

UNITARY REPRESENTATIONS AND A GENERAL VANISHING THEOREM FOR $(0, r)$ -COHOMOLOGY

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1. Introduction

Let $X=G/K$ be a Hermitian symmetric space where K is a maximal compact subgroup of a connected non-compact semisimple Lie group G . We assume that G has a finite center. Let $H \subset K$ be a Cartan subgroup of G , let g, k, h be the complexifications of the Lie algebras g_0, k_0, h_0 of G, K, H , let $\Delta = \Delta(g, h)$ be the set of non-zero roots of (g, h) , and let $\Delta^+ \subset \Delta$ be a system of positive roots compatible with the G -invariant complex structure on X . That is, if $g_0 = k_0 + p_0$ is a Cartan decomposition of g_0 then the splitting of the complexified tangent space $p = p^g$ of X at the origin is given by

$$(1.1) \quad p = p^+ \oplus p^- \quad \text{where} \quad p^\pm = \sum_{\alpha \in \Delta_\pm^+} g_{\pm\alpha}$$

for g_β the root space of $\beta \in \Delta$ and $\Delta_n^+ = \Delta^+ \cap \Delta_n$, Δ_n = the set of noncompact roots in Δ . Let $\Delta_k^+ = \Delta^+ \cap \Delta_k$ where Δ_k = the set of compact roots in Δ and let $\langle Q \rangle$ be the sum of roots in Q for $Q \subset \Delta$. In particular we set $2\delta = \langle \Delta^+ \rangle$ as usual, and then we can define the following subset of the dual space h^* of h : for L the character lattice of H :

$$(1.2) \quad F'_0 = \{ \text{integral forms } \Lambda \text{ in } L \mid (\Lambda + \delta, \alpha) \neq 0 \text{ for each} \\ \alpha \text{ in } \Delta \text{ and } (\Lambda + \delta, \alpha) > 0 \text{ for each } \alpha \text{ in } \Delta_k^+ \}.$$

Now let $\tau \in \hat{K}$ be an irreducible unitary representation of K with highest weight Λ relative to the positive system Δ_k^+ for (k, h) . The induced homogeneous vector bundle $E_\tau = G \times_K V_\tau$ over X has a holomorphic structure (here V_τ is the representation space of τ). Let Γ be a fixed torsion free, co-compact, discrete subgroup of G . Then given $\tau \in \hat{K}$, there is a natural sheaf $\theta_\tau(\Gamma)$ over $X_\Gamma \stackrel{\text{def.}}{=} \Gamma \backslash X$ generated by the presheaf: $U \mapsto$ abelian group of Γ -invariant holomorphic sections of E_τ on the inverse image of U under the map $X \rightarrow X_\Gamma$, where $U \subset X_\Gamma$ is an open set. The cohomology groups $H^*(X_\Gamma, \theta_\tau(\Gamma))$ of X_Γ with coefficients in

$\theta_\tau(\Gamma)$ were introduced (and described somewhat differently) and studied by Matsushima and Murakami in [2], [3]. In [4] they obtained the formula

$$(1.3) \quad \dim H^r(X_\Gamma, \theta_\tau(\Gamma)) = \sum_{\substack{\pi \in \hat{G} \\ \pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1}} m_\pi(\Gamma) \dim \text{Hom}_K(H_\pi, \Lambda^r \mathfrak{p}^+ \otimes V_\tau)$$

where $m_\pi(\Gamma)$ is the multiplicity of π in $L^2(\Gamma \backslash G)$, Ω is the Casimir operator of G , H_π is the space of K -finite vectors in the representation space of π , and $(,)$ is the Killing form of g ; also see [1].

The purpose of this paper is two-fold: (i) We find the most general and precise vanishing theorem possible for the cohomology $H^*(X_\Gamma, \theta_\tau(\Gamma))$, for any $\tau \in \hat{K}$ with highest weight $\Lambda \in F'_0$; (ii) We describe precisely the unitary representations π and the integers r which contribute to the formula (1.3)—i.e., those $\pi \in \hat{G}$ and those integers r such that $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$, $\text{Hom}_K(H_\pi, \Lambda^r \mathfrak{p}^+ \otimes V_\tau) \neq 0$. Actually we do a bit more in Lemma 2.2. The results are formulated in Theorems 1.9 and 1.10 below. Theorem 1.9 solves a problem (one of four problems) raised by M. Harris at the recent 834th meeting of the A.M.S. in New Jersey. If one assumes that the positive root system

$$(1.4) \quad P^{(\Lambda)} \stackrel{\text{def.}}{=} \{ \alpha \in \Delta \mid (\Lambda + \delta, \alpha) > 0 \}$$

(corresponding to the regular element $\Lambda + \delta$) is also compatible with some G -invariant complex structure on X , here we write $X \leftrightarrow P^{(\Lambda)}$, then the results (i), (ii) are already obtained in [12], [14] respectively. Thus the complete generality of this paper amounts to the removal of the assumption $X \leftrightarrow P^{(\Lambda)}$. In view of (1.3), clearly Theorem 1.9 implies Theorem 1.10.

Before stating the main results we introduce a bit more notation. For $\Lambda \in F'_0$ let

$$(1.5) \quad \begin{aligned} Q_\Lambda &= \{ \alpha \in \Delta_n^+ \mid (\Lambda + \delta, \alpha) > 0 \}, \quad Q'_\Lambda = \Lambda_n^+ - Q_\Lambda \\ b_\Lambda &= \mathfrak{h} + \sum_{\alpha \in P^{(\Lambda)}} \mathfrak{g}_\alpha \quad (\text{a Borel subalgebra of } \mathfrak{g}). \end{aligned}$$

If $q = l + u$ is a Levi decomposition of a parabolic subalgebra of \mathfrak{g} with unipotent radical u and a reductive complement l , we let $\Delta(u)$, $\Lambda(l)$ denote the roots of u , l and we set

$$(1.6) \quad q_{u,n} = \text{the set of non-compact roots in } \Delta(u).$$

In particular if θ is the Cartan involution of $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ we shall consider θ -stable parabolic subalgebras $q \supset \mathfrak{h}$ in the sense of [9]. That is, for some $\lambda \in \mathfrak{h}^*$ which is pure-imaginary valued on \mathfrak{h}_0 one has

$$(1.7) \quad \begin{aligned} \Delta(l) &= \{ \alpha \in \Delta \mid (\lambda, \alpha) = 0 \}, \quad \Delta(u) = \{ \alpha \in \Delta \mid (\lambda, \alpha) > 0 \}; \\ l &= \mathfrak{h} + \sum_{\alpha \in \Delta(l)} \mathfrak{g}_\alpha. \end{aligned}$$

One then has $l = (l_0)^{\mathcal{C}}$ where l_0 is the Lie algebra of a connected Lie subgroup L of G . For $2\delta^{(\Lambda)} = \langle P^{(\Lambda)} \rangle$, $2\delta_n^{(\Lambda)} = \langle P_n^{(\Lambda)} \rangle$, $\Lambda + \delta - \delta^{(\Lambda)}$ is $P^{(\Lambda)}$ -dominant since $\Lambda + \delta$ is $P^{(\Lambda)}$ -regular and we assume that the finite-dimensional irreducible g -module F^Λ with $P^{(\Lambda)}$ -highest weight $\Lambda + \delta - \delta^{(\Lambda)}$ integrates to a smooth G -module. In particular if $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$ then the highest weight space of F^Λ is l_0 -invariant and hence is L -invariant (as L is connected). Thus this weight space can be regarded as an $(l, L \cap K)$ module which we shall denote by $\mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}^{1)}$. We let R_q^* denote Zuckerman's parabolic induction functor from the category of $(l, L \cap K)$ modules to the category of (g, K) modules [7]. Then we have [8]

Theorem 1.8. *For a θ -stable parabolic subalgebra $q = l + u$ (as above) and $\Lambda \in F'_0$ with $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$, let $\pi(q) = R_q^s \mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$, where $s = \dim u \cap k$. Then $\pi(q) \in \hat{G}^{2)}$ and $\pi(q)(\Omega) = (\Lambda, \Lambda + 2\delta) 1$. Also $\pi(q)|_K$ contains $\mu(q) \stackrel{\text{def.}}{=} \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle$ as its lowest highest weight (relative to Δ_k^+); see (1.6). For $j \neq s$, $R_q^j \mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}} = 0$.*

Also see [10]. We can now state the main results.

Theorem 1.9. *For $\Lambda \in F'_0$, let $\pi \in \hat{G}$ with $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$, and let $r \geq 0$ be an integer with $\text{Hom}_K(H_\pi, \Lambda^r P^+ \otimes V_\tau) \neq 0$ where (as above) H_π is the space of K -finite vectors in the representation space of π and V_τ is the representation space of $\tau \in \tilde{K}$ with Δ_k^+ -highest weight Λ . Then (i) $\pi = R_q^{\dim u \cap k} \mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$ for some θ -stable parabolic $q = l + u \supset b_\Lambda$ (where as above we assume $F^{(\Lambda)}$ integrates smoothly to G) with $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$ and (ii) $r = |Q'_\Lambda| - |q_{u,n}| + 2|Q_\Delta \cap q_{u,n}|$; ³⁾ see (1.5), (1.6). Conversely let $q = l + u \supset b_\Lambda$ be a θ -stable parabolic subalgebra of g and assume $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$. Let $\pi(q) = R_q^{\dim u \cap k} \mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$ so that $\pi(q) \in \hat{G}$ such that $\pi(q)(\Omega) = (\Lambda, \Lambda + 2\delta) 1$ by Theorem 1.8. Let $r_{q,\Lambda} \stackrel{\text{def.}}{=} |Q'_\Lambda| - |q_{u,n}| + 2|Q_\Delta \cap q_{u,n}|$, and let $\mu(q) = \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle$ be the lowest K -type of $\pi(q)$. Then if, at least, $\mu(q) - \delta_n^{(\Lambda)}$ is Δ_k^+ -dominant, $\text{Hom}_K(H_{\pi(q)} \Lambda^{r_{q,\Lambda}} p^+ \otimes V_\tau) \neq 0$ and in fact is one-dimensional.*

Theorem 1.10. *Let $\tau \in \tilde{K}$ be an irreducible representation of K with Δ_k^+ -highest weight $\Lambda \in F'_0$. Then the sheaf cohomology $H^r(X_\Gamma, \theta_\tau(\Gamma))$ vanishes unless $r = |Q'_\Lambda| - |q_{u,n}| + 2|Q_\Delta \cap q_{u,n}|$ for some θ -stable parabolic subalgebra $q = l + u \supset b_\Lambda$ of g with $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$.*

As observed earlier the first half of Theorem 1.9 immediately implies Theorem 1.10, given the formula (1.3).

1). $\mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$ is unitary
 2). $\pi \in \hat{G}$ and its (g, K) module are denoted by the same symbol.
 3). $|S|$ is the cardinality of a set S .

2. Proof of the main lemma

In this section G/K need *not* be assumed to be Hermitian. We assume only $\text{rank}_R G = \text{rank}_R K$. We shall make use of the $1/2$ -spin modules S^\pm for k . Its weights are of the form $\delta_n - \langle T \rangle$, where $T \subset \Delta_n^+$, $(-1)^{|T|} = \pm 1$. If $V_{\Lambda + \delta_n}$ is the irreducible k -module with Δ_k^+ -highest weight $\Lambda + \delta_n$, $\Lambda \in F'$, then $S^\pm \otimes V_{\Lambda + \delta_n}$ integrates¹⁾ smoothly to a K -module, and we have the following main lemma. Here W, W_k are the Weyl groups of $(g, h), (k, h)$ and for $(w, \tau) \in W \times W_k$

$$(2.1) \quad \Phi_w^{(\Lambda)} = w(-P^{(\Lambda)}) \cap P^{(\Lambda)}, \quad \Phi_\tau = \tau(-\Delta_k^+) \cap \Delta_k^+.$$

Lemma 2.2. *Let μ be a common K -type (relative to Δ_k^+) of $\pi|_K, S^\pm \otimes V_{\Lambda + \delta_n}$, where $\pi \in \hat{G}$ such that $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$. Then (even without the assumption that G/K is Hermitian symmetric) there is a θ -stable parabolic subalgebra $q = l + u \supset b_\Lambda$ such that $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0, \mu = \mu(q)$ and $\pi = R_q^s \mathbf{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$, $s = \dim u \cap k$. q can be chosen such that $(P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}} = \Phi_{\tau^{-1}w}^{(\Lambda)}$ for some $(w, \tau) \subset W \times W_k$ with $wP^{(\Lambda)} \supset \Delta_k^+$ (see above notation). Conversely let $q = l + u \supset b_\Lambda$ be a θ -stable parabolic subalgebra of g with $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$, let $\pi = \pi(q) = R_q^{\dim u \cap k} \mathbf{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$, and let $\mu = \mu(q)$. Then $\pi \in \hat{G}$ such that $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$ and if (at least) $\mu - \delta_n^{(\Lambda)}$ is Δ_k^+ -dominant then μ is a common K -type of $\pi|_K, S^\pm \otimes V_{\Lambda + \delta_n}$ for $\det \sigma(-1)^{|P^{(\Lambda)} - q_{u,n}|} = \pm 1$ where $\sigma \in W$ is the unique element $\ni P^{(\Lambda)} = \sigma \Delta^+$.*

Proof. The hypotheses on Λ, μ, π are exactly those of Theorem 2.8 of [15]. Thus by that theorem μ has the form

$$(2.2) \quad \mu = \Lambda + \delta_n + \tau^{-1}(w\delta^{(\Lambda)} - \delta_k)$$

for some $(w, \tau) \in W \times W_k$ with $\Delta_k^+ \subset wP^{(\Lambda)}$. We re-interpret (2.3) as follows. Let P be the positive root system $\tau^{-1}wP^{(\Lambda)}$. Thus (as $\Delta_k^+ \subset wP^{(\Lambda)}$), $\tau^{-1}\Delta_k^+ \subset P \Rightarrow P = \tau^{-1}\Delta_k^+ \cup P_n, P_n \stackrel{\text{def.}}{=} \Delta_n \cap P, \Rightarrow \delta(P) (= \frac{1}{2}$ the sum of roots in $P) = \tau^{-1}\delta_k + \delta_n(P)$, where $2\delta_n(P) = \langle P_n \rangle$. On the other hand $\delta(P) = \tau^{-1}w\delta^{(\Lambda)}$; i.e. $\tau^{-1}(w\delta^{(\Lambda)} - \delta_k) = \delta_n(P)$. Then since $\delta - \delta^{(\Lambda)} = \delta_n - \delta_n^{(\Lambda)}$ for $2\delta_n^{(\Lambda)} = \langle P_n^{(\Lambda)} \rangle$, (2.3) can be written

$$(2.4) \quad \mu = (\Lambda + \delta - \delta^{(\Lambda)}) + \delta_n^{(\Lambda)} + \delta_n(P).$$

Thus by Kumaresan's 2nd lemma, Proposition 5.16 of [9], there is a θ -stable parabolic subalgebra $q = l + u \supset b_\Lambda$ of g such that

$$(2.6) \quad \begin{aligned} \mu &= \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle \stackrel{\text{def.}}{=} \mu(q) \\ &= \Lambda + \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle, \\ (\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) &= 0. \end{aligned}$$

Thus $\pi|_K$ contains the lowest highest weight $\mu(q)$, and since $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$

1). Since $\Lambda \in L$ implies $(\Lambda + \delta_n) + \delta_n = \Lambda + 2\delta_n \in L$.

$=(\Lambda + \delta - \delta^{(\Lambda)}, \Lambda + \delta - \delta^{(\Lambda)} + 2\delta^{(\Lambda)}) 1$ we may also conclude from Proposition 6.1 of [9] that $\pi = R_q^{\dim u \cap k} \mathbf{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$. From (2.3), (2.6), $\tau^{-1}(w\delta^{(\Lambda)} - \delta_k) = -\delta_n^{(\Lambda)} + \langle q_{u,n} \rangle$. Then by (2.1), $\langle (P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}} \rangle = 2\delta_n^{(\Lambda)} - \langle q_{u,n} \rangle + (\delta_k - \tau^{-1}\delta_k) = \delta_n^{(\Lambda)} - \tau^{-1}w\delta^{(\Lambda)} + \tau^{-1}\delta_k + (\delta_k - \tau^{-1}\delta_k) = \delta^{(\Lambda)} - \tau^{-1}w\delta^{(\Lambda)} = \langle \Phi_{\tau^{-1}w}^{(\Lambda)} \rangle$, from whence we may conclude that $(P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}} = \Phi_{\tau^{-1}w}^{(\Lambda)}$. Conversely let $q = l + u \supset b_\Delta$ be a θ -stable parabolic subalgebra of g with $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$, let $\pi = \pi(q) = R_q^s \mathbf{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$, $s = \dim u \cap k$, and let $\mu = \mu(q)$. Then $\pi \in \hat{G}$ such that $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$ by Theorem 1.8, and by that same theorem μ occurs in π . It suffices therefore to show that μ occurs in $S^\pm \otimes V_{\Lambda + \delta_n}$, at least if $\mu - \delta_n^{(\Lambda)}$ is Δ_k^+ -dominant.

Define $Q = P_n^{(\Lambda)} - q_{u,n} \subset P_n^{(\Lambda)}$ so that $\mu \stackrel{(iv)}{=} \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle = \Lambda + \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle = \Lambda + \delta_n + \delta_n^{(\Lambda)} - \langle Q \rangle = \Lambda + \delta + \delta_n^{(\Lambda)} - \langle Q \rangle - \delta_k$. That is, π, μ, Q satisfy the conditions of Theorem 2.6 of [15]. By that theorem $\exists (w, \tau) \in W \times W_k$ therefore such that $\Phi_w^{(\Lambda)} = Q \cup \Phi_\tau, \tau^{-1}wP^{(\Lambda)} \supset \Delta_k^+, w(\Lambda + \delta - \delta^{(\Lambda)}) = \Lambda + \delta - \delta^{(\Lambda)}$. Let $P = \tau^{-1}wP^{(\Lambda)}$ so that $\delta_n(P) = \tau^{-1}w\delta^{(\Lambda)} - \delta_k$ by the same argument preceding (2.4). Note also that for $\alpha \in \Delta_k^+ \subset \tau^{-1}wP^{(\Lambda)}, w^{-1}\tau\alpha \in P^{(\Lambda)} \Rightarrow (\Lambda + \delta - \delta^{(\Lambda)}, w^{-1}\tau\alpha) \geq 0$; i.e. $0 \leq (\tau^{-1}w(\Lambda + \delta - \delta^{(\Lambda)}), \alpha) = (\tau^{-1}(\Lambda + \delta - \delta^{(\Lambda)}), \alpha)$; i.e. both $\Lambda + \delta - \delta^{(\Lambda)}$ and $\tau^{-1}(\Lambda + \delta - \delta^{(\Lambda)})$ are Δ_k^+ -dominant $\Rightarrow \Lambda + \delta - \delta^{(\Lambda)} \stackrel{(v)}{=} \tau^{-1}(\Lambda + \delta - \delta^{(\Lambda)})$ (since $\tau \in W_k$). Now $\delta^{(\Lambda)} - w\delta^{(\Lambda)} = \langle \Phi_w^{(\Lambda)} \rangle = \langle Q \rangle + \delta_k - \tau\delta_k \Rightarrow \tau^{-1}\langle Q \rangle = \tau^{-1}\delta_n^{(\Lambda)} - \tau^{-1}w\delta^{(\Lambda)} + \delta_k = \tau^{-1}\delta_n^{(\Lambda)} - \delta_n(P) \Rightarrow \mu = \Lambda + \delta - \delta^{(\Lambda)} + 2\delta_n^{(\Lambda)} - \langle Q \rangle = \tau[\Lambda + \delta - \delta^{(\Lambda)} + \tau^{-1}(\delta_n^{(\Lambda)} - \langle Q \rangle)] + \delta_n^{(\Lambda)}$ (by (iv), (v)) $= \tau[\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)] + \delta_n^{(\Lambda)} \Rightarrow \mu - \delta_n^{(\Lambda)} = \tau[\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)]$, where $\delta_n(P)$ is Δ_k^+ -dominant since $P \supset \Delta_k^+$. If we assume that $\mu - \delta_n^{(\Lambda)}$ is Δ_k^+ -dominant then we see that both $\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)$ and $\tau[\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)]$ are both Δ_k^+ -dominant with $\tau \in W_k$. Hence $\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P) \stackrel{(vi)}{=} \tau[\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)] \Rightarrow \mu - \delta_n^{(\Lambda)} = \Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P) = \Lambda + \delta_n - \delta_n^{(\Lambda)} + \delta_n(P) \Rightarrow \mu = \Lambda + \delta_n + \delta_n(P)$. But (v) and (vi) imply that $\delta_n(P) = \tau\delta_n(P) = w\delta^{(\Lambda)} - \tau\delta_k$; i.e., $\tau^{-1}w\delta^{(\Lambda)} - \delta_k = w\delta^{(\Lambda)} - \tau\delta_k \Rightarrow \langle \Phi_{\tau^{-1}w}^{(\Lambda)} \rangle = \langle \Phi_w^{(\Lambda)} \rangle - \langle \Phi_\tau \rangle = \langle Q \rangle \Rightarrow \Phi_{\tau^{-1}w}^{(\Lambda)} \stackrel{a.}{=} \stackrel{def.}{=} Q = P_n^{(\Lambda)} - q_{u,n}$. As $\Delta_k^+ \subset P = \tau^{-1}w\sigma\Delta^+, \delta_n(P)$ is a k -type of S^\pm for $\det \tau^{-1}w\sigma = \pm 1$; i.e. $\mu = \Lambda + \delta_n + \delta_n(P)$ is a K -type of $S^\pm \otimes V_{\Lambda + \delta_n}$ for $\det \sigma(-1)|^{P_n^{(\Lambda)} - q_{u,n}} (= \det \sigma \det \tau^{-1}w \text{ by } a.) = \pm 1$, which concludes the proof of Lemma 2.2.

REMARKS. Lemma 2.2 refines and completes the results in [11], [13]. In the converse statement in Lemma 2.2 the preceding proof shows that $P_n^{(\Lambda)} - q_{u,n} = \Phi_w^{(\Lambda)}$ for some $w \in W$ such that $\Delta_k^+ \subset wP^{(\Lambda)}$. Also since the K -type $\mu(q)$ occurs exactly once in $\pi(q)$ one has

$$(2.7) \quad \dim \text{Hom}_K(H_{\pi(q)}, S^\pm \otimes V_{\Lambda + \delta_n}) = 1$$

for $\det \sigma(-1)|^{P_n^{(\Lambda)} - q_{u,n}} = \pm 1$.

3. Proof of Theorem 1.9.

Assume now, as above, G/K has a G -invariant complex structure com-

patible with the positive system Δ^+ . Then (see 1.1)

$$(3.7) \quad \begin{aligned} \Sigma \oplus \Lambda^{n-j} P^+ &= S^\pm \otimes V_{\delta_n} \\ (-1)^j &= \pm 1 \end{aligned}$$

for $n = |\Delta_n^+|$. Here $\dim V_{\delta_n} = 1$ by Weyl's dimension formula (since $(\delta_n, \Delta_k^+) = 0$) and for $(-1)^{n-r} = \pm 1$ we have $r = n - (n-r) \Rightarrow \Lambda^r P^+ \otimes V_\Delta \stackrel{(i)}{\subset} S^\pm \otimes V_{\delta_n} \otimes V_\Delta = S^\pm \otimes V_{\Delta+\delta_n}$. Suppose $\text{Hom}_K(H_\pi, \Lambda^r P^+ \otimes V_\tau) \neq 0$ as in the statement of Theorem 1.9. Then there is a common K -type μ of $\pi|_K, \Lambda^r P^+ \otimes V_\tau$. Since Λ is the highest weight of $\tau, \mu = \Lambda + \langle T \rangle$, where $T \subset \Delta_n^+, |T| = r$, and since $\Lambda^r P^+ \otimes V_\tau \subset S^\pm \otimes V_{\Delta+\delta_n}$ (by (i)) Lemma 2.2 gives $\mu = \mu(q), \pi = R_q^s \mathbf{C}_{\Delta+\delta-\delta^{(\Lambda)}}$, $s = \dim u \cap k$, for some θ -stable parabolic $q = l + u \supset b_\Delta$ with $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$. We claim that

$$(3.8) \quad r = 2|q_{u,n} \cap Q_\Delta| + |Q'_\Delta| - |q_{u,n}|.$$

To see this we use the following set-theoretic observation.

Lemma 3.9. *There is a bijection b of the subsets of $P_n^{(\Lambda)}$ onto the subsets of Δ_n^+ given by $bQ = (Q'_\Delta \cap -Q) \cup (Q_\Delta - Q) \subset \Delta_n^+$ for $Q \subset P_n^{(\Lambda)}, b^{-1}T = (Q_\Delta - T) \cup [-(T \cap Q'_\Delta)] \subset P_n^{(\Lambda)}$ for $T \subset \Delta_n^+$. b satisfies $\langle bQ \rangle = \langle Q_\Delta \rangle - \langle Q \rangle$ and $|Q| = |bQ| - 2|(bQ) \cap Q_\Delta| + |Q_\Delta|$ for $Q \subset P_n^{(\Lambda)}$.*

In the following application we need only to know that b is onto. Namely, we can write $T = bQ_1$ for some $Q_1 \subset P_n^{(\Lambda)}$ with $\langle T \rangle = \langle Q_\Delta \rangle - \langle Q_1 \rangle$, and

$$(3.10) \quad |Q_1| = |T| - 2|T \cap Q_\Delta| + |Q_\Delta|.$$

By Lemma 2.2 we may also assume that $(P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}} \stackrel{(ii)}{=} \Phi_{\tau^{-1}w}^{(\Lambda)}$ for a suitable $(w, \tau) \in W \times W_k$. As $\mu = \Lambda + \langle T \rangle$ and $\mu = \mu(q) \stackrel{\text{def.}}{=} \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle = \Lambda + \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle$, we have $\langle T \rangle = \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle$. But also $\langle T \rangle = \langle Q_\Delta \rangle - \langle Q_1 \rangle = \delta_n + \delta_n^{(\Lambda)} - \langle Q_1 \rangle$ and hence $\langle Q_1 \rangle = 2\delta_n^{(\Lambda)} - \langle q_{u,n} \rangle = \langle P_n^{(\Lambda)} \rangle - \langle q_{u,n} \rangle \Rightarrow \langle Q_1 \cup \Phi_{\tau^{-1}} \rangle = \langle \Phi_{\tau^{-1}w}^{(\Lambda)} \rangle$ (by (ii)) $\Rightarrow Q_1 \cup \Phi_{\tau^{-1}} = \Phi_{\tau^{-1}w}^{(\Lambda)} = (P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}}$ (again by (ii)) $\Rightarrow Q_1 = P_n^{(\Lambda)} - q_{u,n}$. Finally since $T = bQ_1, T \cap Q_\Delta = Q_\Delta - Q_1$ by definition of b ; i.e. $T \cap Q_\Delta = Q_\Delta \cap q_{u,n}$ by (iii), so that $r = |T| = |Q_1| + 2|Q_\Delta \cap q_{u,n}| - |Q_\Delta|$ (by (3.10)) $= |Q'_\Delta| - |q_{u,n}| + 2|Q_\Delta \cap q_{u,n}|$, by (1.5) and (iii), which proves (3.8) and which proves the first half of Theorem 1.9. Conversely let $q = l + u \supset b_\Delta$ be a θ -stable parabolic subalgebra of g with $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$, let $\pi = \pi(q) = R_q^s \mathbf{C}_{\Delta+\delta-\delta^{(\Lambda)}}$, $s = \dim u \cap k$, and let $r_{q,\Delta} = |Q'_\Delta| - |q_{u,n}| + 2|Q_\Delta \cap q_{u,n}|$. As observed in Lemma 2.2, $\pi \in \hat{G}$ such that $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$ and for $\det \sigma (-1)^{|P_n^{(\Lambda)} - q_{u,n}|} = \pm 1, \mu = \mu(q)$ is a common K -type of $\pi|_K, S^\pm \otimes V_{\Delta+\delta_n}$ at least if $\mu - \delta_n^{(\Lambda)}$ is Δ_k^+ -dominant, which we assume. The point therefore is to show that μ is a K -type of $\Lambda^r P^+ \otimes V_\Delta$ for $r = r_{q,\Delta}$. Since V_μ occurs in $S^\pm \otimes V_{\Delta+\delta_n}, V_\mu$ occurs in $\Lambda^{n-j} P^+ \otimes V_\Delta$ for some j

with $(-1)^j = \pm 1$ (by (3.7)) according as $(-1)^{|P_n^{(\Lambda)} - q_{u,n}|} \det \sigma = \pm 1$. Write $\mu = \Lambda + \langle T \rangle$ where $T \subset \Delta_n^+$, $|T| = n - j$. Also write $T = bQ$ by Lemma 3.9, where $Q \subset P_n^{(\Lambda)}$, $|Q| = |T| - 2|Q_\Delta - Q| + |Q_\Delta|$ (since $bQ \cap Q_\Delta = Q_\Delta - Q$), and $\langle T \rangle = \langle Q_\Delta \rangle - \langle Q \rangle$. By the remarks following the proof of Lemma 2.2 we can write $P_n^{(\Lambda)} - q_{u,n} = \Phi_w^{(\Lambda)}$ for a suitable $w \in W$. Since $\mu = \mu(q) = \Lambda + \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle$ we have $\langle T \rangle = \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle = \delta_n + \delta_n^{(\Lambda)} - (2\delta_n^{(\Lambda)} - \langle q_{u,n} \rangle) = \langle Q_\Delta \rangle - \langle \Phi_w^{(\Lambda)} \rangle$; i.e. $\langle Q_\Delta \rangle - \langle \Phi_w^{(\Lambda)} \rangle = \langle Q_\Delta \rangle - \langle Q \rangle \Rightarrow \langle Q \rangle = \langle \Phi_w^{(\Lambda)} \rangle \Rightarrow Q = \Phi_w^{(\Lambda)} \Rightarrow$ (by (vii)) $n - |q_{u,n}| = |\Phi_w^{(\Lambda)}| = |T| - 2|Q_\Delta - \Phi_w^{(\Lambda)}| + |Q_\Delta| = n - j - 2|Q_\Delta \cap q_{u,n}| + |Q_\Delta| \Rightarrow n - j = 2|Q_\Delta \cap q_{u,n}| + |Q'_\Delta| - |q_{u,n}| = r_{q,\Delta}$; i.e. $V_\mu \subset \Lambda^{r_{q,\Delta}} P^+ \otimes V_\Delta$, as desired. By (2.7) $\dim \text{Hom}_K(H_{\mu(q)}, \Lambda^{r_{q,\Delta}} P^+ \otimes V_\Delta) = 1$, which completes the proof of Theorem 1.9.

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