LIE GROUPS AS FRAMED BOUNDARIES

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1. Introduction

Let G be a compact Lie group. We consider the problem of finding a parallelizable compact manifold V whose boundary ∂V is G. Our candidates are the disc bundles $D(\lambda)$ of the canonical (complex or quaternionic) line bundles λ over G/S, where S is a closed subgroup of G isomorphic to the group of unit complex numbers or quaternions. It turns out, however, that in many cases this family of bounding manifolds of G does not contain a parallelizable representative.

Some motivation is provided by the problem of identifying the elements represented in the stable homotopy groups of spheres by a Lie group with various framings; intuition suggests that not many elements will arise in this manner, but few explicit calculations exist. Some examples of stable homotopy classes which are not representable on Lie groups are noted in [9]. In [4], K. Knapp exhibited homotopy classes which lie outside the image of the bi-stable J-homomorphism J'. Such a class is not representable by a Lie group with an S^1-invariant framing, since elements obtained in this way are in the image of the S^1-transfer and hence in Im J'; see [4]. The last remark also shows that any Lie group is framed cobordant to the boundary of a parallelizable manifold. Our aim was to further clarify this situation by giving specific framed null cobordisms of Lie groups where possible.

We summarize briefly the contents of this paper. In §2 we show that if $D(\lambda)$ is as above and λ is stably trivial as a real vector bundle, then $D(\lambda)$ is parallelizable. In the next two sections we compute the Stiefel-Whitney and Pontrjagin classes of λ and give some examples of simple Lie groups and subgroups S producing parallelizable disc bundles $D(\lambda)$. These examples include the simply connected Lie groups $Spin(n)$, $SU(n)$, $Sp(n)$, the special orthogonal groups $SO(n)$, the projective orthogonal group $PSO(8)$, and groups covered by the special unitary groups $SU(p^k)$, where p is an odd prime. Finally, in §5, we give examples of framings of certain groups G which extend over some $D(\lambda)$. Some of the formulae in this section have appeared in [5] and [9].

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2. A family of bounding manifolds

We first establish some notation. As in §1, we let G be a compact Lie group. Let $i: S \rightarrow G$ be the inclusion of a closed subgroup S, isomorphic either to S^1 or to S^3. The sphere bundle $G \rightarrow G/S$ has an associated (complex or quaternionic) line bundle λ. The total space of the disc bundle $D(\lambda)$ will be denoted by V; that of the sphere bundle $S(\lambda)$ is G itself, so $\partial V = G$. In this section we address the problem of finding conditions under which V is parallelizable.

Let $\rho: S^1 \rightarrow U(1)$ be the standard complex representation of S^1. If no confusion is likely, we shall denote also by ρ its realification $S^1 \rightarrow SO(2)$ and their stabilizations $S^1 \rightarrow U$ and $S^1 \rightarrow SO$. Similarly, we use the standard representation $\sigma: S^3 \rightarrow SU(2)$ and its realification $S^3 \rightarrow SO(4)$.

If H is a closed subgroup of G, and ϑ is a (real or complex) representation of H, then the vector bundle associated via ϑ to a principal H-bundle $X \rightarrow X/H$ will be denoted by $\alpha(\vartheta)$. This defines well known homomorphisms

$$RO(H) \rightarrow KO(X/H) \quad \text{and} \quad RU(H) \rightarrow KU(X/H)$$

between Grothendieck rings of (real or complex) representations and vector bundles. The following elementary fact will be useful.

Lemma 2.1. *Let ϑ be a representation of H, and consider the vector bundle $\alpha(\vartheta)$ associated to $G \rightarrow G/H$. If there exists a virtual representation ϑ' of G such that the restriction of ϑ' to H is equivalent to ϑ, then $\alpha(\vartheta)$ is stably trivial.*

Proof. This follows (in the real case) from the commutative diagram below, where the homomorphism $RO(G) \rightarrow RO(H)$ is restriction.

$$
\begin{array}{ccccc}
RO(G) & \longrightarrow & RO(H) & \longrightarrow & RO(H) \\
\Big\downarrow{\alpha} & & \Big\downarrow{\alpha} & & \Big\downarrow{\alpha} \\
KO(BG) & \longrightarrow & KO(BH) & \longrightarrow & KO(G/H)
\end{array}
$$

The bottom line is induced from the fibration $G/H \rightarrow BH \rightarrow BG$, so it is zero. The complex case is similar.

Let S be a closed subgroup of G isomorphic to S^1 or to S^3. The adjoint

representation $\mathrm{Ad}(G)$ of G decomposes on restriction to S as

$$\mathrm{Ad}(G)|_S = \mathrm{Ad}(S) \oplus \mathrm{Ad}(G, S)$$

where $\mathrm{Ad}(G, S)$ has the property that $\alpha(\mathrm{Ad}(G, S))$ is isomorphic to the tangent bundle $\tau(G/S)$ of G/S; see [6]. Hence, if S is isomorphic to S^1, then G/S is stably parallelizable by 2.1. If S is isomorphic to S^3, the second exterior power $\Lambda^2(\sigma)$ and $\mathrm{Ad}(S) \oplus 3$ are isomorphic as real representations and by 2.1 we obtain that $\tau(G/S) \oplus \Lambda^2(\lambda)$ is stably trivial.

Let $p : V = D(\lambda) \rightarrow G/S$ be the disc bundle projection. Then

$$\tau(V) \cong p^*(\tau(G/S) \oplus \lambda)$$

and, as p is a homotopy equivalence, we obtain:

Proposition 2.2.
a)   *Let $S \cong S^1$. Then V is parallelizable if and only if λ is stably trivial as a real vector bundle.*
b)   *Let $S \cong S^3$. Then V is parallelizable if and only if $\lambda - \Lambda^2(\lambda)$ is stably trivial as a real vector bundle. In particular, if λ is itself stably trivial, then V is parallelizable.*

Observe also that, if S is isomorphic to $O(1)$, if G is connected, and if λ is the canonical real line bundle associated to the covering $G \rightarrow G/S$, then an argument similar to (a) above shows that $V = D(\lambda)$ is not orientable (and hence certainly not parallelizable).

3.   The case S isomorphic to S^1

Throughout this section S will denote a closed subgroup of G isomorphic to S^1, and λ the vector bundle over G/S associated with the principal S-bundle $G \rightarrow G/S$. According to 2.2, the disc bundle $D(\lambda) = V$ is parallelizable if and only if λ is stably trivial. We first observe that if S is a central subgroup of G isomorphic to S^1, then G is diffeomorphic to $(G/S) \times S$, and so $V \cong (G/S) \times D^2$ is parallelizable. Because of this, we shall assume G is semisimple from now on.

Proposition 3.1.   *Let G be a semisimple compact Lie group. Then the Stiefel-Whitney classes of λ vanish if and only if the class $[S]$ represented by S in $\pi_1(G)$ is not halvable.*

Proof. Except for w_1 and w_2, the Stiefel-Whitney classes of λ are zero for dimensional reasons, and, in fact, $w_1 = 0$ as λ is orientable. Consider the Gysin sequence in mod 2 cohomology for the bundle $G \rightarrow G/S$:

$$0 \rightarrow H^1(G/S) \rightarrow H^1(G) \rightarrow H^0(G/S) \rightarrow H^2(G/S) \rightarrow \cdots$$

where $H^1(G/S) \to H^1(G)$ is induced by the projection $p: G \to G/S$ and $H^0(G/S) \to H^2(G/S)$ is given by multiplication by $w_2(\lambda)$. From this, $w_2 = 0$ if and only if p^* is not surjective on H^1. The result now follows by observing that p_* is not injective on H_1 if and only if $[S]$ is not halvable.

Note that the condition "$[S]$ in $\pi_1(G)$ not halvable" implies that $[S]$ must have even order in $\pi_1(G)$. Hence, if G is simply connected or $\pi_1(G)$ is an odd torsion group, then a parallelizable bounding manifold may never be obtained by this method. Nevertheless, several series of classical groups do have 2-torsion in their fundamental groups, namely $SO(n)$, the semispinor groups $Ss(4n)$, the projective orthogonal groups $PSO(2n)$, the projective symplectic groups $PSp(n)$, and quotients of $SU(2n)$ by central subgroups of even order. Note also that λ is never trivial, otherwise $G \cong (G/S) \times S$ and $\pi_1(G)$ contains an infinite cyclic summand, which contradicts the semisimplicity of G. This suggests that the problem of deciding when λ is stably trivial is not as straightforward as may at first appear. In fact, the computation of the first Pontrjagin class $p_1(\lambda)$ will require a little more work.

To compute $p_1(\lambda)$ we shall fix a maximal torus of G and consider only circle subgroups of G lying in this maximal torus. So assume, as before, that G is a compact connected semisimple Lie group, say of rank m. Let T be a maximal torus of G and select coordinates on T, i.e., an isomorphism of Lie groups $T \cong S^1 \times \cdots \times S^1$. This is equivalent to selecting polynomial generators $x_1, \cdots, x_m$ for $H^*(BT)$. A circle subgroup S of T is determined by degrees $d_1, \cdots, d_m$ such that

$$ S = \{(\exp(2\pi\sqrt{-1}\, d_1 t), \cdots, \exp(2\pi\sqrt{-1}\, d_m t)): t \in \mathbf{R}\} $$

where the greatest common divisor of $d_1, \cdots, d_m$ is one, and S determines these degrees up to a uniform change of sign. Our results on $p_1(\lambda)$ are as follows.

Theorem 3.2. *The class $p_1(\lambda)$ is a torsion element of $H^4(G/S)$ and its order may be computed as follows. Assume that G is simple, so that the third Betti number of G is one. Then the subgroup $H^4(BT)^W$ of elements of $H^4(BT)$ invariant under the action of the Weyl group of G is infinite cyclic. Let $g(x_1, \cdots, x_m)$ generate $H^4(BT)^W$. Then the order of $p_1(\lambda)$ is the absolute value of $g(d_1, \cdots, d_m)$.*

Theorem 3.3. *Let ϑ be a complex representation of T invariant under the Weyl group, and let $\alpha(\vartheta)$ be the corresponding vector bundle on BT. We may define an integer $a(\vartheta)$ by $c_2(\alpha(\vartheta)) = a(\vartheta)\, g(x_1, \cdots, x_m)$.*

a) *Let p be an odd prime that does not divide $a(\vartheta)\, g(d_1, \cdots, d_m)$. Then λ is stably trivial at p, i.e., $\lambda - 2 = 0$ in $\mathbf{Z}_{(p)} \otimes \widetilde{KO}(G/S)$. In particular, if $p_1(\lambda) = 0$ and ϑ can be found with $a(\vartheta) = 1$ or -1, then the order of $\lambda - 2$ in $\widetilde{KO}(G/S)$ is a power of 2.*

b) *If ϑ is the sum of another representation and its conjugate, and $a(\vartheta)\, g(d_1, \cdots, d_m)$*

is odd, then 2λ is stably trivial at 2, i.e., $2(\lambda-2)=0$ in $\mathbf{Z}_{(2)}\otimes\tilde{K}O(G/S)$. In particular, if $p_1(\lambda)=0$ and such a ϑ can be found with $a(\vartheta)=1$ or -1, then $2(\lambda-2)=0$ in $\tilde{K}O(G/S)$.

To prove 3.2 we need the following technical lemmas.

Lemma 3.4. *Let $p: E \to B$ be a fibration with connected fibre F and simply connected base. Then, for any coefficient group π,*

a) *if $H^i(B; H^j(F; \pi))=0$ when $0<i\leq r$ or $0<j\leq s$ then there is an exact sequence*

$$\to H^{r+s}(F; \pi) \to H^{r+s+1}(B; \pi) \to H^{r+s+1}(E; \pi) \to H^{r+s+1}(F; \pi),$$

and

b) *if $H^i(B; H^j(F; \pi))=0$ when $0<i<r$ or $0<j\leq s$, and $H^{r+s}(F; \pi)=0$, then $p^*: H^{r+s+1}(B; \pi) \to H^{r+s+1}(E; \pi)$ is a monomorphism.*

Part (a) of 3.4 is of course just the classical Serre exact sequence of a fibration, and part (b) is a slight generalization of it.

Lemma 3.5. *Let G be a compact, connected (not necessarily semisimple) Lie group and T a maximal torus of G. Then, in integral cohomology, the map $BT \to BG$ induces an isomorphism*

$$H^4(BG) \to H^4(BT)^W.$$

Proof. By 3.4 (b), taking $r=2$ and $s=1$, we have that $H^4(BG; \pi) \to H^4(BT; \pi)$ is a monomorphism for every π, since the fibre G/T can be given a cell decomposition consisting only of cells of even dimension. Now consider the diagram

$$
\begin{array}{ccccc}
H^4(BG; \mathbf{Z}) & \longrightarrow & H^4(BG; \mathbf{Q}) & \longrightarrow & H^4(BG; \mathbf{Q}/\mathbf{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow H^4(BT; \mathbf{Z}) & \longrightarrow & H^4(BT; \mathbf{Q}) & \longrightarrow & H^4(BT; \mathbf{Q}/\mathbf{Z})
\end{array}
$$

By a classical result of Borel, the middle vertical arrow is an isomorphism onto the group $H^4(BT; \mathbf{Q})^W$ of invariants. With this in mind, a simple diagram chasing argument ends the proof of 3.5.

Note that the same proof applies to show that $H^2(BT)^W$ is isomorphic to $H^2(BG)$.

Proof of 3.2. Recall that a model for BS^1 is $\mathbf{CP}$, the infinite dimensional complex projective space. We have a fibration

$$G/S \xrightarrow{f} \mathbf{CP} \xrightarrow{Bj} BG$$

where f classifies λ and j is the inclusion of S in G. By 3.4(a) and 3.5 we have a diagram with exact row where i is the inclusion of S in T.

$$H^4(BT)^W$$

$$\cong \nearrow \qquad \downarrow (Bi)^*$$

$$H^4(BG) \xrightarrow{(Bj)^*} H^4(\boldsymbol{CP}) \xrightarrow{f^*} H^4(G/S)$$

Here we use the assumption that G is semisimple, to satisfy the hypothesis of 3.4(a). Since we are assuming that G is simple, the third Betti number of G is one, and since $H^4(BG) \cong H^3(G)$ rationally, the group $H^4(BT)^W$ of invariants is infinite cyclic. Let ξ be the universal complex line bundle on $\boldsymbol{CP}$, and let $x=c_1(\xi)$. Then $H^*(\boldsymbol{CP}) \cong \boldsymbol{Z}[x]$ and $p_1(\xi)=x^2$, and so $p_1(\lambda)=f^*p_1(\xi)=f^* x^2$, so that the order of $p_1(\lambda)$ is the index of $\ker f^*$ in $H^4(\boldsymbol{CP})$. By the diagram, $\ker f^*$ equals $\mathrm{im}(Bi)^*$, and this we can compute, as follows. Since $i: S^1 \to T$ is given by

$$i(\exp(2\pi\sqrt{-1}\,t)) = (\exp(2\pi\sqrt{-1}\,d_1\,t), \cdots, \exp(2\pi\sqrt{-1}\,d_m\,t)) ,$$

we have $(Bi)^*\,x_j=d_j\,x$, and so

$$(Bi)^*\,g(x_1, \cdots, x_m) = g(d_1x, \cdots, d_mx) = g(d_1, \cdots, d_m)\,x^2 ,$$

which proves 3.2.

Next we state a technical lemma that we need to prove 3.3. Let $u=\xi-1$. Then $K(\boldsymbol{CP}) \cong \boldsymbol{Z}[[u]]$, and if $r: K(\boldsymbol{CP}) \to KO(\boldsymbol{CP})$ is the map that forgets the complex structure of a bundle, then $KO(\boldsymbol{CP}) \cong \boldsymbol{Z}[[r(u)]]$. Let s_2 be the second Newton polynomial in the (universal) Chern classes, so that $s_2=c_1^2-2c_2$, and $p_1(\Theta)=s_2(\Theta)$ if Θ is a complex bundle.

Lemma 3.6. *If $\Theta \in \tilde{K}(\boldsymbol{CP})$ then $r(\Theta) \equiv \langle s_2(\Theta), [\boldsymbol{CP}^2]\rangle r(u)$ modulo $r(u)^2$.*

Proof. Note first that $r(u^2)=r(u)^2+2r(u)$. Indeed, if E is a complex vector bundle and F is a real vector bundle, there is a natural isomorphism of $r(E \otimes_c (C \otimes_R F))$ with $r(E) \otimes_R F$. Taking $E=\xi$ and $F=r(\xi)$ we obtain $r(\xi)^2= r(\xi^2 \oplus 1)$, from which our assertion follows. Write

$$\Theta = a_1\,u+a_2\,u^2+a_3\,u^3+\cdots$$

Then $r(\Theta)=(a_1+2a_2)\,r(u)+a_2\,r(u)^2+r(a_3\,u^3+\cdots)$. It is clear from the Atiyah-Hirzebruch spectral sequence for $KO(\boldsymbol{CP})$ that all elements of filtration six or greater are multiples of $r(u)^2$. Since for $i \geq 3$ the filtration of u^i is at least six, we have

$$r(\Theta) \equiv (a_1+2a_2)\,r(u) \quad \text{modulo} \quad r(u)^2 .$$

To end the proof note that $s_2(u)=x^2$ and $s_2(u^2)=2x^2$, while $s_2(u^i)=0$ if $i\geqq3$. This means that $s_2(\Theta)=(a_1+2a_2)\,x^2$, and this proves 3.6.

Proof of 3.3. Consider the diagram

$$
\begin{array}{ccc}
R(T) & \longleftarrow & R(G) \\
\alpha\downarrow & & \downarrow \\
K(BT) & \longleftarrow & K(BG) \\
(Bi)^*\downarrow & & \downarrow \\
K(\boldsymbol{CP}) & \longleftarrow & K(BG) \\
r\downarrow & & (Bj)^*\downarrow \\
KO(G/S) \xleftarrow{\;f^*\;} KO(\boldsymbol{CP}) & \xleftarrow{(Bj)^*} & KO(BG)
\end{array}
$$

Let $\Theta=(Bi)^*\,\alpha(\vartheta)$-dim $\alpha(\vartheta)$ in $\tilde{K}(\boldsymbol{CP})$. Since the image of $R(G)$ is the subgroup of invariants $R(T)^W$, and since $f^*(Bj)^*$ vanishes on $\tilde{K}O(BG)$, it is clear from the diagram that $f^*\,r(\Theta)=0$. We have

$$
\begin{aligned}
s_2(\Theta) &= (Bi)^*s_2\,\alpha(\vartheta) = -2(Bi)^*c_2\,\alpha(\vartheta) \\
&= -2a(\vartheta)\,g(d_1,\cdots,d_m)\,x^2
\end{aligned}
$$

where we have noted that $c_1\alpha(\vartheta)=0$; this is because $c_1\alpha(\vartheta)$ lies in $H^2(BT)^W$, which is isomorphic to $H^2(BG)$, which is trivial since G is semisimple. Thus, by 3.6 we have

$$(3.7)\qquad r(\Theta) = r(u)\,(-2a(\vartheta)\,g(d_1,\cdots,d_m)+Ar(u))$$

where $A\in KO(\boldsymbol{CP})$. To prove part (a) of 3.3, if p does not divide $2a(\vartheta)\,g(d_1,\cdots,d_m)$, then the second factor of the right hand side of 3.7 is invertible in $KO(\boldsymbol{CP})$ after localization at p. Then, from $f^*r(\Theta)=0$, we conclude that $f^*r(u)=0$, and this is the content of the statement of 3.3(a), since f classifies λ. The proof of part (b) is identical, except that in this case the element A in 3.7 is even, so that 3.7 may be written

$$r(\Theta) = 2r(u)\,(-a(\vartheta)\,g(d_1,\cdots,d_m)+(A/2)\,r(u))\,,$$

so we conclude $2f^*r(u)=0$ at the prime 2 when $a(\vartheta)\,g(d_1,\cdots,d_m)$ is odd.

We close this section with explicit calculations of $p_1(\lambda)$ and $w_2(\lambda)$ for the classical groups. We saw in 3.1 that $w_2(\lambda)$ is nonzero if and only if $[S]$ is halvable in $\pi_1(G)$. To decide whether or not $[S]$ is halvable we shall make use of the following elementary lemma.

Lemma 3.8. *Let G be a compact, connected semisimple Lie group. Let T be a maximal torus of G and $\tilde{T}$ the restriction to T of the universal cover $\tilde{G}$ of G. Then $\tilde{T}$ is a maximal torus of $\tilde{G}$ and*

$$0 \to \pi_1(\tilde{T}) \to \pi_1(T) \to \pi_1(G) \to 0$$

is an exact sequence.

Our calculations for the classical groups follow.

Example 3.9. The groups $Sp(m)$ and $PSp(m)$.

As a maximal torus for $Sp(m)$ choose the set T_{Sp} of diagonal matrices with complex entries. The diagonal entries give coordinates $x_1, \cdots, x_m$ for T_{Sp}, which we identify with polynomial generators for $H^*(BT)$ as usual, and the Weyl group acts on these generators by permutations and sign changes. Then $x_1^2 + \cdots + x_m^2$ lies in $H^4(BT_{Sp})^W$ and clearly it must be a generator. By 3.2 the order of $p_1(\lambda)$ for a circle subgroup S of T_{Sp} given by degrees $d_1, \cdots, d_m$ is $d_1^2 + \cdots + d_m^2$. Thus $p_1(\lambda)$ can vanish only when d_i is 1 or -1 for some i and d_j is zero for all other indices j, i.e., when S is one of the natural circle factors of T_{Sp}. To apply 3.3 let ϑ be the representation $(z_1, z_1^{-1}, \cdots, z_m, z_m^{-1})$ of T_{Sp}. Then $\alpha(\vartheta)$ is the bundle $H_1 \oplus H_1^{-1} \oplus \cdots \oplus H_m \oplus H_m^{-1}$, where H_j is the complex line bundle on BT with first Chern class x_j, and we have $c_2(\alpha(\vartheta)) = -(x_1^2 + \cdots + x_m^2)$ so that $a(\vartheta) = -1$. Thus for the group $Sp(m)$ the vanishing of $p_1(\lambda)$ implies $2(\lambda - 2) = 0$ in $\tilde{K}O(Sp(m)/S)$, and since $w_2(\lambda) \neq 0$, because $Sp(m)$ is simply connected, the order of $\lambda - 2$ in this case is exactly 2.

The centre Z of $Sp(m)$ is cyclic of order two, generated by minus the identity matrix; the projective symplectic group $PSp(m)$ is the quotient $Sp(m)/Z$. Let $T_{PSp} = T_{Sp}/Z$. This is a maximal torus of $PSp(m)$. Since Z acts on T_{Sp} by multiplying each diagonal entry by minus one, the homomorphism of T_{Sp} into a product of m copies of S^1 defined by sending $(z_1, \cdots, z_m)$ to $(z_1^2, z_2 z_1^{-1}, \cdots, z_m z_1^{-1})$ gives an induced isomorphism of T_{PSp} with $S^1 \times \cdots \times S^1$, which determines generators $y_1, \cdots, y_m$ of $H^2(BT_{PSp})$. Under the homomorphism of cohomology groups induced by $BT_{Sp} \to BT_{PSp}$, the generator y_1 goes to $2x_1$, while y_i goes to $x_i - x_1$ for $i > 1$. Then $y_1 + 2y_i$ goes to $2x_i$, if $i > 1$, and so $y_1^2 + (y_1 + 2y_2)^2 + \cdots + (y_1 + 2y_m)^2$ lies in $H^4(BT_{PSp})^W$. Call this element Y. Then

$$Y = my_1^2 + 4(y_1 y_2 + y_2^2) + \cdots + 4(y_1 y_m + y_m^2)$$

so we see that $H^4(BT_{PSp})^W$ is generated by Y if m is odd, by $(1/2)Y$ if $m \equiv 2$ mod 4, and by $(1/4)Y$ if $m \equiv 0$ mod 4. Then for a circle subgroup of T_{PSp} given by degrees $d_1, \cdots, d_m$, the order of $p_1(\lambda)$ is $Y(d_1, \cdots, d_m)$ if m is odd, $(1/2)Y(d_1, \cdots, d_m)$ if $m \equiv 2$ mod 4, and $(1/4)Y(d_1, \cdots, d_m)$ if $m \equiv 0$ mod 4. So, if m is odd, then $p_1(\lambda)$ vanishes only when $m = 1$ and $d_1 = 1$ or -1. If $m \equiv 2$ mod 4, then $p_1(\lambda)$ vanishes only in the following cases: when $m = 2$ and (d_1, d_2) is one of $(1, 0)$, $(-1, 0)$, $(1, -1)$ or $(-1, 1)$. If $m \equiv 0$ mod 4, then $p_1(\lambda)$ vanishes only when $d_1 = 2$ or -2 and $d_i = -(1/2)d_1$ for $i > 1$; when $d_1 = 0$ and $d_i = 1$ or -1 for some $i > 1$ and $d_j = 0$ for all other indices j; and when $m = 4$ and $(d_1, \cdots, d_4)$ is such that

$d_1^2=1$ and $(d_1+2d_i)^2=1$ for $i=2, 3, 4$. To compute $w_2(\lambda)$ we use 3.8. It is trivial to compute $\pi_1(T_{Sp}) \to \pi_1(T_{PSp})$ and conclude that the first generator y_1 of $\pi_1(T_{PSp})$ goes to the generator of $\pi_1(PSp(m)) \cong Z_2$, while other generators y_i go to zero. Since $[S]=d_1 y_1+\cdots+d_m y_m$ in $\pi_1(T_{PSp})$ we see that $[S]$ is halvable in $\pi_1(PSp(m))$ if and only if d_1 is even. Thus $w_2(\lambda)=0$ if and only if d_1 is odd, by 3.1. So we see that there are only very few examples of circles in a maximal torus of $PSp(m)$ with both $p_1(\lambda)=0$ and $w_2(\lambda)=0$, namely, one circle when $m=1$, two circles when $m=2$, and eight circles when $m=4$. We can apply 3.3 with ϑ the representation

$$\left(z_1 z_1^{-1},\ z_1 z_2^2,\ z_1^{-1} z_2^{-2},\ \cdots,\ z_1 z_m^2,\ z_1^{-1} z_m^{-2}\right).$$

Then $\alpha(\vartheta)$ is $L_1 \oplus L_1^{-1} \oplus \cdots \oplus L_m \oplus L_m^{-1}$, where L_1 is the complex line bundle on BT with first Chern class y_1 and L_i is the line bundle with first Chern class y_1+2y_i, for $i>1$. Then $c_2(\alpha(\vartheta))=-Y$, so we obtain $a(\vartheta)=-1$, -2 or -4, according to whether m is odd, two mod 4 or zero mod 4. Now, $PSp(1) \cong SO(3)$ and $PSp(2) \cong SO(5)$, and the relevant circles are the naturally embedded copies of $SO(2)$. By 2.1, with the standard representations of $SO(3)$ and $SO(5)$, we have of $\lambda-2=0$. In the remaining case, theorem 3.3 shows that the order of $\lambda-2$ in $\tilde{K}O(PSp(4)/S)$ is a power of 2.

EXAMPLE 3.10.   The groups $SO(n)$, $PSO(2m)$, $Spin(n)$ and $Ss(2m)$.

We exclude from consideration the group $SO(2)$, which is not semisimple, and the groups covered by $Spin(4)$ which are not simple; the reader should have no difficulty dealing with these groups.

Consider $SO(n)$ with $n=2m$ or $2m+1$. The rank of $SO(n)$ is m and the standard maximal torus T_{SO} consists of all matrices formed with two by two blocks, each of them in $SO(2)$, along the diagonal (together with a 1 in the (n, n) entry when $n=2m+1$). This gives coordinates $x_1, \cdots, x_m$ for T_{SO}, and the Weyl group acts on these coordinates by permutations and sign changes, as in the case of T_{Sp}, but with the condition that there be an even number of sign changes when $n=2m$. Then the formulae given by 3.2 and the conclusions derived from them and from 3.3 are the same for $SO(2m)$ and $SO(2m+1)$ as for $Sp(m)$. Further details about $p_1(\lambda)$ are unnecessary for the case of $SO(n)$, except to note that if $p_1(\lambda)=0$, then the corresponding circle subgroup is just a natural factor in T_{SO}, and by 2.1 we obtain that λ is stably trivial.

The centre Z of $SO(2m)$ is cyclic of order two, generated by minus the identity matrix; the projective orthogonal group $PSO(2m)$ is $SO(2m)/Z$. The calculations of $p_1(\lambda)$ for $PSO(2m)$ are identical with those for $PSp(m)$.

The calculation of $w_2(\lambda)$, however, is not the same for $SO(n)$ and $PSO(2m)$ as for $Sp(m)$ and $PSp(m)$, since $SO(n)$ is not simply connected. We shall describe in the next paragraph the double covering of T_{SO} by T_{Spin}, i.e., the restriction

to T_{SO} of the covering of $SO(n)$ by $Spin(n)$. With this description in hand it is easy to compute with 3.8 and show that, for a circle given by degrees $d_1, \cdots, d_m$, the class $w_2(\lambda)$ is nonzero if and only if $d_1 + \cdots + d_m$ is even, in the case of $SO(n)$, while, in the case of $PSO(2m)$, if m is even then $w_2(\lambda)$ is nonzero if and only if both d_1 and $d_2 + \cdots + d_m$ are even, and if m is odd then $w_2(\lambda)$ is nonzero if and only if d_1 is even. From all this we can conclude that only in the case $PSO(8k)$ can one have circles with $p_1(\lambda)$ and $w_2(\lambda)$ both zero. In fact, for $k > 1$, such circles lift to a natural factor of T_{SO} and, using an Atiyah transfer argument, one can show that $2(\lambda - 2) = 0$ in $\widetilde{KO}(PSO(8k)/S)$. Finally we remark that since $PSO(8)$ is diffeomorphic to $\mathbf{RP}^7 \times SO(7)$, and the inclusion of $SO(7)$ into $PSO(8)$ obtained from this diffeomorphism is a homomorphism, one then has circles in $PSO(8)$ coming from $SO(7)$ for which λ is stably trivial.

Let T_{Spin} be the inverse image under the double covering $Spin(n) \to SO(n)$ of the standard maximal torus T_{SO}. Then T_{Spin} is of course a maximal torus of $Spin(n)$, and it can be given coordinates in such a way that $T_{Spin} \to T_{SO}$ sends $(z_1, \cdots, z_m)$ to $(z_1^2 z_2 \cdots z_m, z_2, \cdots, z_m)$; see, for example, [7]. Let $u_1, \cdots, u_m$ be the corresponding generators of $H^2(BT_{Spin})$. Then the generator $x_1^2 + \cdots + x_m^2$ of $H^4(BT_{SO})^W$ goes to the element $(2u_1 + u_2 + \cdots + u_m)^2 + u_2^2 + \cdots + u_m^2$ which is therefore in $H^4(BT_{Spin})^W$. Thus, for a circle in T_{Spin} given by degrees $d_1, \cdots, d_m$, the order of $p_1(\lambda)$ is $(1/2)\,((2d_1 + d_2 + \cdots + d_m)^2 + d_2^2 + \cdots + d_m^2)$. Of course $w_2(\lambda)$ is nonzero, since $Spin(n)$ is simply connected.

Finally we consider the semispinor groups $Ss(2m)$, defined when m is even. If m is even, the element ω in T_{Spin} with coordinates $((-1)^{1+m/2}, -1, \cdots, -1)$ lies in the centre of $Spin(2m)$ and $Ss(2m)$ is the quotient of $Spin(2m)$ by the cyclic subgroup of order two generated by ω. We take $T_{Ss} = T_{Spin}/(\omega)$ as maximal torus for $Ss(2m)$. Then the homomorphism of T_{Spin} into a product of copies of S^1 that sends $(z_1, \cdots, z_m)$ to $(z_1^2, z_2 z_1^{-1}, \cdots, z_m z_1^{-1})$ if $m/2$ is even, and to $(z_1, z_2^2, z_3 z_2^{-1}, \cdots, z_m z_2^{-1})$ if $m/2$ is odd, induces coordinates on T_{Ss}. The remaining details of the calculation of the order of $p_1(\lambda)$ for circles in T_{Ss} are analogous to those in the case of $PSp(m)$. We leave this and the calculation of $w_2(\lambda)$ to the reader. The conclusion is that only $Ss(8)$ admits circles with both $w_2(\lambda)$ and $p_1(\lambda)$ zero. However, since $Ss(8)$ is isomorphic to $SO(8)$, under the triality automorphism of $Spin(8)$, this case has already been covered.

EXAMPLE 3.11. The group $SU(n)$ and its quotient groups.

Take as maximal torus of the special unitary group $SU(n)$ the set T_{SU} of n by n unitary diagonal matrices with determinant one. Then T_{SU} has coordinates $x_1, \cdots, x_n$ such that $x_1 + \cdots + x_n = 0$, i.e., we view $H^*(BT_{SU})$ as the polynomial ring with generators $x_1, \cdots, x_n$ and the one relation $x_1 + \cdots + x_n = 0$. The rank of $SU(n)$ is $n - 1$. The Weyl group permutes $x_1, \cdots, x_n$, so the second elementary symmetric polynomial, which coincides with minus $(1/2)\,(x_1^2 + \cdots + x_n^2)$

because $x_1+\cdots+x_n=0$, generates $H^4(BT_{SU})^W$. Thus, for a circle subgroup of T_{SU} given by degrees $d_1, \cdots, d_n$ such that $d_1+\cdots+d_n=0$ the order of $p_1(\lambda)$ is $(1/2)(d_1^2+\cdots+d_n^2)$.

The centre of $SU(n)$ is cyclic of order n; it consists of all diagonal matrices $(\omega, \cdots, \omega)$, where ω is an n-th root of unity. Let k divide n and let W be the central subgroup of order k. We consider the group $SU(n)/W$. Let $T_{SU/W}=T_{SU}/W$. The homomorphism of a product P of n copies of S^1 into itself that sends $(z_1, \cdots, z_n)$ to $(z_1^k, z_2\,z_1^{-1}, \cdots, z_n\,z_1^{-1})$ induces an inclusion of $T_{SU/W}$ in P as the subgroup of all elements $(w_1, \cdots, w_n)$ such that $w_1^{n/k}\,w_2\cdots w_n=1$. This gives us coordinates for $T_{SU/W}$; we view $H^*(BT_{SU/W})$ as the polynomial ring with generators $y_1, \cdots, y_n$ and the one relation $(n/k)\,y_1+y_2+\cdots+y_n=0$. The homomorphism of cohomology rings induced by $BT_{SU}\to BT_{SU/W}$ sends y_1 to kx_1, and y_i to x_i-x_1 if $i>1$. From this, it is quite straightforward to show that for a circle subgroup of $T_{SU/W}$ determined by degrees $d_1, \cdots, d_n$ such that $(n/k)d_1+d_2+\cdots+d_n=0$ the order of $p_1(\lambda)$ is $(1/2a)(d_1^2+(d_1+kd_2)^2+\cdots+(d_1+kd_n)^2)$, where a is the greatest common divisor of $n(n-1)/2$ and k^2; we leave these details and the computation of $w_2(\lambda)$ to the interested reader. We just point out that only $SU(2)/\boldsymbol{Z_2}$ and $SU(4)/\boldsymbol{Z_2}$ admit circles with $w_2(\lambda)$ and $p_1(\lambda)$ both zero. However these groups are isomorphic to $SO(3)$ and $SO(6)$, respectively.

4. The case S isomorphic to S^3

In this section we shall seek subgroups S isomorphic to S^3 such that the 4-dimensional real vector bundle λ associated to $G\to G/S$ is stably trivial. By 2.2 such a λ produces a parallelizable disc bundle bounding G.

Proposition 4.1. *Let G be a compact connected simple classical Lie group. Assume further that either G is simply connected or G is not of any of the following types: $PSO(8n+4)$, $Ss(8n+4)$, $PSp(2n+1)$ or a quotient of $SU(4n+2)$ by a central subgroup of even order. Then the Stiefel-Whitney and Pontrjagin classes of λ vanish if and only if the class represented by $[S]$ is a generator of $\pi_3(G)$.*

Proof. Assume first that G is simply connected. Then $H^4(G/S)\cong\pi_3(G/S)\cong\pi_3(G)/\mathrm{im}(i_*)$, where i_* is the homomorphism induced on π_3 by the inclusion $i\colon S\to G$. The Gysin sequence (with integral coefficients)

$$0 \to H^3(G) \to H^0(G/S) \to H^4(G/S) \to 0$$
$$ 1 \quad\longmapsto\quad c_2(\lambda)$$

shows that $c_2(\lambda)$ generates $H^4(G/S)$. Thus, if $[S]$ generates $\pi_3(G)$, all Chern classes of λ vanish, implying that all the Pontrjagin and Stiefel classes of λ also vanish. Conversely, if the Stiefel-Whitney and Pontrjagin classes of λ are trivial, then $p_1(\lambda)=-2c_2(\lambda)=0$ and $w_4(\lambda)=0$. Since $w_4(\lambda)$ is the mod 2 reduction

of $c_2(\lambda)$, it follows that $c_2(\lambda)=0$. Hence, i_* is surjective on π_3 and $[S]$ is a generator.

If G is not simply connected we shall denote by $\tilde{G}$ its universal cover. The inclusion $i\colon S\to G$ lifts to an inclusion $\tilde{i}\colon S\to\tilde{G}$ which is also a homomorphism and the vector bundle over $\tilde{G}/S$ associated to $\tilde{i}$ is the pull-back of λ. If the Stiefel-Whitney and Pontrjagin classes of λ vanish, then by the previous case, $[S]$ generates $\pi_3(\tilde{G})\cong\pi_3(G)$. If, on the other hand, $[S]$ generates $\pi_3(G)$ the assertion is easily deduced from the Gysin sequence

$$H^3(G)\to H^0(G/S)\to H^4(G/S)\to H^4(G)\to 0$$
$$1\quad\mapsto c_2(\lambda)$$

and from the table below. There are two cases to be considered. If G is isomorphic to $SO(n)$, $PSO(8n)$, $Ss(8n)$, $PSp(2n)$ or to a quotient of $SU(n)$ by a central subgroup of order m, with either m odd or n/m even, then $H^4(G/S)\cong H^4(G)$ and therefore $c_2(\lambda)=0$. If G is isomorphic to $PSO(4n+2)$, then $c_2(\lambda)\in H^4(G/S)=\mathbf{Z}_4$ is the element of order 2, showing that $w_4(\lambda)$ and $p_1(\lambda)$ both are zero. To see that $p_2(\lambda)=c_2(\lambda)^2=0$, write $c_2(\lambda)=2x$. Then $c_2(\lambda)^2=4x^2=(4x)x=0$. The case of quotients of $SU(4n)$ by central subgroups of order m, with $4n/m$ odd, is analogous.

The following table lists all compact connected simple non-simply connected classical Lie groups, except for $SO(3)$. Hore G/S denotes the quotient of G by a subgroup S isomorphic to S^3 with $[S]$ generating $\pi_3(G)$. To obtain $H^4(G/S)$, note that the covering $\tilde{G}/S\to G/S$ is classified by a map $G/S\to B(\pi_1 G)$ which is a 3-equivalence. As $H^4(G/S)$ is a torsion subgroup, we get $H^4(G/S)\cong H_3(G/S)\cong H_3(B\pi_1 G)$. To obtain $H^4(G)$ the results of [2] and [3] may be used.

G	$H^4(G/S)$	$H^4(G)$
$SO(n)$, $n\geqq 5$.	$\mathbf{Z}_2$	$\mathbf{Z}_2$
$PSO(8n)$, $n\geqq 1$.	$\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2$
$PSO(8n+4)$, $n\geqq 1$.	$\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$
$PSO(4n+2)$, $n\geqq 1$.	$\mathbf{Z}_4$	$\mathbf{Z}_2$
$Ss(8n)$, $n\geqq 1$.	$\mathbf{Z}_2$	$\mathbf{Z}_2$
$Ss(8n+4)$, $n\geqq 1$.	$\mathbf{Z}_2$	0
$PSp(2n)$, $n\geqq 1$.	$\mathbf{Z}_2$	$\mathbf{Z}_2$
$PSp(2n+1)$, $n\geqq 1$.	$\mathbf{Z}_2$	0
$SU(n)/\mathbf{Z}_m$, $m>1$, $n>2$.		
i) m odd or n/m even.	$\mathbf{Z}_m$	$\mathbf{Z}_m$
ii) n/m odd, $m=2k$.	$\mathbf{Z}_{2k}$	$\mathbf{Z}_k$

Observe that the arguments of 4.1 imply:

Proposition 4.2. *Let G be a compact connected simple Lie group which is isomorphic to $PSO(8n+4)$, $Ss(8n+4)$, $PSp(2n+1)$ or $SU(4n+2)/A$, where A is a central subgroup of even order. Then, for any subgroup S isomorphic to S^3, at least one of $w_4(\lambda)$ or $p_1(\lambda)$ is not zero.*

According to 2.2(b), the stable triviality of λ may not be necessary for the parallelizability of the disc bundle V; rather, one must deal with $\lambda - \Lambda^2(\lambda)$. But since $w_4(\lambda - \Lambda^2(\lambda)) = w_4(\lambda)$ and $p_1(\lambda - \Lambda^2(\lambda)) = -p_1(\lambda)$ one obtains:

Corollary 4.3.

a) *Let G be as in 4.1 and let S be a subgroup isomorphic to S^3. If V is parallelizable, then $[S]$ generates $\pi_3(G)$.*

b) *Let G be as in 4.2. Then V is not parallelizable for any subgroup S isomorphic to S^3.*

Next we give some examples of simple groups G and subgroups S isomorphic to S^3 producing parallelizable disc bundles. By 4.3(a), we must consider subgroups S generating $\pi_3(G)$. Some of these subgroups can be described as follows. For $SU(n)$ and $Sp(n)$ one may take $S = SU(2)$ and $S = Sp(1)$, respectively. For $SO(n)$ we may consider the representation $\sigma: SU(2) \to SO(4) \subseteq SO(n)$. This representation factors through a representation $\tilde{\sigma}: SU(2) \to Spin(4) \subseteq Spin(n)$. Note that the standard inclusion $Spin(3) \subseteq Spin(n)$ represents twice a generator of $\pi_3(Spin(n))$, so we disregard this.

Theorem 4.4. *Let G be isomorphic to $SU(n)$ with $n \geq 2$, to $Sp(n)$ with $n \geq 1$, or to $SO(n)$ or $Spin(n)$ with $n \geq 4$, and let S be the subgroup isomorphic to S^3 described above. Then the 4-dimensional real vector bundle λ over G/S is stably trivial. Hence, the associated disc bundle $D(\lambda)$ is parallelizable.*

Proof. This follows from 2.1 using the (realifications of the) standard representations of these classical groups.

We shall finally consider groups G whose universal cover is $SU(p^r)$, where p is an odd prime. Let S be the image of $SU(2)$ under the covering projection. By 4.1 we have $c_2(\lambda) = 0$ and hence the homomorphism $q^*: H^*(G/S) \to H^*(G)$ induced by the projection $q: G \to G/S$ is injective. First we take G to be to the projective unitary group $PSU(p^r)$. In [8] it is shown that the Atiyah-Hirzebruch spectral sequence for $K(PSU(p^r))$ collapses and that the extensions are trivial. Therefore, the projection also induces a monomorphism $q^*: K^*(PSU(p^r)/S) \to K^*(PSU(p^r))$. But $q^*(\lambda)$ is trivial, being the vector bundle associated to the trivial S-bundle over $PSU(p^r)$. Hence λ is stably trivial as a complex bundle. Next, if G is any group covered by $SU(p^r)$, there is then a covering projection $G/S \to PSU(p^r)/S$ and the vector bundle over G/S is the pull-back of the vector bundle over $PSU(p^r)/S$. Therefore, we obtain the following result.

Theorem 4.5. *Let G be a Lie group with universal cover $SU(p^r)$, let p be an odd prime, and let S be the image of $SU(2)$ under the covering projection. Then the vector bundle λ over G/S is stably trivial as a complex bundle and therefore the disc bundle $V=D(\lambda)$ is parallelizable.*

5. Explicit formulae

Having established examples of parallelizable manifolds with $\partial V=G$, we shall describe some specific framings of G which extend over V, that is, framings which via the Pontrjagin-Thom construction represent zero in the stable homotopy group $\pi^s_{\dim G}$. Almost all the formulae which we obtain here have appeared elsewhere, but it is our intention to give a common treatment based on the preceding discussion.

For general information on the Pontrjagin-Thom construction we refer to [9] and the references therein. To introduce notation, however, we summarize a few facts. A *framing* of a vector bundle E over G is a bundle equivalence of E with the trivial bundle of the appropriate dimension over G. If G is embedded in some Euclidean space with trivial normal bundle, the the Pontrjagin-Thom construction associates to a framing of the normal bundle a homotopy class in $\pi^s_{\dim G}$. If G is oriented, a framing of the tangent bundle $L: \tau G \to G \times R^{\dim G}$ may be defined by $L_g: \tau_g(G) \to \tau_e(G)$, where L_g is induced by left multiplication by g^{-1}, and where $\tau_e(G)$ is identified with $R^{\dim G}$ through the orientation. Furthermore, a map $f: G \to SO(n)$ may be used to twist L to give a framing $L^f: \tau(G) \oplus (G \times R^n) \to G \times R^{n+\dim G}$, and we do this by the formula

$$L^f(v, x) = (L(v), f(g)^{-1} x),$$

where $v \in \tau(G)$ and $x \in R^n$, producing an element $[G, L^f] \in \pi^s_{\dim G}$. Different conventions are possible, which affect the sign of $[G, L^f]$, but we shall not digress on this subject as we are interested only in showing that $[G, L^f]=0$ under certain conditions. We remark that a convention may be chosen so that, when f is a representation, $[G, L^f]$ is the element $[G,f]$ of [9].

The following simple result is proved in §2 of [9].

Proposition 5.1. *Let φ be a representation of G on R^n such that some point $x \in R^n$ has trivial isotropy. Then G is embedded in R^n as the orbit of x, and the normal bundle may be framed via φ in such a way that the resulting element of $\pi^s_{\dim G}$ is $[G, \varphi]$.*

It is remarked in [9] that if $\vartheta: SU(n) \to SO(2n)$ is the standard representation of $SU(n)$, one may take $f=(n-1)\vartheta$ in 5.1, so that $SU(n)$ is embedded in $C^{n(n-1)}$ as the subset of $(n-1)$-tuples $(x_1, \cdots, x_{n-1})$, with $x_i \in C^n$, $|x_i|=1$, and $x_i \cdot x_j^t =0$, if $i \neq j$. Moreover, if the equation $|x_{n-1}|=1$ is replaced by $|x_{n-1}| \leq 1$, a

submanifold V of $C^{n(n-1)}$ is obtained with $\partial V = SU(n)$ and the framing of $SU(n)$ extends to V. Thus, $[G, (n-1)\vartheta] = 0$. This procedure applies to other standard embeddings, and we list below the formulae which arise; ϑ denotes the (realification of the) appropriate standard representation, and the corresponding embedding is noted in parentheses.

Proposition 5.2.
a) $[O(n), n\vartheta] = 0$, $(O(n) \subseteq R^{n^2})$.
$[U(n), n\vartheta] = 0$, $(U(n) \subseteq C^{n^2})$.
$[Sp(n), n\vartheta] = 0$, $(Sp(n) \subseteq H^{n^2})$.
b) $[SO(n), (n-1)\vartheta] = 0$, $(SO(n) \subseteq R^{n(n-1)})$.
$[SU(n), (n-1)\vartheta] = 0$, $(SU(n) \subseteq C^{n(n-1)})$.

The construction of V is valid only when the equations defining the embedding of G do not involve the determinant. That is why the embeddings $SO(n) \subseteq R^n$, and $SU(n) \subseteq C^n$ do not appear.

Examining, for example, the case $G = SU(n)$ in more detail, we see that $V = SU(n) \times [0, 1]/\sim$, where $\sim$ is the equivalence relation defined by $(X, 0) \sim (XZ, 0)$ for all Z in $SU(2)$. But this is just $SU(n) \times_{SU(2)} D^4$, where the action of $SU(2)$ is given by the formula $Z \cdot (X, Y) = (XZ, \sigma(Z)^{-1}Y)$. Thus, V is just an embedding of the bounding manifold constructed in §2, i.e., the disc bundle associated to the S^3-bundle $SU(n) \to SU(n)/SU(2)$. A similar analysis applies to the other examples in 5.2, where we take $O(1) \subseteq O(n)$, $U(1) \subseteq U(n)$, $SO(2) \subseteq SO(n)$, and $Sp(1) \subseteq Sp(n)$. One natural embedding is missing above, namely $\sigma: S^3 \to SO(4) \subseteq SO(n)$. This may be dealt with by the following general result of [5], which describes the behaviour of framings under this construction, and is proved by direct calculation.

Proposition 5.3. *Let S be a closed subgroup of G which is isomorphic to either S^1 or S^3. Let $f: G \to SO(n)$ be a representation of G such that $f|_S \cong \rho \oplus \text{Ad}(G, S) \oplus (\text{trivial representation})$ if S is isomorphic to S^1, or $f|_S \cong \sigma \oplus \text{Ad}(G, S) \oplus (\text{trivial representation})$ if S is isomorphic to S^3. Then the framing L^f extends over V and hence $[G, f] = 0$.*

For the standard examples $S = U(1) \subseteq U(n)$, $S = Sp(1) \subseteq Sp(n)$, $S = SO(2) \subseteq SO(n)$ and $S = SU(2) \subseteq SU(n)$ the representation $\text{Ad}(G, S)$ turns out to be, up to trivial summands, $(n-1)\rho$, $(n-1)\sigma$, $(n-2)\rho$ and $(n-2)\sigma$ respectively. Taking f in 5.3 to be the appropriate multiple of the standard representation the formulae of 5.2 follow (excluding the trivial case $S = O(1) \subseteq O(n)$). For the subgroup $S \subseteq SO(n)$ given by σ, where $n \geq 4$, one has

$$\text{Ad}(SO(n))|_S \cong \Lambda^2(\sigma \oplus (n-4)) \cong \text{Ad}(S^3) \oplus (n-4)\,\sigma \oplus ((n-4)\,(n-5)/2) \oplus 3 \,,$$

hence $\text{Ad}(SO(n), S) \cong (n-4)\sigma \oplus ((n-4)\,(n-5)/2) \oplus 3$ and 5.3 applies to give the

formula

$$[SO(n), (n-3)\vartheta] = 0$$

for $n \geq 4$, as noted in [5].

In summary, we have seen two situations which tend to produce a framed nullcobordism of G:

a) Existence of a representation with trivial isotropy at some point.

b) Existence of a representation which restricts to S in the manner prescribed in 5.3.

Both criteria are satisfied if G admits a faithful representation which restricts (up to trivial summands) to ρ, when S is isomorphic to S^1, or to σ, when S is isomorphic to S^3; this is the case for all our examples. A related result is that of P. Löffler and L. Smith [6], who noted the existence of a framing F of G/H provided by a representation φ with $\varphi|_H \cong \mathrm{Ad}(G, H) \oplus (\text{trivial representa-tion})$ for some subgroup H of G, and proved that the complex e-invariant of $[G/H, F]$ is zero under certain conditions.

Finally we note the following modification of 5.3.

Corollary 5.4. *Let $S \cong S^1$ or S^3 be a closed subgroup of G. Let $\varphi \colon G \to SO(n)$ be a representation of G such that $\varphi|_S \cong \rho \oplus (\text{trivial representation})$ if $S \cong S^1$, or $\varphi|_S \cong \sigma \oplus (\text{trivial representation})$ if $S \cong S^3$. Then $[G, \psi] = 0$, where $\psi = \mathrm{Ad}(G) \oplus \varphi$ if $S \cong S^1$ and $\psi = \mathrm{Ad}(G) \oplus (n-3)\varphi - \Lambda^2\varphi$ if $S \cong S^3$.*

Proof. Let $S \cong S^1$. Then $\mathrm{Ad}(G) \oplus \varphi$ satisfies the hypothesis of 5.3, since $\mathrm{Ad}(S^1)$ is a trivial representation. Let $S \cong S^3$. Then $\mathrm{Ad}(G)|_S \cong \mathrm{Ad}(G, S) \oplus \mathrm{Ad}(S^3)$, and $\mathrm{Ad}(S^3) \cong \Lambda^2\varphi \oplus ((n-4)(n-3)/2) - ((n-4)\varphi \oplus 3)|_S$. Hence $\mathrm{Ad}(G, S)$ is the restriction to S of the representation $\mathrm{Ad}(G) \oplus (n-4)\varphi \oplus 3 - (\Lambda^2\varphi \oplus ((n-4)(n-3)/2))$.

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