution of (10.1)–(10.3) is exponentially asymptotic stable in L_2, i.e., there exist constants $K\geq 1$ and $\varepsilon_0>0$ such that

$$\int_0^1 |x(t, \xi; g)|^2 d\xi \leq Ke^{-\varepsilon_0 t}\left(\int_0^1 |g^0(\xi)|^2 d\xi + \int_0^1\int_{-h}^0 |g^1(s, \xi)|^2 d\xi ds\right), \quad t\geq 0,\ g\in M_2 \tag{10.7}$$

provided that

$$\sup\{\mathrm{Re}\,\lambda: \lambda\in\sigma_d(A)\}<0\,. \tag{10.8}$$

For the system (10.1)–(10.3), in view of Proposition 7.1, the condition (10.8) is replaced by that, for $\mathrm{Re}\,\lambda\geq 0$,

$$\lambda g^0 - \frac{\partial}{\partial\xi}\left(a(\xi)\frac{\partial g^0}{\partial \xi}\right) - b(\xi)g^0 - \sum_{r=1}^m e^{-\lambda h_r}\cdot a_r(\xi)g^0 - \int_{-h}^0 e^{\lambda s}a_I(s, \xi)g^0 ds = 0\,, \tag{10.9}$$
$$g^0\in D(A_0) \text{ implies } g^0=0 \text{ in } L_2\,.$$

EXAMPLE 10.2. In this example, we consider the special equation

$$\frac{\partial x(t, \xi)}{\partial t} = a\frac{\partial^2 x(t, \xi)}{\partial \xi^2} + bx(t, \xi) + \sum_{r=1}^m a_r x(t-h_r, \xi)\,, \qquad t>0, \quad \xi\in(0, 1) \tag{10.10}$$

of (10.1) with the same mixed conditions (10.2) and (10.3), where $a>0$, b, a_r are real constants. For the system (10.10), (10.2), (10.3) we have easily that

$$\mu_n = -an^2\pi^2+b\,, \quad \Psi_n = \sqrt{2}\,\sin n\pi\xi\,, \qquad n=1, 2, \cdots,$$

so that the spectrum $\sigma(A)$ is given by

$$\sigma(A) = \{\lambda\in \boldsymbol{C}^1: \lambda + an^2\pi^2 - b - \sum_{r=1}^m a_r e^{-\lambda h_r} = 0 \quad \textit{for some } n=1, 2, \cdots\}\,.$$

It is evident that $\sigma(A)$ is countable and each $\lambda\in\sigma(A)$ has finite multiplicity (may be $\neq 1$). The asymptotic stability condition (10.9) is now reduced to a verifiable condition that

$$(10.11)\qquad \begin{cases} \text{all roots of the transcendental equations} \\ \qquad \lambda=-an^2\pi^2+b+\sum_{r=1}^{m} a_r e^{-\lambda h_r}, \qquad n=1,2,\cdots \\ \text{have negative real parts}. \end{cases}$$

A simple sufficient condition for (10.11) is $\sum_{r=1}^{m}|a_r|<a\pi^2-b$, which is shown by direct calculations using contradiction. Recently Lenhart and Travis [28, Corollary 1.2] have proved that (10.11) holds for all $h_r\geq 0$ if and only if $\sum_{r=1}^{m}|a_r|<a\pi^2-b$ and $\sum_{r=1}^{m} a_r<a\pi^2-b$. Set $\Delta_n(\lambda)=\lambda+an^2\pi^2-b-\sum_{r=1}^{m} a_r e^{-\lambda h_r}$. Let $\{\lambda_{nj}\}_{j=1}^{\infty}$ be the set of roots of $\Delta_n(\lambda)=0$ and let k_{nj} be the multiplicity of λ_{nj}. Then the retarded resolvent $R(\lambda;A_0,\eta)$ is given by

$$R(\lambda;A_0,\eta)z=\sum_{n=1}^{\infty}\frac{2}{\Delta_n(\lambda)}\langle z,\sin n\pi\xi\rangle_{L_2}\sin n\pi\xi\,,\ \lambda\in C^1-\{\lambda_{nj}: n,j=1,2,\cdots\}\,;$$

the basis of the generalized eigenspace $\mathcal{M}_{\lambda_{nj}}$ corresponding to $\lambda_{nj}\in\sigma(A)$ is given by

$$\{e^{\lambda_{nj}s}\sin n\pi\xi,\cdots,s^{k_{nj}-1}e^{\lambda_{nj}s}\sin n\pi\xi\}.$$

If $a_m\neq 0$, this system of generalized eigenfunctions is complete in M_2. But if $a_m=0$, the completeness does not hold in general (cf. [29, 35]).

Acknowledgments

I would like to express my sincere gratitude to Professor H. Tanabe for his helpful comments and suggestions in accomplishing this work. I have learned much about functional analysis and partial differential equations from his excellent books [42,43]. I also express my gratitude to the referee for his comments.

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