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A REMARK ON THE PERCOLATION FOR THE 2D ISING MODEL

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1. Introduction

Let $\boldsymbol{Z}^2$ denote the square lattice and put $\Omega=\{-1,+1\}^{\boldsymbol{Z}^2}$. We call Ω the space of spin configurations on $\boldsymbol{Z}^2$ and equip it with the usual topological Borel σ-algebra $\mathcal{F}$. The Hamiltonian H_V^ω of our model in a finite set V with boundary configuration $\omega\in\Omega$ is a function from $\{-1,+1\}^V$ to $\boldsymbol{R}$ defined by

$$(1)\qquad H_V^\omega(\sigma)=-\sum_{\substack{x,y\in V\\|x-y|=1}}\sigma(x)\sigma(y)-h\cdot\sum_{x\in V}\sigma(x)-\sum_{\substack{x\in V,y\in V^c\\|x-y|=1}}\sigma(x)\omega(y)$$
$$\sigma\in\{-1,+1\}^V,$$

where h is a real parameter called the external field. A Gibbs state for the system of Hamiltonians $\{H_V^\omega;\ \omega\in\Omega,\ V$ is a finite subset of $\boldsymbol{Z}^2\}$ and at the inverse temperature $\beta>0$, is a probability measure μ on Ω satisfying the DLR equation:

$$(2)\qquad \mu(\{\omega(x)=\sigma(x),\ x\in V\}\,|\,\mathcal{F}_{V^c})(\omega)=(Z_V^\omega)^{-1}\exp\{-\beta\cdot H_V^\omega(\sigma)\}$$

for μ—a.a. ω, for every finite $V\subset\boldsymbol{Z}^2$ and $\sigma\in\{1,+1\}^V$, where

$$Z_V^\omega:=\sum_{\sigma\in\{-1,+1\}^V}\exp\{-\beta\cdot H_V^\omega(\sigma)\}\ ,$$

$\mathcal{F}_W$ denotes the σ-algebra generated by $\{\omega(x);\ x\in W\}$ for $W\subset\boldsymbol{Z}^2$, and $\mu(\cdot\,|\mathcal{F}_W)(\omega)$ is the regular conditional probability distribution of μ conditioned on $\mathcal{F}_W$. It is well known that this model exibits the phase transition:

(i) If $h\neq 0$, then for every $\beta>0$ the Gibbs state for the parameter (β,h) is unique.

(ii) There exists a critical value β_c such that the number of Gibbs states for $(\beta,0)$ is more than one if $\beta>\beta_c$, and the Gibbs state for $(\beta,0)$ is unique if $\beta\leq\beta_c$.

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We are interested in the case where $\beta<\beta_c$; for such β and every h we have an unique Gibbs state, which we will denote by $\mu_{\beta,h}$. This $\mu_{\beta,h}$ has several nice properties as listed below.

[a] (*Spacial Symmetry*) $\mu_{\beta,h}$ is invariant under $\boldsymbol{Z}^2$-translations, rotations by right angles and reflections with respect to (w.r.t.) x^1- and x^2-axes.

[b] (*Symmetry w.r.t. Spin Reversing*) If $R\colon \Omega\to\Omega$ is defined by

$$(R\omega)(x) = -\omega(x) \qquad \text{for every} \quad x\in\boldsymbol{Z}^2\,,$$

then $\quad \mu_{\beta,h}\circ R = \mu_{\beta,-h}\,,$

in particular, $\mu_{\beta,0}$ is invariant under R.

[c] (*FKG Inequality*) Introduce the order $\leq$ in Ω by the componentwise inequality: $\omega\leq\eta$ iff $\omega(x)\leq\eta(x)$ for every $x\in\boldsymbol{Z}^2$; and say that a function f on Ω is *increasing* if $f(\omega)\leq f(\eta)$ whenever $\omega\leq\eta$. Then $\mu_{\beta,h}$ satisfies

$$E_{\beta,h}(f\cdot g)\geq E_{\beta,h}(f)\cdot E_{\beta,h}(g)\,. \tag{3}$$

for bounded increasing functions f and g, where $E_{\beta,h}$ denotes the expectation w.r.t. $\mu_{\beta,h}$.

[c′] (*Monotonicity*) For any bounded increasing function f on Ω, if $h\leq h'$ then

$$E_{\beta,h}(f)\leq E_{\beta,h'}(f)\,. \tag{4}$$

[d] (*Tail Triviality*) If we define the tail σ-algebra $\mathcal{F}_\infty$ by

$$\mathcal{F}_\infty = \bigcap_{V\,;\,\text{finite}\subset\boldsymbol{Z}^2} \mathcal{F}_{V^c}\,,$$

then $\quad \mu_{\beta,h}(A) = 0$ or $1 \qquad$ for every $\quad A\in\mathcal{F}_\infty\,.$

[e] (*Strongly Mixing Property*) $\mu_{\beta,h}$ is strongly mixing in each direction with mixing coefficient $\psi_{\beta,h}(n)$: i.e. there exists a sequence $\psi_{\beta,h}(n)$, $n=1, 2, \cdots$, decreasing to zero as n goes to ∞, such that for each $n\geq 1$

$$|\mu_{\beta,h}(A\cap B)-\mu_{\beta,h}(A)\,\mu_{\beta,h}(B)|\leq\psi_{\beta,h}(n) \tag{5}$$

for every $A\in\mathcal{F}^i_{-,0}$ and $B\in\mathcal{F}^i_{+,n}$, $i=1, 2$, where for $i=1, 2$, and for an integer k, $\mathcal{F}^i_{+,k}$ (or $\mathcal{F}^i_{-,k}$) is the σ-algebra generated by

$$\{\omega(x);\, x=(x^1, x^2),\, x^i\geq k \text{ (or } x^i\leq -k\text{, resp.)}\}\,.$$

REMARK 1.1. The strong mixing property [e] is studied in [11], where it is proved that the mixing coefficient $\psi_{\beta,h}(n)$ decays exponentially in n. This exponential decay is based on the exponential decay of the covariance function (so-called the truncated pair correlation function)

$$\rho^T_{\beta,h}(0, x) = E_{\beta,h}(\omega(0)\,\omega(x))-E_{\beta,h}(\omega(0))^2\,.$$

Since by GHS inequality [8], the covariance function is increasing in h if $h\leq 0$, and decreasing in h if $h\geq 0$, it takes the maximum value when $h=0$. Further, if $h=0$ and $\beta<\beta_c$ then the single spin expectation $E_{\beta,0}(\omega(0))$ is equal to zero, and the covariance function is equal to the correlation function $E_{\beta,0}(\omega(0)\omega(x))$. By Griffiths' inequality, this correlation function is increasing in β (see for example [7]). Hence for any $\beta<\beta_1<\beta_c$ and any $h\in\boldsymbol{R}$, the covariance function $\rho^T_{\beta,h}(0,x)$ is dominated by the covariance function $\rho^T_{\beta_1,0}(0,x)$, which decays exponentially. Therefore every $\mu_{\beta,h}$ from $\{\mu_{\beta,h}\}_{h\in\boldsymbol{R}\,:\,\beta<\beta_1}$ is strongly mixing in each direction with the same coefficient $\psi_{\beta_1,0}(n)$, which decays exponentially in n.

Now we introduce our notations and terminologies for the percolation problem, Let $\mathcal{L}$ be the planer graph whose vertex set is just equal to $\boldsymbol{Z}^2$, and whose edge set is the collection of all nearest neighbour pairs $\{x,y\}$ of $\boldsymbol{Z}^2$, i.e. $|x^1-y^1|+|x^2-y^2|=1$. We denote by $\mathcal{L}^*$ the matching graph of $\mathcal{L}$: the vertex set of $\mathcal{L}^*$ is also $\boldsymbol{Z}^2$, and the edge set of $\mathcal{L}^*$ is the collection of all $*$nearest neighbour pairs $\{x,y\}$ of $\boldsymbol{Z}^2$, i.e. $\max\{|x^1-y^1|, |x^2-y^2|\}=1$. Let $\mathcal{G}=\mathcal{L}$ or $\mathcal{L}^*$. We call a sequence of points $\{x_1,x_2,\cdots,x_n\}\subset\boldsymbol{Z}^2$ a (self-avoiding) *$\mathcal{G}$-path* if (a) $x_i\neq x_j$ whenever $i\neq j$, and (b) for every k with $0\leq k\leq n-1$, x_k and x_{k+1} are joined by an edge of $\mathcal{G}$. A $\mathcal{G}$-path $\gamma=\{x_0,x_1,\cdots,x_n\}$ is called a *$\mathcal{G}$-circuit* if $\gamma'=\{x_1,x_2,\cdots,x_n,x_0\}$ is also a $\mathcal{G}$-path. A set $V\subset\boldsymbol{Z}^2$ is said *$\mathcal{G}$-connected* if for every pair $\{x,y\}\subset V$ there exists a $\mathcal{G}$-path $\gamma=\{x_0,x_1,\cdots,x_n\}$ in V such that $x_0=x$, and $x_n=y$.

For $\omega\in\Omega$ and $x\in\boldsymbol{Z}^2$, we denote by $C_x^+(\mathcal{G})=C_x^+(\mathcal{G};\omega)$ the maximal $\mathcal{G}$-connected component of $\omega^{-1}(+1):=\{y\in\boldsymbol{Z}^2;\,\omega(y)=+1\}$ containing the point x. Also $C_x^-(\mathcal{G})$ is defined in the same way for $\omega^{-1}(-1)$.

Our main concern is to investigate how the probability

$$\mu_{\beta,h}(\# C_0^{\varepsilon}(\mathcal{G})=\infty)\quad(\varepsilon=+\text{ or }-)$$

changes as we vary parameter (β,h) under the condition $\beta<\beta_c$, where for $V\subset\boldsymbol{Z}^2$, $\# V$ denotes the cardinality of V. The first answer to this problem was given by Coniglio et al. [5].

Theorem ([5]).

$$\mu_{\beta,0}(\# C_0^{\varepsilon}(\mathcal{L})=\infty)=0\qquad\textit{for both}\quad\varepsilon=+\textit{ and }-,\quad\textit{for}\quad\beta\leq\beta_c\,.$$

REMARK 1.2. The first result of this type is in the work of Miyamoto [17], where the bond percolation problem is considered w.r.t. a probability measure on $\{-1,+1\}^{\boldsymbol{B}^2}$ ($\boldsymbol{B}^2$ is the set of all bonds in $\boldsymbol{Z}^2$), having properties corresponding to [a]$\sim$[d]. Both the above theorem and the result in [17] are dependent versions of Harris' result [10] for the Bernoulli percolation.

On the contrary, it can be shown that for sufficiently small β, there exists a positive $h_c(\beta)$ such that

$$\mu_{\beta,h}(\# C_0^{\varepsilon}(\mathcal{L}^*) = \infty) > 0 \qquad \text{for both} \quad \varepsilon = + \text{ and } - , \tag{6}$$

if $|h| < h_c(\beta)$ (see for example [12, 15]).

Our original problem is the following:

Problem. Can one find a positive $h_c(\beta)$ for every $\beta < \beta_c$ such that (6) holds if $|h| < h_c(\beta)$?

In order to handle this problem, it may be helpful if we have an RSW-type theorem, since in the Bernoulli percolation case it was so powerful to analyze the two-dimensional percolation problem. The idea is to introduce "sponge percolation probabilities".

For each $n \geq 1$, $L \geq 1$, let $V_{n,L}$ be the rectangle defined by

$$V_{n,L} := \{x = (x^1, x^2) \in \boldsymbol{Z}^2;\ |x^1| \leq n,\ |x^2| \leq L\} .$$

For $\varepsilon = +$ or $-$, $\mathcal{G} = \mathcal{L}$ or $\mathcal{L}^*$ and for n, $L \geq 1$, let $A_{n,L}^{\varepsilon}(\mathcal{G})$ be the event that the two horizontal sides of $V_{n,L}$ (i.e. $\{(x^1, x^2) \in V_{n,L};\ x^2 = L\}$ and $\{(x^1, x^2) \in V_{n,L}:\ x^2 = -L\}$) are connected by a $\mathcal{G}$-path lying entirely in the set $V_{n,L} \cap \omega^{-1}(\varepsilon)$, i.e.

$$A_{n,L}^{\varepsilon}(\mathcal{G}) := \left\{ \omega \in \Omega;\ \begin{array}{l} \text{there exists a } \mathcal{G}\text{-path } \gamma \subset V_{n,L} \cap \omega^{-1}(\varepsilon) \\ \text{intersecting both horizontal sides of } V_{n,L} \end{array} \right\}$$

where of course $\omega^{-1}(\varepsilon) = \omega^{-1}(+1)$ if $\varepsilon = +$, and $\omega^{-1}(-1)$ if $\varepsilon = -$.

Finally, if $\mathcal{G} = \mathcal{L}$ or $\mathcal{L}^*$ then we denote by $\mathcal{G}'$ the matching graph of $\mathcal{G}$: i.e. if $\mathcal{G} = \mathcal{L}$ then $\mathcal{G}' = \mathcal{L}^*$, and if $\mathcal{G} = \mathcal{L}^*$ then $\mathcal{G}' = \mathcal{L}$.

DEFINITION. For a given probability measure μ which satisfies [a], [c] and [d], we say that the RSW theorem holds for μ if for each $\varepsilon \in \{+, -\}$ and for each $\mathcal{G} \in \{\mathcal{L}, \mathcal{L}^*\}$, the following five statements are equivalent.

$$\mu(\# C_0^{\varepsilon}(\mathcal{G}) = \infty) > 0 , \tag{7}$$

$$\lim_{n \to \infty} \mu(A_{3n,n}^{\varepsilon}(\mathcal{G})) = 1 , \tag{8}$$

$$\lim_{n \to \infty} \mu(A_{n,n}^{\varepsilon}(\mathcal{G})) = 1 , \tag{9}$$

$$\lim_{n \to \infty} \mu(A_{n,3n}^{\varepsilon}(\mathcal{G})) = 1 , \tag{10}$$

$$E_{\mu}(\# C_0^{-\varepsilon}(\mathcal{G}')) < \infty , \tag{11}$$

where $-\varepsilon$ is the opposite sign of ε, $\{\varepsilon, -\varepsilon\} = \{+, -\}$.

The famous result of Russo [18], Seymour and Welsh [19] is then rephrased in the following way:

Theorem ([18, 19]). *For every $p \in [0, 1]$, the RSW theorem holds for ν_p, where ν_p is the Bernoulli probability measure with density p.*

In this paper, we show that we can assume that the RSW theorem holds for $\mu_{\beta,h}$ ($h \in \boldsymbol{R}$, $\beta < \beta_c$), so far as our original problem is concerned. Namely, our result is the following:

Theorem 1. *Let $\beta < \beta_c$. Then either one of the following statements holds:*
(*i*) *There exists a positive $h_c(\beta)$ such that* (6) *holds for* $|h| < h_c(\beta)$,

or

(*ii*) *the RSW theorem holds for $\mu_{\beta,h}$ for every $h \in \boldsymbol{R}$.*

REMARK 1.3. Recently, two remarkable results were obtained by J.T. Chayes-L. Chayes-Shonman [3] and by Gandolfi [6] related to the percolation for the Ising model. In [3], it was proved that the connectivity function $\mu^+_{\beta,0}(x \in C_0^-(\mathcal{L}^*))$ decays exponentially as $|x| \to \infty$ for $\beta > \beta_c$, where $\mu^+_{\beta,0}$ is the extremal Gibbs state obtained from the+boundary condition. (In [14], the author was trying to prove this, but succeeded *only* with the logarithmic factor:

$$\mu^+_{\beta,0}(x \in C_0^-(\mathcal{L}^*)) \leq \exp[-\Delta|x|/\log|x|], \quad |x| \to \infty$$

for some positive constant Δ.) With this exponential decay of the connectivity function, we can discuss the similar phenomena as in [9] (see [15]). This is well explained in the case of Bernoulli percolation in [4]. In [6], it is shown that the number of the infinite connected components of $\omega^{-1}(+1)$ is at most one a.s. with respect to any stationary Gibbs state corresponding to some class of interactions in $\boldsymbol{Z}^d$, $d \geq 2$ (for example finite range). This is a nice extension of the independent case result [2].

2. Sponge percolation for strongly mixing random fields with FKG inequality

In this section, we consider general relation between the statements (7)$\sim$(11), mainly under the condition that μ is strongly mixing in each direction with the coefficient $\psi(n)$ decreasing to zero as n tends to ∞.

Lemma 2.1 ([18], Proposition 1). *If μ is invariant under $\mathbf{Z}^2$-translation, then* (11) *implies* (7) *for each $\varepsilon \in \{-1, +1\}$ and for each $\mathcal{G} \in \{\mathcal{L}, \mathcal{L}^*\}$.*

Lemma 2.2. *Assume that μ satisfies* [*a*], [*c*] *and that μ is strongly mixing in each direction* ([*e*]). *If*

$$\liminf_{n\to\infty} \mu(A^{\varepsilon}_{n,3n}(\mathcal{G})) = \alpha > 0\,, \tag{12}$$

then for every $\xi > 0$, there exists $M > 0$ such that

$$\liminf_{n\to\infty} \mu(B^{\varepsilon}_n(M;\mathcal{G})) \geq 1 - \xi\,, \tag{13}$$

where $B^{\varepsilon}_n(M, \mathcal{G})$ is the event that there exists a $\mathcal{G}$-circuit surrounding the origin in $\omega^{-1}(\varepsilon) \cap V_{Mn,Mn} \backslash V_{n,n}$. (See Fig. 1.)

REMARK 2.1. This is a variant of Lemma 4.2 of [15]. The new part of this lemma is only that we need no additional condition for the decay rate of $\psi(n)$ as $n\to\infty$.

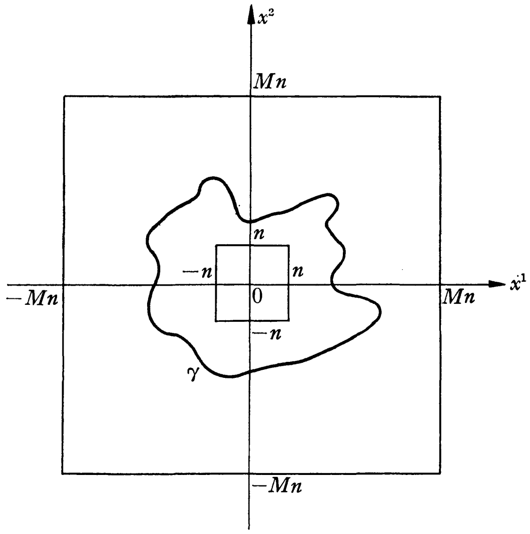

Fig. 1 event $B_n^{\varepsilon}(M;\mathcal{G})$
The curve γ is a $\mathcal{G}$-circuit in $\omega^{-1}(\varepsilon)\cap V_{Mn,Mn}\setminus V_{n,n}$ surrounding the origin.

Before we go into the proof, we prepare notations. For $x\in\boldsymbol{Z}^2$ let $\tau(x)$ denote the x-translation, i.e.

(14-a) $$(\tau(x)\omega)(y)=\omega(x+y)\quad y\in\boldsymbol{Z}^2$$

and

(14-b) $$\tau(x)A=\{\tau(x)\omega;\ \omega\in A\}\quad A\in\mathcal{G}$$

The rotation by the right angle is denoted by rot: $\Omega\to\Omega$, i.e.

(15-a) $$(\mathrm{rot}\ \omega)(x^1,x^2)=\omega(-x^2,x^1)\quad (x^1,x^2)\in\boldsymbol{Z}^2,\ \omega\in\Omega\,,$$

and

(15-b) $$\mathrm{rot}\,A=\{\mathrm{rot}\,\omega;\ \omega\in A\}\quad A\in\mathcal{F}\,.$$

We call that an event A is *increasing* (*decreasing*) if its indicator function $I_A(\omega)$ is increasing (decreasing).

Proof of Lemma 2.2. Fix $n\geq 1$ arbitrarily, and set $M(k)=3^{2k}n$ for $k\geq 1$. Define

$$E_{k,i}=(\mathrm{rot})^i\,(\tau(2M(k),0)\,A^{\varepsilon}_{M(k),3M(k)}(\mathcal{G}))$$

for $0\leq i\leq 3$ (Fig. 2),

$F_k = \bigcap_{i=0}^{3} E_{k,i}$, and $G_k = \bigcup_{j=1}^{k} F_j$. Then

$$\begin{aligned}\mu(G_k^c) &= \mu(G_{k-1}^c \cap F_k^c)\\ &= \mu(G_{k-1}^c \cap [(E_{k,0} \cap E_{k,2})^c \cup (E_{k,1} \cap E_{k,3})^c])\\ &= \mu(G_{k-1}^c \cap (E_{k,0} \cap E_{k,2})^c) +\\ &\quad + \mu(G_{k-1}^c \cap (E_{k,1} \cap E_{k,3})^c) -\\ &\quad - \mu(G_{k-1}^c \cap (E_{k,0} \cap E_{k,2})^c \cap (E_{k,1} \cap E_{k,3})^c)\,.\end{aligned}$$

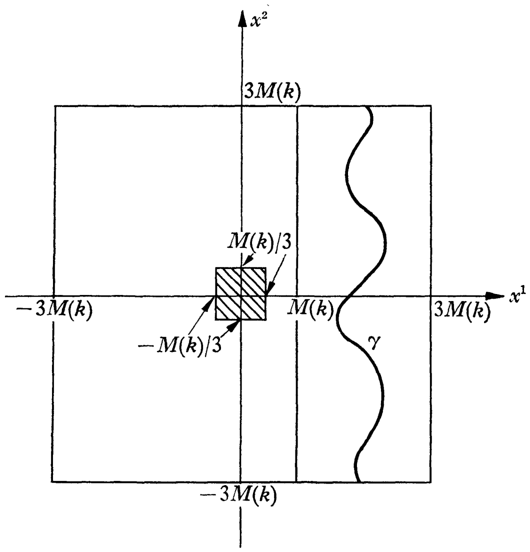

Fig. 2 Event $E_{k,0}$ and event G_{k-1}

G_{k-1} is the event occuring in the shaded square. The curve γ connecting the top and the bottom sides of the rectangle $[M(k), 3M(k)] \times [-3M(k), 3M(k)]$ is a $\mathcal{G}$-path in $\omega^{-1}(\varepsilon)$.

Note that either $\{G_j\}_{1\le j\le k}$, $\{E_{j,i}\}_{0\le i\le 3\,;\,1\le j\le k}$ are all increasing or all decreasing events. Therefore we can use FKG inequality [c] so that we obtain

$$\begin{aligned}\mu(G_k^c) &\le \mu(G_{k-1}^c \cap (E_{k,0} \cap E_{k,2})^c) +\\ &\quad + \mu(G_{k-1}^c \cap (E_{k,1} \cap E_{k,3})^c) -\\ &\quad - \mu(G_{k-1}^c)\, \mu((E_{k,0} \cap E_{k,2})^c)\, \mu((E_{k,1} \cap E_{k,3})^c)\,.\end{aligned}$$

The strong mixing property [e] implies that

$$\begin{aligned}&\mu(G_{k-1}^c \cap (E_{k,0} \cap E_{k,2})^c)\\ &\le \mu(G_{k-1}^c)\,(1-\mu(E_{k,0})\,\mu(E_{k,2})) + 2\psi(2M(k)/3)\,,\end{aligned}$$

(see Fig. 2). Similarly, we have

$$\mu((E_{k,0} \cap E_{k,2})^c) \ge 1 - \mu(E_{k,0})\,\mu(E_{k,2}) - \psi(2M(k))$$

Combining these inequalities with the translation invariance [a], we obtain

$$(16)\qquad \mu(G_k^c) \leq \mu(G_{k-1}^c)\,[1-\mu(A^{\varepsilon}_{M(k),3M(k)}(\mathcal{G}))^4]+ \\ +6\psi(2M(k)/3)\,.$$

For any given $\xi>0$, we take $k\geq 1$ large enough so that

$$(17)\qquad 2\{1-(\alpha/2)^4\}^k \leq \xi\,.$$

Depending on this k, we take $n_0\geq 1$ large enough so that

$$(18\text{-a})\qquad \mu(A^{\varepsilon}_{n,3n}(\mathcal{G}))\geq \alpha/2 \qquad \text{for every} \quad n\geq n_0\,.$$

and

$$(18\text{-b})\qquad 6\psi(6n)\leq \{1-(\alpha/2)^4\}^{k+1} \qquad \text{for every} \quad n\geq n_0\,.$$

Starting from some $n\geq n_0$, we define $M(k)$ as before, and set $M=M(k)$. Then from (16)$\sim$(18), we obtain

$$\begin{aligned} \mu(G_k^c) &\leq \mu(G_{k-1}^c)\ \{1-(\alpha/2)^4\}+6\psi(2M(k)/3) \\ &\leq \{1-(\alpha/2)^4\}^k+6\sum_{j=1}^{k}\{1-(\alpha/2)^4\}^{k-j}\cdot\psi(2M(j)/3) \\ &\leq 2\{1-(\alpha/2)^4\}^k\,. \end{aligned}$$

Since $B_n^{\varepsilon}(M;\mathcal{G})\supset G_k$, we obtain from the above estimate

$$\mu(B_n^{\varepsilon}(M;\mathcal{G}))\geq \mu(G_k)\geq 1-2\{1-(\alpha/2)^4\}^k\geq 1-\xi$$

for every $n\geq n_0$. □

Lemma 2.3. *Assume that μ satisfies* [*a*], [*c*] *and* [*e*]. *If in addition* (8) *and* (12) *hold for μ, then* (9) *and* (10) *hold for μ, too.*

Proof. Let $\xi>0$ be given arbitrarily. From (12) and Lemma 2.2, we can find $M>0$ and $n_0\geq 1$ such that

$$(19)\qquad \mu(B_n^{\varepsilon}(M;\mathcal{G}))\geq 1-\xi \qquad \text{for every } n\geq n_0\,.$$

On the other hand, the condition (8) implies that for any $\delta>0$ we can choose $n_1\geq 1$ such that

$$(20)\qquad \mu(A^{\varepsilon}_{3n,n}(\mathcal{G}))>1-\delta \qquad \text{for every } n\geq n_1\,.$$

We divide the lower side of $V_{3n,n}$ into $3M$ pieces $\{I_1, I_2, \cdots, I_{3M}\}$ such that the length $|I_j|$ of each I_j satisfy

$$2[n/M]\leq |I_j|\leq 4[n/M]\,,$$

where $[u]$ denotes the integer part of $u\in\boldsymbol{R}$. This can be simply done when we take $|I_j|=2[n/M]$ for $j=1, 2, \cdots, 3M-1$, and $|I_{3M}|=6n-2(3M-1)[n/M]$, if n is so large that $2[n/M]\geq 6M$. Let $n_2=n_2(M):=\min\{n;\ 2[n/M]\geq 6M\}$, and $n\geq n_2$. For $j=1, 2, \cdots, 3M$, we define the event $D^\varepsilon_{n,j}(\mathcal{G})$ by

$$D^\varepsilon_{n,j}(\mathcal{G}) = \left\{\omega\in\Omega;\ \begin{matrix} I_j \text{ is connected to the upper side of} \\ V_{3n,n} \text{ by a } \mathcal{G}\text{-path in } \omega^{-1}(\varepsilon)\cap V_{3n,n}\end{matrix}\right\}.$$

Then the collection of the events $\{D^\varepsilon_{n,j}(\mathcal{G})\}_{1\leq j\leq 3N}$ are all increasing if $\varepsilon=+$, and are all decreasing if $\varepsilon=-$. It follows from (20) that there is at least one k for which

$$\mu(D^\varepsilon_{n,k}(\mathcal{G}))>1-\delta^{1/3M}, \tag{21}$$

for otherwise, from FKG inequality [c] we have

$$\mu((A^\varepsilon_{3n,n}(\mathcal{G}))^c) = \mu(\bigcap_{j=1}^{3M}(D^\varepsilon_{n,j}(\mathcal{G}))^c)\geq \prod_{j=1}^{3M}\mu((D^\varepsilon_{n,j}(\mathcal{G}))^c)\geq\delta,$$

which contradicts (20). Let k be the number for which (21) holds, and let $\tilde{D}^\varepsilon_{n,k}(\mathcal{G})$ be the reflection of the event $D^\varepsilon_{n,k}(\mathcal{G})$ w.r.t. the line $\{x^2=-n-|I_k|/2\}$. Consider the event

$H_{n,k}=D^\varepsilon_{n,k}(\mathcal{G})\cap\tilde{D}^\varepsilon_{n,k}(\mathcal{G})\cap\tau(x(n,k))\,B^\varepsilon_{|I_n|/2}(M;\mathcal{G})$, where $x(n,k)=(-3n+\sum_{j=1}^{k-1}|I_j|+|I_k|/2,\ -n-|I_k|/2)$. It is easy to see that if $\omega\in H_{n,k}$ then the two

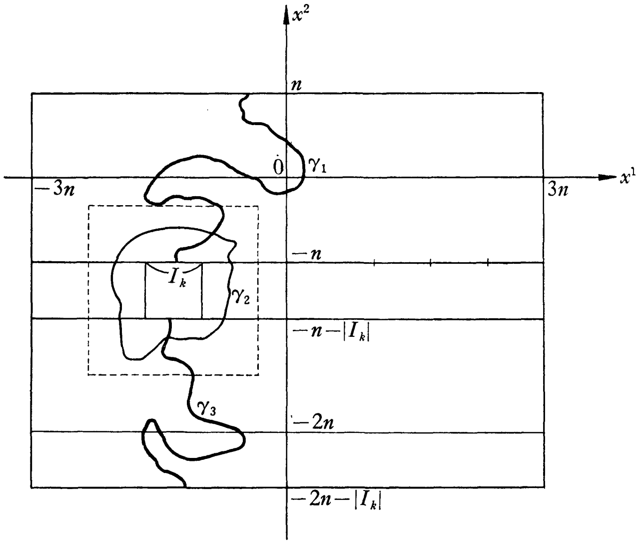

Fig. 3 event $H_{n,k}$

The curves γ_1 and γ_3 are $\mathcal{G}$-paths in $\omega^{-1}(\varepsilon)$, γ_2 is a $\mathcal{G}$-circuit in $\omega^{-1}(\varepsilon)$ surrounding I_k in the dotted square.

horizontal sides of the rectangle $\Lambda=\{-3n\leq x^1\leq 3n,\ -3n\leq x^2\leq n\}$ are connected by a $\mathcal{G}$-path in $\omega^{-1}(\varepsilon)\cap\Lambda$. (Fig. 3) Using FKG inequality [c] and the spacial symmetry [a], we obtain that

$$\begin{aligned}\mu(H_{n,k})&\geq\mu(D^{\varepsilon}_{n,k}(\mathcal{G}))^2\quad\mu(B^{\varepsilon}_{|I_k|/2}(M;\mathcal{G}))\\&\geq(1-\delta^{1/3M})^2\,(1-\xi)\end{aligned}$$

for every $n\geq Mn_0+n_1+n_2$. Since δ can be chosen arbitrarily small, we can make the right hand side of the above inequality greater than $1-2\xi$ if n_1 is sufficiently large.

The translation invariance of μ implies that

$$\begin{aligned}\mu(A^{\varepsilon}_{3n,2n}(\mathcal{G}))&=\mu\begin{pmatrix}\text{two horizontal side sides of }\Lambda\text{ are connected}\\ \text{by a }\mathcal{G}\text{-path in }\omega^{-1}(\varepsilon)\cap\Lambda\end{pmatrix}\\&\geq\mu(H_{n,k}),\end{aligned}$$

and hence we obtain

$$\mu(A^{\varepsilon}_{3n,2n}(\mathcal{G}))\geq 1-2\xi$$

for $n\geq Mn_0+n_1+n_2$. Since ξ is arbitrary, we end up with

$$\lim_{n\to\infty}\mu(A^{\varepsilon}_{3n,2n}(\mathcal{G}))=1\,. \tag{22}$$

Starting with (19) and (22) again, the same argument leads us to

$$\lim_{n\to\infty}\mu(A^{\varepsilon}_{3n,4n}(\mathcal{G}))=1\,. \tag{23}$$

Therefore repeating this argument several times, we obtain

$$\lim_{n\to\infty}\mu(A^{\varepsilon}_{3n,2k_n}(\mathcal{G}))=1\qquad\text{for every }k\geq 1\,. \tag{24}$$

It is now easy to deduce (9) and (10) from (24). □

Corollary 2.4. *Let μ satisfy* [*a*], [*c*] *and* [*e*]. *If* (7) *and* (12) *hold for μ, then* (8)~(10) *hold for μ, as well.*

Proof. It is sufficient to show that (8) holds for μ, because of the above lemma. But this is proven in [18], Lemma 2, and in [16], Lemma 5.1, where the independence is not acutally employed. Both proofs are quite simple, and here is presented one of them for the convenience of readers.

In our situation, it is simpler to quote Kesten's proof [16]. Assume that (7) holds for μ. Then for any $\xi>0$ we can take $n_0\geq 1$ such that

$$\mu\begin{pmatrix}\text{there exists a }\mathcal{G}\text{-path }\gamma\text{ in}\\ \omega^{-1}(\varepsilon)\cap V_{3n,3n}\backslash V_{n,n}\text{ connecting}\\ \partial V_{n,n}\text{ with }\partial V_{3n,3n}\end{pmatrix}\geq 1-\xi \tag{25}$$

for every $n \geq n_0$, where $\partial V_{k,k} = V_{k,k} \setminus V_{k-1,k-1}$, because the event that there exists an unbounded $\mathcal{G}$-path in $\omega^{-1}(+1)$ is invariant under the $\boldsymbol{Z}^2$-translation and [e] implies that μ is ergodic in each direction. It is easy to see that if a $\mathcal{G}$-path γ connects $\partial V_{n,n}$ with $\partial V_{3n,3n}$ in the annulus $V_{3n,3n} \setminus V_{n,n}$, then there is at least one rectangle Λ among

$$\Lambda_0 = \{-3n \leq x^1 \leq 3n,\ n \leq x^2 \leq 3n\}\ ,$$
$$\Lambda_1 = \{-3n \leq x^1 \leq -n,\ -3n \leq x^2 \leq 3n\}\ ,$$
$$\Lambda_2 = \{-3n \leq x^1 \leq 3n,\ -3n \leq x^2 \leq -n\}\ ,$$

and

$$\Lambda_3 = \{n \leq x^1 \leq 3n,\ -3n \leq x^2 \leq 3n\}\ ,$$

such that $\gamma \cap \Lambda$ connects (in $\mathcal{G}$) two longer sides of Λ (Fig. 4).

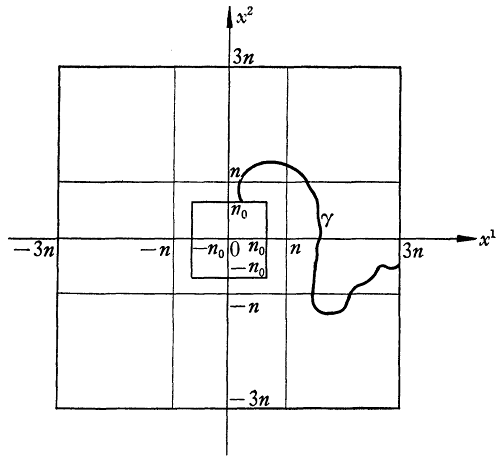

Fig. 4

The curve γ is a $\mathcal{G}$-path connecting V_{n_0,n_0} with $\partial V_{3n,3n}$. In the above picture, γ connects two longer sides of Λ_3.

This implies that

$$\mu\Big(\bigcup_{i=0}^{3} (\mathrm{rot})^i\, \tau(0, 2n)\, A^{\xi}_{3n,n}(\mathcal{G})\Big) \geq \textit{the L.H.S. of } (25)$$
$$> 1 - \xi$$

for every $n \geq n_0$. Using the spacial symmetry [a] and the FKG inequality [c], as in the proof of Lemma 2.2, we obtain

$$\mu(A^{\xi}_{3n,n}(\mathcal{G})) > 1 - \xi^{1/4} \qquad \text{for every } n \geq n_0\,. \quad \square$$

By appealing to the rescaling argument in [1], we can obtain even a stronger result than Lemma 2.3. The cost we have to pay for it is that we need a little more delicate argument.

Lemma 2.5. *Assume that μ satisfies [a], [c] and [e]. If we assume in addition* (12) *and that*

$$\liminf_{n\to\infty} \mu(A^{-\varepsilon}_{n,kn}(\mathcal{G}')) = 0 \tag{26}$$

for some $k\geq 1$, then we also have (8)$\sim$(10) *for μ.*

Proof. First we observe that

$$A^{-\varepsilon}_{n,kn}(\mathcal{G}') \cup \operatorname{rot}(A^{\varepsilon}_{kn,n}(\mathcal{G})) = \Omega\,, \quad \text{and}$$
$$A^{-\varepsilon}_{n,kn}(\mathcal{G}') \cap \operatorname{rot}(A^{\varepsilon}_{kn,n}(\mathcal{G})) = \phi\,,$$

because every $\mathcal{G}'$-path γ' in $V_{n,kn}$ connecting two horizontal sides of $V_{n,kn}$ intersects any $\mathcal{G}$-path γ in $V_{n,kn}$ connecting two vertical sides of $V_{n,kn}$. By the rotation invariance of μ, (26) therefore implies that

$$\limsup_{n\to\infty} \mu(A^{\varepsilon}_{kn,n}(\mathcal{G})) = 1\,. \tag{27}$$

The same argument as in the proof of Lemma 2.3 can be applied to conclude from (12) and (27) that

$$\limsup_{n\to\infty} \mu(A^{\varepsilon}_{n,kn}(\mathcal{G})) = 1\,. \tag{28}$$

On the other hand, it is easy to see that

$$\tau(0,(k-1)\,n)\,A^{\varepsilon}_{n,kn}(\mathcal{G}) \cap \operatorname{rot} A^{\varepsilon}_{n,n}(\mathcal{G}) \cap \tau(0,-(k-1)\,n)\,A^{\varepsilon}_{n,kn}(\mathcal{G})$$

is a subset of $A^{\varepsilon}_{n,(2k-1)n}(\mathcal{G})$. Hence from FKG inequality [c], we obtain

$$\mu(A^{\varepsilon}_{n,(2k-1)n}(\mathcal{G}) \geq \mu(A^{\varepsilon}_{n,n}(\mathcal{G}))\;\mu(A^{\varepsilon}_{n,kn}(\mathcal{G}))^2\,.$$

Using this inequality twice, we obtain

$$\begin{aligned}\mu(A^{\varepsilon}_{n,9n}(\mathcal{G})) &\geq \mu(A^{\varepsilon}_{n,n}(\mathcal{G}))^3\;\mu(A^{\varepsilon}_{n,3n}(\mathcal{G}))^4\\ &\geq \mu(A^{\varepsilon}_{n,3n}(\mathcal{G}))^7\end{aligned} \tag{29}$$

for every $n\geq 1$. Since

$$A^{\varepsilon}_{3n,9n}(\mathcal{G}) \supset \tau(2n,0)\,A^{\varepsilon}_{n,9n}(\mathcal{G}) \cup \tau(-2n,0)\,A^{\varepsilon}_{n,9n}(\mathcal{G})\,,$$

the strong mixing property [e] implies that

$$\mu(A^{\varepsilon}_{3n,9u}(\mathcal{G})) \geq 2\mu(A^{\varepsilon}_{n,9n}(\mathcal{G})) - \mu(A^{\varepsilon}_{n,9n}(\mathcal{G}))^2 - \psi(2n)\,. \tag{30}$$

The inequality (29) and the fact that $2u-u^2$ is increasing in $u\in([0,1]$, imply that the right hand side of (30) is not smaller than

$$2\mu(A^{\varepsilon}_{n,3n}(\mathcal{G}))^7 - \mu(A^{\varepsilon}_{n,3n}(\mathcal{G}))^{14} - \psi(n)\,.$$

Let $f(u)=2u^7-u^{14}$, and u^* be the unique solution of $u=f(u)$ in the interval $(0, 1)$. Take $\delta>0$ small enough so that the equation $u+\delta=f(u)$ has two solutions $\alpha<\alpha'$ in $(u^*, 1)$. Choose $n_0\geq 1$ large enough so that $\psi(n)<\delta$ for every $n\geq n_0$ and $\mu(A^{\varepsilon}_{n_0,3n_0}(\mathcal{G}))>\alpha$. The latter inequality is possible because of (28). Putting $n_k=3^k n_0$ and $a_k=\mu(A^{\varepsilon}_{n_k,3n_k}(\mathcal{G}))$, we obtain

$$a_{k+1}\geq f(a_k)-\psi(n_k) \qquad \text{for every } k\geq 1\,. \tag{31}$$

Taking the inferior limits of the both sides, we obtain

$$\gamma := \liminf_{k\to\infty} a_k \geq f(\gamma)\,. \tag{32}$$

But since $a_0>\alpha$ and $\psi(n_k)<\delta$ for every $k\geq 1$, (31) also implies that $\gamma\geq f(\alpha')-\delta=\alpha'$, and hence we have $\gamma=1$, i.e.

$$\lim_{k\to\infty} a_k = 1\,. \tag{33}$$

Again by the argument in the beginning of this proof, we can obtain from (33),

$$\lim_{k\to\infty} \mu(A^{\varepsilon}_{n_k,9n_k}(\mathcal{G})) = 1\,, \tag{34}$$

which is sufficient to prove (10). □

Lemma 2.6. *Assume* [*a*], [*c*] *and* [*e*] *with the mixing coefficient* $\psi(n)$ *satisfying*

$$\int^{\infty} t\psi(t)\,dt<\infty\,, \tag{35}$$

where $\psi(u)$ *is a step function on* $[0, \infty)$ *induced from* $\psi(n)$. *Then* (10) *implies* (11).

Proof. From (10), for any $\xi>0$ we can find $n_0\geq 1$ such that

$$\mu(A^{\varepsilon}_{n,3n}(\mathcal{G}))\geq 1-\xi \qquad \text{for every } n\geq n_0\,.$$

Let $n\geq n_0$ be arbitrarily fixed and let $E_{k,i}$, $0\leq i\leq 3$, F_k and G_k be as in the proof of Lemma 2.2, with $M(k)=3^{2k}n$.

Let

$$\pi^{\varepsilon}(m; \mathcal{G}) = \mu(C^{\varepsilon}_0(\mathcal{G})\cap\partial V_{m,m}\neq\phi)\,. \tag{36}$$

Then it is easy to see that

$$\pi^{-\varepsilon}(M(k); \mathcal{G}')\leq\mu(G^c_k)\,. \tag{37}$$

Noting that $E_\mu(\#C^{-\varepsilon}_0(\mathcal{G}'))\leq \sum_{m=1}^{\infty} 8m\cdot\pi^{-\varepsilon}(m; \mathcal{G}')$, we know that the only thing we have to do is to ensure the convergence of the series in the right hand side of the above inequality. For $M(k-1)\leq m\leq M(k)$, we have

$$\begin{aligned}
&m\cdot\pi^{-\varepsilon}(m;\mathcal{G}')\\
&\leq M(k)\cdot\mu(G_k^c)\\
&\leq M(k)\,[\{1-(1-\xi)^4\}^k+\\
&\qquad+6\sum_{j=1}^{k}\{1-(1-\xi)^4\}^{k-j}\,\psi(2M(j)/3))]\,.
\end{aligned}$$

Since $M(k)=3^{2k}\cdot n$, $E_\mu(C_0^{-\varepsilon}(\mathcal{G}'))$ is finite if both

$$\sum_{k=1}^{\infty}3^{4k}\{1-(1-\xi)^4\}^k<\infty\,, \tag{38}$$

and

$$\sum_{k=1}^{\infty}3^{4k}\sum_{j=1}^{k}\{1-(1-\xi)^4\}^{k-j}\,\psi(2\cdot 3^{2j-1}n)<\infty\,. \tag{39}$$

But since $\xi>0$ can be chosen arbitrarily small, (38) is possible. Changing the order of the summation in the left hand side of (39), we obtain

$$\sum_{j=1}^{\infty}3^{4j}\,\psi(2\cdot 3^{2j-1}n)\,[1-3^4\{1-(1-\xi)^4\}]^{-1}\,,$$

which is convergent if

$$\int^{\infty}9^{2u}\cdot\psi(9^u\cdot(2n/3))\,du<\infty\,.$$

This is equivalent to (35) by the change of the variables $t=9^u$. □

3. Proof of Theorem 1

The symmetry property [b] implies that if RSW theorem holds for $\mu=\mu_{\beta,h}$, then it also holds for $\mu=\mu_{\beta,-h}$. Therefore we can assume that $h\geq 0$. We start with the special case where $h=0$, $\mathcal{G}=\mathcal{L}^*$.

Lemma 3.1. *Let* $0<\beta<\beta_c$, $\mathcal{G}=\mathcal{L}^*$ *and* $\varepsilon\in\{+,-\}$. *Then* (7)$\sim$(11) *are equivalent for* $\mu=\mu_{\beta,0}$.

Proof. In [13], the author has proved that under the assumption of the lemma, (12) holds for $\mu=\mu_{\beta,0}$ with $\alpha=2^{-23}$. Lemma 2.3 assures that (8)$\sim$(10) are equivalent under (12). Further, from Corollary 2.4, we know that (7) implies (8)$\sim$(10). Since $\beta<\beta_c$, $\mu=\mu_{\beta,0}$ satisfies [e] with exponentially decreasing mixing coefficient $\psi_{\beta,0}(n)$. Therefore we can use Lemma 2.6 to conclude that (10) implies (11).

Finally, (11) implies (7) by Lemma 2.1. □

Corollary 3.2. *Let* $0<\beta<\beta_c$, $h\geq 0$, $\mathcal{G}=\mathcal{L}^*$ *and* $\varepsilon=+$. *Then* (7)$\sim$(11)

are equivalent for $\mu=\mu_{\beta,h}$.

Proof. As we have seen in the proof of Lemma 3.1, (12) assures the equivalence of (7)$\sim$(11). Since (12) holds for $\mu=\mu_{\beta,0}$, $\mathcal{G}=\mathcal{L}^*$, $\varepsilon=+$, the monotonicity [c'] implies that (12) holds for $\mu=\mu_{\beta,h}$, $\mathcal{G}=\mathcal{L}^*$ and $\varepsilon=+$, if $h\geq 0$. Therefore (7)$\sim$(11) are equivalent for $\mu=\mu_{\beta,h}$, $h\geq 0$, $\mathcal{G}=\mathcal{L}^*$ and $\varepsilon=+$. □

The key observation is the following proposition, which we will prove in the next section.

Proposition 3.3. *Let* $0<\beta<\beta_c$. *If we have*

(7)′
$$\mu_{\beta,0}(\# C_0^-(\mathcal{L}^*)=\infty)>0\,,$$

then there exists some $h_c(\beta)>0$ *such that* (6) *holds if* $|h|<h_c(\beta)$.

Lemma 3.4. *Let* $0<\beta<\beta_c$, $h\geq 0$, $\mathcal{G}\in\{\mathcal{L},\mathcal{L}^*\}$, *and* $\varepsilon=+$. *If we assume that* (7)′ *does not hold, then* (12) *holds for* $\mu=\mu_{\beta,h}$ *and* (7)$\sim$(11) *are equivalent for* $\mu=\mu_{\beta,h}$.

Proof. Since we have the equivalence of (7)$\sim$(11) when $\mathcal{G}=\mathcal{L}^*$, the only thing we have to do is to show the equivalence of (7)$\sim$(11) when $\mathcal{G}=\mathcal{L}$. As we have mentioned in the proof of Corollary 3.2, it is sufficient to show (12) for $\mu=\mu_{\beta,0}$, $\mathcal{G}=\mathcal{L}$ and $\varepsilon=+$. To do this, first we use Lemma 2.5 for $\mathcal{G}=\mathcal{L}^*$, $\mu=\mu_{\beta,0}$ and $\varepsilon=-$. As was already noted in the proof of Lemma 3.1, (12) is true in this case, i.e. we have

$$\liminf_{n\to\infty}\ \mu_{\beta,0}(A^-_{n,3n}(\mathcal{L}^*))\geq 2^{-23}\,.$$

Therefore if we assume that

(26)′
$$\liminf_{n\to\infty}\ \mu_{\beta,0}(A^+_{n,3n}(\mathcal{L}))=0\,,$$

then from Lemma 2.5 we obtain for example

(8)′
$$\lim_{n\to\infty}\ \mu_{\beta,0}(A^-_{n,n}(\mathcal{L}^*))=1\,,$$

which, by Lemma 3.1, contradicts our assumption that (7)′ does not hold. □

To finish the proof of Theorem 1, it is, in view of Proposition 3.3, enough to show that all the statements (7)$\sim$(11) are vacant for $\mu=\mu_{\beta,h}$, $0<\beta<\beta_c$, $h\geq 0$, $\mathcal{G}\in\{\mathcal{L},\mathcal{L}^*\}$ and $\varepsilon=-$, provided that (7)′ does not hold. Assume that (7)′ does not hold. Since the event $\{\# C_0^-(\mathcal{L}^*)=\infty\}$ is decreasing, the monotonicity [c′] implies that

$$\mu_{\beta,h}(\# C_0^-(\mathcal{L}^*)=\infty)=0\,,$$

i.e. (7) does not hold for $\mu=\mu_{\beta,h}$, $0<\beta<\beta_c$, $h\geq 0$, $\mathcal{G}=\mathcal{L}^*$ and $\varepsilon=-$. By Lemma 3.4, we know that (12) holds for $\mu=\mu_{\beta,0}$, $\mathcal{G}=\mathcal{L}$ and $\varepsilon=+$, from the assumption that (7)′ does not hold.

Therefore we know by duality that all the statements (8)$\sim$(10) do not hold for $\mu=\mu_{\beta,0}$, $0<\beta<\beta_c$, $\mathcal{G}=\mathcal{L}^*$, $\varepsilon=-$.

Finally, by Lemma 2.1, (11) fails (since (7) fails) for $\mu=\mu_{\beta,0}$, $\mathcal{G}=\mathcal{L}^*$ and $\varepsilon=-$. Thus, (7)$\sim$(11) are all vacant for $\mu=\mu_{\beta,0}$, $\mathcal{G}=\mathcal{L}^*$ and $\varepsilon=-$. By monotonicity [c′], this implies that (7)$\sim$(11) are all vacant for $\mu=\mu_{\beta,h}$, $h\geq 0$, $\mathcal{G}=\mathcal{L}$ and $\varepsilon=-$. $\square$

4. Something more about the RSW theorem

First, we prove Proposition 3.3. This is done by using our previous result in [14]. To explain this, we need some notations. Let μ be strongly mixing in each direction with mixing coefficiant $\psi(n)$, decreasing exponentially as $n\to\infty$. Then we can find some $C>0$ and $N_0\geq 1$ such that

$$n^{2d-1}\,\psi(n)\leq e^{-Cn} \qquad \text{for every } n\geq N_0\,. \tag{40}$$

As in [14], section 1, we take $N_1\geq 2N_0$ large enough so that

$$\begin{aligned}\log(N_1-1)\geq \max_{t\geq 1}\,\{2\log t-t\log 2\}+3\log 10+\\ +2\log\{3(2N_1-1)/(2N_1-3)\}\,.\end{aligned} \tag{41}$$

For $n\geq N_1$, let $\pi^{\varepsilon}(n;\mathcal{G})$ denote the μ-probability of the event that the origin is $\mathcal{G}$-connected to the boundary $\partial V_{n,n}=\{(x^1,x^2)\in\boldsymbol{Z}^2;\ \max\{|x^1|,|x^2|\}=n\}$ in $\omega^{-1}(\varepsilon)$. Finally we define the subsequence $\{n(k)\}$ by

$n(1)=N_1$, $n(k+1)=2n(k)+[2n(k)/k]$ $k\geq 1$. Then we have the following fact:

Proposition 4.1. (11) *holds if there exists* $k_0\geq 1$ *such that*

$$\pi^{-\varepsilon}(n(k_0);\mathcal{G}')\cdot(2n(k_0+1)+1)<10^{-2}\,. \tag{42}$$

This is a direct consequence of Corollary 2 of [14]. Note that (42) follows automatically from (11). From this we can obtain the following:

Theorem 2. *The parameter region*

$$\mathcal{D}(\mathcal{G};\varepsilon)=\{(\beta,h);\,0<\beta<\beta_c,\ \ (7)\sim(11)\ \textit{holds for}\ \ \mu=\mu_{\beta,h}\}$$

is an open set in $\boldsymbol{R}^2$ *for each* $\mathcal{G}\in\{\mathcal{L},\mathcal{L}^*\}$ *and for each* $\varepsilon\in\{+,-\}$.

Proof. Let $(\beta,h)\in\mathcal{D}(\mathcal{G};\varepsilon)$, and $\beta_1\in(\beta,\beta_c)$. Then by Remark 1.1, every μ from the family $\{\mu_{\beta,h};\,0<\beta<\beta_1,\ h\in\boldsymbol{R}\}$ is strongly mixing with the same mixing coefficient $\psi_{\beta_1}(n)$, which decays exponentially as $n\to\infty$. Therefore we define N_0, N_1 for $\psi(n)=\psi_{\beta_1}(n)$ as above.

Since $(\beta, h)\in\mathcal{D}(\mathcal{G}, \varepsilon)$, we have (11) for $\mu=\mu_{\beta,h}$. In particular, we have

$$\lim_{n\to\infty} \pi^{-\varepsilon}(n; \mathcal{G}')\cdot n = 0 \qquad \text{for} \quad \mu = \mu_{\beta,h}\,. \tag{43}$$

Therefore there exists $k_0\geq 1$ such that

$$\pi^{-\varepsilon}(n(k_0); \mathcal{G}')\cdot(2n(k_0+1)+1)<10^{-3} \tag{44}$$

for $\mu=\mu_{\beta,h}$. Since $\pi^{-\varepsilon}(n, \mathcal{G}')$ is an expectation w.r.t. $\mu=\mu_{\beta,h}$ of a bounded continuous function on Ω, and since $\mu_{\beta,h}$ is continuous in $(\beta, h)\in(0, \beta_c)\times\boldsymbol{R}$, we can find $\delta>0$ such that (42) holds for $\mu=\mu_{\tilde{\beta},\tilde{h}}$ whenever $|\beta-\tilde{\beta}|+|h-\tilde{h}|<\delta$. Hence by Proposition 4.1, we obtain (11) for $\mu=\mu_{\tilde{\beta},\tilde{h}}$ if $|\beta-\tilde{\beta}|+|h-\tilde{h}|<\delta$. But from Theorem 1 of [15], (11) implies (10). Combining this with Lemma 2.1, we obtain that all the conditions (7)$\sim$(11) hold for $\mu=\mu_{\tilde{\beta},\tilde{h}}$, $|\beta-\tilde{\beta}|+|h-\tilde{h}|<\delta$. □

Corollary 4.2. *Let $\mathcal{G}\in\{\mathcal{L}, \mathcal{L}^*\}$, $\varepsilon\in\{+, -\}$ and $\mathcal{R}=\{(\beta, h); 0<\beta<\beta_c$, the RSW theorem holds for $\mu=\mu_{\beta,h}\}$. Then we have*

(*i*) $\{(\beta, 0); 0<\beta<\beta_c\}\subset\mathcal{R}$,

and

(*ii*) *if we define $\mathcal{R}_0(\mathcal{G}; \varepsilon)$ by*

$$\mathcal{R}_0(\mathcal{G}; \varepsilon) = \{(\beta, h)\in\mathcal{R};\quad (7)\ \textit{holds for} \quad \mu = \mu_{\beta,h}\}\,,$$

then $\mathcal{R}_0(\mathcal{G}; \varepsilon)$ is an open set in the space $\mathcal{R}$ in the sense of the relative topology in $\boldsymbol{R}^2$.

Proof. (i) is a consequence of Lemma 3.1, Lemma 3.4 and the symmetry [b] for $\mu=\mu_{\beta,0}$. (ii) is obvious from Theorem 2. □

Proof of Proposition 3.3. Immediate from Corollary 4.2, (i) and Theorem 2. □

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