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ON CHARACTER CORRESPONDENCES IN π-SEPARABLE GROUPS

Dedicated to Professor Tuyosi Oyama on his 60th birthday

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1. Introduction

Let A and G be finite groups and suppose A acts on G by automorphisms. We denote by $\mathrm{Irr}(G)$ the set of ordinary (complex) irreducible characters of G. For a prime p, $\mathrm{IBr}_p(G)$ denotes the set of all irreducible Brauer characters of G with respect to p. If φ is a class function of G and $a \in A$, φ^a, defined by $\varphi^a(g^a)=\varphi(g)$ for $g \in G$, is again a class function. For a set S of class functions of G which is stable under the action of A, we write S_A to denote the set of all A-invariant elements of S. Let π be a set of prime numbers and let π' be the set of primes complementary to π. For $\chi \in \mathrm{Irr}(G)$, we denote by $\hat{\chi}$ the restriction of χ to the set $\hat{G}$ of all π-elements of G. If $\hat{\chi}$ can not be written in the form $\hat{\chi}=\hat{\zeta}+\hat{\psi}$ with ordinary characters ζ, ψ of G, then we say that χ is π-irreducible and that $\hat{\chi}$ is a π-irreducible character of G. We denote the set of all π-irreducible characters of G by $\mathrm{I}_\pi(G)$. We say that G is π-separable if every composition factor of G is either a π-group or a π'-group.

For a π-separable group G, Isaacs [8] considered the vector space c.f. $(\hat{G})$ of all complex-valued class functions defined on $\hat{G}$ and showed that $\mathrm{I}_\pi(G)$ is a basis of c.f. $(\hat{G})$ which has the following properties.

(1) If $\chi \in \mathrm{Irr}(G)$, then $\hat{\chi}$ is a nonnegative integer linear combination of elements of $\mathrm{I}_\pi(G)$.

(2) If $\varphi \in \mathrm{I}_\pi(G)$, then $\varphi=\hat{\chi}$ for some $\chi \in \mathrm{Irr}(G)$. These imply that $\mathrm{I}_\pi(G)$ behaves as a π-generalization of Brauer characters.

Now assume that A acts on G by automorphisms and $(|A|, |G|)=1$. Under the assumption that A is solvable, Glauberman [2] established a natural bijection from $\mathrm{Irr}(G)_A$ onto $\mathrm{Irr}(C_G(A))$. If A is non-solvable, then $|A|$ is even by the Odd-order Theorem and hence $|G|$ is odd. In that case, Isaacs [4] showed that there also exists a similar bijection from $\mathrm{Irr}(G)_A$ onto $\mathrm{Irr}(C_G(A))$.

On the other hand, Uno [10] studied a character correspondence between Brauer characters. He proved that if G is p-solvable, then there exists a bijection from $\mathrm{IBr}_p(G)_A$ onto $\mathrm{IBr}_p(C_G(A))$ and this has similar properties as those of

Glauberman and Isaacs.

The purpose of this paper is to generalize this result to π-separable groups by applying Isaacs's π-generalization of Brauer characters. Namely, we have the following theorem.

Theorem. *Let A act on G such that $(|G|, |A|)=1$. Suppose G is π-separable. Then there exists a natural bijection*

$$\tilde{\Pi}(G, A): \mathrm{I}_\pi(G)_A \to I_\pi(C_G(A))$$

and the following hold.

(1) *If $B \trianglelefteq A$, then $\tilde{\Pi}(G, A)=\tilde{\Pi}(G, B)\tilde{\Pi}(C_G(B), A/B)$.*

(2) *If A is a q-group for a prime q and $\psi \in I_\pi(G)_A$, then $(\psi)\tilde{\Pi}(G, A)$ is a unique π-irreducible constituent of $\psi_{C_G(A)}$ with multiplicity prime to q.*

A p-solvable group is just a $\{p\}'$-separable group, and Uno's methods work mostly in our case, with slight modifications. However we shall reproduce them for the completeness. Also, we shall start with a review of the character correspondences of Glauberman and Isaacs, and collect, in section 3, some facts on characters of π-separable groups ([8], [9]) that will be needed to prove the above theorem. In section 4, we shall consider a correspondence of π-irreducible characters of π-separable groups and prove the theorem.

Concerning our terminologies and notations, we refer to Gorenstein [3] and Isaacs [6].

2. Character Correspondences of Glauberman and Isaacs

Here we summalize some properties of the character correspondences of Glauberman and Isaacs.

HYPOTHESIS 2.1. A acts on G such that $(|A|, |G|)=1$. Put $C=C_G(A)$ and let $\Gamma=GA$ be the semidirect product of G by A.

Theorem 2.2. *Assume Hypothesis* 2.1. *Then there is a natural bijection*

$$\Pi(G, A)\colon \mathrm{Irr}(G)_A \to \mathrm{Irr}(C)$$

and the following hold.

(1) *If $B \trianglelefteq A$ and $D=C_G(B)$, then $\Pi(G, B)$ maps $\mathrm{Irr}(G)_A$ onto $\mathrm{Irr}(D)_A$.*

(2) *In the situation of* (1), $\Pi(G, A)=\Pi(G, B)\Pi(D, A/B)$.

(3) *If A is a p-group and $\chi \in \mathrm{Irr}(G)_A$, then $(\chi)\Pi(G, A)$ is a unique irreducible constituent of χ_C such that*
$p \nmid [\chi_C, (\chi)\Pi(G, A)]$.

(4) *If α is an automorphism of Γ which leaves G and A invarint, then C is α-invariant and $(\chi^\alpha)\Pi(G, A)=\{(\chi)\Pi(G, A)\}^\alpha$ for every $\chi \in \mathrm{Irr}(G)_A$.*

(5) *Let* $\chi\in\mathrm{Irr}(G)_A$ *and* $\xi=(\chi)\Pi(G, A)$. *Then* $\boldsymbol{Q}(\chi)=\boldsymbol{Q}(\xi)$, *where* $\boldsymbol{Q}(\chi)$ *is the field obtained by adjoining all the values of* χ *to the rational field* $\boldsymbol{Q}$.

Proof. See Corollary 5.2 of [11], Theorem 2.2 and Lemma 4.2 of [10].

Theorem 2.3. *Assume Hypothesis* 2.1 *and let* N *be an* A*-invariant normal subgroup of* G. *Let* $\chi\in\mathrm{Irr}(G)_A$, $\theta\in\mathrm{Irr}(N)_A$, $T=I_G(\theta)$, $\psi=(\chi)\Pi(G, A)$ *and* $\phi=(\theta)\Pi(N, A)$ *where* $I_G(\theta)$ *denotes the inertia group of* θ *in* G. *Then the following hold.*

(1) $[\chi_N, \theta]\neq 0$ *if and only if* $[\psi_{N\cap C}, \phi]\neq 0$.

(2) $T\cap C=I_C(\phi)$ *and* $(\varphi^G)\Pi(G, A)=\{(\varphi)\Pi(T, A)\}^C$ *for every* $\varphi\in\mathrm{Irr}(T\mid\theta)$.

Proof. See Lemma 2.5 of [12].

Assume Hypothesis 2.1. Then for $\chi\in\mathrm{Irr}(G)_A$, there exists a unique extension χ' of χ to Γ such that $A\leqq\ker(\det\chi')$. We call χ' the canonical extension of χ (cf. [6], Chapter 13).

Theorem 2.4. *Assume Hypothesis* 2.1 *and let* A *is cyclic. Let* $\chi\in\mathrm{Irr}(G)_A$ *and* χ' *be the canonical extension of* χ *to* Γ. *Then there exists* $\varepsilon=\pm 1$ *such that* $\chi'(ca)=\varepsilon(\chi)\Pi(G, A)(c)$ *for all* $c\in C$ *and all generators* a *of* A.

Proof. See Theorem 3.3 of [2].

3. Results of Isaacs on $B_\pi(G)$ and $I_\pi(G)$

Suppose that G is a p-solvable group. By the Fong-Swan Theorem (cf. [1], Theorem 72.1), each $\varphi\in\mathrm{IBr}_p(G)$ is the form $\varphi=\hat{\chi}$ for some $\chi\in\mathrm{Irr}(G)$, where $\hat{\chi}$ denotes the restriction of χ to the set of p'-elements of G. From this point of view, Isaacs [7] defined a characteristic subset $\mathcal{Y}(G)\subseteq\mathrm{Irr}(G)$ such that the restriction map $\chi\to\hat{\chi}$ defines a bijection from $\mathcal{Y}(G)$ onto $\mathrm{IBr}_p(G)$. Later he generalized its construction to π-separable groups and constructed a subset $B_\pi(G)\subseteq\mathrm{Irr}(G)$ such that the restriction to the set of π-elements induces a bijection from $B_\pi(G)$ onto $I_\pi(G)$. Moreover $B_\pi(G)$ is stable under the natural action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$.

We do not mention here the definition of $B_\pi(G)$, but the main result of Isaacs [8] can be stated as follows:

Theorem 3.1. *Let* G *be* π*-separable. Then the following hold.*

(1) *The restriction map* $\psi\to\hat{\psi}$ *defines a bijection from* $B_\pi(G)$ *onto* $I_\pi(G)$. *In partiqular* $I_\pi(G)$ *is a basis of c.f.* $(\hat{G})$.

(2) *There exist nonnegative integers* $d_{\chi\varphi}$ *for* $\chi\in\mathrm{Irr}(G)$ *and* $\varphi\in B_\pi(G)$ *such that* $\hat{\chi}=\sum_{\varphi\in B_\pi(G)} d_{\chi\varphi}\hat{\varphi}$.

Proof. See Corollary 10.1 (a) and Corollary 10.2 of [8].

Let us write $\boldsymbol{Q}_\pi$ to denote the field obtained by adjoining all the complex n-th roots of unity to $\boldsymbol{Q}$, for all π-numbers n.

The following two results are concerned with the characterizations of characters in $B_\pi(G)$.

Theorem 3.2. *Let G be π-separable and let $\chi \in \mathrm{Irr}(G)$.*

(1) *If $\chi \in B_\pi(G)$, then $\chi(g) \in \boldsymbol{Q}_\pi$ for all $g \in G$.*

(2) *Assume $2 \in \pi$ or $2 \nmid |G|$. Let χ have values in $\boldsymbol{Q}_\pi$ and suppose that $\hat{\chi} \in I_\pi(G)$. Then $\chi \in B_\pi(G)$.*

Proof. See Corollary 12.1 and Theorem 12.3. of [8].

Theorem 3.3. *Let G be π-separable and let $N \trianglelefteq G$.*

(1) *If $\chi \in B_\pi(G)$, then every irreducible constituent of χ_N belongs to $B_\pi(N)$.*

(2) *If G/N is a π-group and $\psi \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G \mid \psi)$, then $\psi \in B_\pi(N)$ if and only if $\chi \in B_\pi(G)$.*

(3) *If G/N is a π'-group, then for $\psi \in B_\pi(N)$, there exists a unique irreducible constituent χ of ψ^G which belongs to $B_\pi(G)$. If ψ is G-invariant, then χ is an extension of ψ.*

Proof. See Theorem 6.2, Corollary 6.3 and Theorem 7.1 of [8].

In the rest of this section, we give facts about π-irreducible characters. The next result is an analog of Clifford's character correspondence.

Theorem 3.4. *Let G be π-separable and $N \trianglelefteq G$. Let $\theta \in I_\pi(N)$ and $T = I_G(\theta)$. Then induction defines a bijection from $I_\pi(T \mid \theta)$ onto $I_\pi(G \mid \theta)$.*

Proof. See Proposition 3.2 of [9].

Now we show the following character correspondence which can be considered as a π-generalization of Theorem 3.1 of [5], and the proof there also works in our case. However we give here a rather short proof.

Theorem 3.5. *Let G be π-separable and N be a normal subrgoup of G such that G/N is a π-group. Let $\psi \in \mathrm{Irr}(N)$ and assume*

(1) $\hat{\psi} \in I_\pi(N)$

(2) *$\psi^g = \psi$ for those $g \in G$ with $\hat{\psi}^g = \hat{\psi}$.*

Then $\wedge$ defines a one to one correspondence between $\mathrm{Irr}(G \mid \psi)$ and $I_\pi(G \mid \hat{\psi})$.

Proof. Let $\psi' \in B_\pi(N)$ be such that $\hat{\psi}' = \hat{\psi}$, $S = \mathrm{Irr}(G \mid \psi)$, $T = I_\pi(G \mid \hat{\psi})$ and $U = \{\varphi \in B_\pi(\mathrm{G});\ [\varphi_N, \psi'] \neq 0\}$. Note that the map $\wedge$ gives a one to one correspondence between T and U. Write $\psi^G = \sum_{\chi \in S} a_\chi \chi$, $\psi'^G = \sum_{\varphi \in U} b_\varphi \varphi$ with positive integers a_χ, b_φ and $\hat{\chi}=\sum_{\hat{\varphi}\in T} d_{\chi\varphi}\hat{\varphi}$ with $d_{\chi\varphi}\geq 0$. Since $\hat{\psi}^G=\hat{\psi}'^G$, we have $b_\varphi=\sum_{\chi\in S} a_\chi d_{\chi\varphi}$. On the other hand, we get $I_G(\psi)=I_G(\hat{\psi})=I_G(\hat{\psi}')$ from the assumption. If we denote this common group by I, then we have

$$\sum_{\chi\in S} a_\chi^2 = [\psi^G, \psi^G] = |I: N| = [\psi'^G, \psi'^G] = \sum_{\varphi\in U} b_\varphi^2 .$$

So

$$\sum_{\varphi\in U} (\sum_{\chi\in S} a_\chi d_{\chi\varphi})^2 = \sum_{\chi\in S} a_\chi^2 .$$

Expanding the left-hand side of the above equation, we find easily

$$\sum_{\varphi\in U} d_{\chi\varphi}^2 = 1 \quad \text{and} \quad d_{\chi\varphi} d_{\eta\varphi} = 0 \qquad \text{for } \eta\neq\chi \text{ in } S .$$

Thus $d_{\chi\varphi}=1$ for some $\varphi\in U$, while $d_{\chi\varphi'}=0$ if $\varphi'\neq\varphi$. Therefore $\hat{\chi}=\hat{\varphi}\in T$. If, for $\eta\in S$ different from χ, $\hat{\eta}=\hat{\varphi}'$ with $\varphi'\in U$, then we get $\varphi\neq\varphi'$ from the above. Hence $\hat{\eta}\neq\hat{\chi}$ and thus $\wedge$ defines an injection from S into T. If $\hat{\varphi}\in T$ with $\varphi\in U$, then $b_\varphi\neq 0$ and hence $d_{\chi\varphi}\neq 0$ for some $\chi\in S$. This yields that $\hat{\chi}=\hat{\varphi}$. Therefore $\wedge$ is a bijection.

4. Correspondences of π-Irreducible Characters

Let Ω be a subset of $\mathrm{Irr}(G)$. For a subgroup H of G, we set $\Omega(H)=\{\psi\in\mathrm{Irr}(H); [\chi_H, \psi]\neq 0 \text{ for some } \chi\in\Omega\}$.

DEFINITION. Let G be π-separable and assume that A acts on G. Let $G=G_0\trianglerighteq G_1\trianglerighteq\cdots\trianglerighteq G_n=\{1\}$ be a normal series of G and let Ω be a subset of $\mathrm{Irr}(G)$. If Ω satisfies the following two conditions:

(1) Ω is A-invariant

(2) the map $\wedge$ is a bijection from $\Omega(G_i)$ onto $I_\pi(G_i)$ for each i, $0\leq i\leq n$,

then we say, following Uno [10], that Ω has the π-lifting property with respect to A and $\{G_i\}_{i=0}^n$. If Ω has the π-lifting property with respect to A and every normal series of G, then we simply say that Ω has the π-lifting property with respect to A. If Ω has π-lifting property with respect to A, then Ω has π-lifting property with respect to B for any subgroup B of A. Furthermore, we note that $B_\pi(G)$ has the π-lifting property with respect to A.

The next lemma is a generalization of Lemma 3.4 of Uno [10].

Lemma 4.1. *Assume A acts on G. Let $G=G_0\trianglerighteq G_1\trianglerighteq\cdots\trianglerighteq G_n=\{1\}$ be a normal series of G. If $\Omega\subseteq\mathrm{Irr}(G)$ has the π-lifting property with respect to A and $\{G_i\}_{i=0}^n$, then the following hold.*

(1) *If G_i/G_{i+1} is a π-group, then $\mathrm{Irr}(G_i|\theta)\subseteq\Omega(G_i)$ for every $\theta\in\Omega(G_{i+1})$.*

(2) *If G_i/G_{i+1} is a π'-group, then $\mathrm{Irr}(G_i|\theta)\cap\Omega(G_i)$ consists of a single element for every $\theta\in\Omega(G_{i+1})$. If θ is G_i-invariant, then the element of*

$\mathrm{Irr}(G_i|\theta)\cap\Omega(G_i)$ *is an extension of* θ.

Proof. (1) Let $\theta\in\Omega(G_{i+1})$ and $\chi\in\mathrm{Irr}(G_i|\theta)$. By the π-lifting property of Ω, it follows that $\hat{\theta}\in I_\pi(G_{i+1})$ and $I_{G_i}(\theta)=I_{G_i}(\hat{\theta})$. So by Theorem 3.5. $\hat{\chi}\in I_\pi(G_i)$. Again by the π-lifting property of Ω, there exists $\psi\in\Omega(G_i)$ such that $\hat{\psi}=\hat{\chi}$. Set $\psi_{G_{i+1}}=\sum_{j=1}^{t}\eta_j$ where each $\eta_j\in\Omega(G_{i+1})$. Then every $\hat{\eta}_j\in I_\pi(G_{i+1})$ and $\hat{\theta}$ is π-irreducible constituent of $\hat{\psi}_{G_{i+1}}$. By Theorem 3.1 (1), $\hat{\theta}=\hat{\eta}_j$ for some j. Since $\wedge$ is a bijection from $\Omega(G_{i+1})$ onto $I_\pi(G_{i+1})$, it follows that $\theta=\eta_j$ and $\psi\in\mathrm{Irr}(G_i|\theta)$. Thus by Theorem 3.5, we conclude that $\chi=\psi\in\Omega(G_i)$.

(2) Let $\theta\in\Omega(G_{i+1})$ and $\chi\in\Omega(G_i)\cap\mathrm{Irr}(G_i|\theta)$. By the π-lifting property of Ω, $\hat{\theta}\in I_\pi(G_{i+1})$ and $\hat{\chi}\in I_\pi(G_i|\hat{\theta})$. Let χ' (resp. θ') be an element of $B_\pi(G_i)$ (resp. $B_\pi(G_{i+1})$) such that $\hat{\chi}'=\hat{\chi}$ (resp. $\hat{\theta}'=\hat{\theta}$). By Theorem 3.3(1), we can write $\chi'_{G_{i+1}}=\sum_j\eta_j$ where $\eta_j\in B_\pi(G_{i+1})$. Thus By Theorem 3.1 (1), $\hat{\chi}'_{G_{i+1}}=\sum_j\hat{\eta}_j$ where $\hat{\eta}_j\in I_\pi(G_{i+1})$. Thus $\hat{\eta}_j=\hat{\theta}'$ for some j and so $\eta_j=\theta'$. Applying Theorem 3.3 (3), we see that $B_\pi(G_i)\cap\mathrm{Irr}(G_i|\theta')$ consists of the single element χ'. Therefore, we obtain from Theorem 3.1 (1) that $I_\pi(G_i|\hat{\theta})=\{\hat{\chi}\}$. If $\zeta\in\Omega(G_i)\cap\mathrm{Irr}(G_i|\theta)$, then $\zeta\in I_\pi(G_i|\hat{\theta})$ and so $\hat{\zeta}=\hat{\chi}$. Thus by the π-lifting property of Ω, $\zeta=\chi$. Therefore $\Omega(G_i)\cap\mathrm{Irr}(G_i|\theta)$ consists of a single element χ. If θ is G_i-invariant, then θ' is also G_i-invariant. Thus by Theorem 3.3(3), $\chi(1)=\theta(1)$. this implies that χ is an extension of θ.

The next lemma corresponds to Proposition 3.6 of Uno [10]. However the proof needs some elabolations in part.

Lemma 4.2. *Assume Hypothesis* 2.1 *and that* G *is* π*-separable. Let* $G=G_0\rhd G_1\rhd\cdots\rhd G_n=\{1\}$ *be an* A*-composition series of* G *and let* $\Omega\subseteq\mathrm{Irr}(G)$ *have the* π*-lifting property with respect to* A *and* $\{G_i\}_{i=0}^{n}$. *Then the image of* Ω_A *by* $\Pi(G,A)$ *has the* π*-lifting property with respect to* $\{1_{Aut(C)}\}$ *and* $\{G_i\cap C\}_{i=0}^{n}$.

Proof. We proceed by induction on $|G|$.

Set $C_i=G_i\cap C$, $0\leq i\leq n$ and let $\Lambda=\{\Omega_A\}\Pi(G,A)$. First we claim that $\Lambda(C_i)=\{\Omega(G_i)_A\}\Pi(G_i,A)$ for each i, $0\leq i\leq n$. Let η be any element of $\Omega(G_i)_A$. If G_{i-1}/G_i is a π-group, then by Lemma 4.1 (1), $\mathrm{Irr}(G_{i-1}|\eta)_A\subseteq\Omega(G_{i-1})_A$ and $\mathrm{Irr}(G_{i-1}|\eta)_A$ is nonempty (cf. [6], Chapter 13). If G_{i-1}/G_i is a π'-group, then by Lemma 4.1 (2), $\Omega(G_{i-1})\cap\mathrm{Irr}(G_{i-1}|\eta)$ has just one element. Since both $\Omega(G_{i-1})$ and $\mathrm{Irr}(G_{i-1}|\eta)$ are A-invariant, $(\Omega(G_{i-1})\cap\mathrm{Irr}(G_{i-1}|\eta))_A=\Omega(G_{i-1})\cap\mathrm{Irr}(G_{i-1}|\eta)$. Therefore $\Omega(G_{i-1})_A\cap\mathrm{Irr}(G_{i-1}|\eta)$ is nonempty in any case. Applying Theorem 2.3 (1) repeatedly, we can find an element $\chi\in\Omega_A\cap\mathrm{Irr}(G|\eta)$ such that $(\chi)\Pi(G,A)\in\mathrm{Irr}(C\,|\,(\eta)\Pi(G_i,A))$. This implies that $\{\Omega(G_i)_A\}\Pi(G,A)\subseteq\Lambda(C_i)$. Conversely, let ξ be any element of $\Lambda(C_i)$ and let $\eta=(\xi)\Pi^{-1}(G_i,A)$. From the definitions of Λ and $\Lambda(C_i)$, there exists $\chi\in\Omega(G)_A$ such that ξ is a constituent of $(\chi)\Pi(G,A)_{C_i}$. By Theorem 2.3 (1), we see $\chi\in\mathrm{Irr}(G|\eta)$. Thus

$\eta\in\Omega(G_i)$. This implies that $\Lambda(C_i)\subseteqq\{\Omega(G_i)_A\}\Pi(G, A)$ and the claim is proved.

Now, if we set $\Omega_1=\Omega(G_1)$, then Ω_1 is A-invariant and $\Omega_1(G_i)=\Omega(G_i)$ for all $i\geq 1$. So, by the inductive hypothesis applied to G_1, we may assume that the restriction map $\wedge$ from $\Lambda(C_i)$ to $I_\pi(C_i)$ is a bijection for each i, $1\leq i\leq n$. So, it suffices to show that $\wedge$ gives a bijection from Λ onto $I_\pi(C)$.

If $C\leqq G_1$, then $\Lambda=\Lambda(C_1)$ and the assertion holds by the inductive hypothesis. Now we assume $C\not\leqq G_1$. Let $\theta_1, \cdots, \theta_k$ be representatives of C-orbits on $\Omega(G_1)_A$ and set $\phi_i=(\theta_i)\Pi(G_1, A)$, $1\leq i\leq k$. By Theorem 2.2 (4),

$$\phi_i^C=\{(\theta_i)\Pi(G_1, A)\}^C=(\theta_i^C)\Pi(G_1, A) \qquad \text{for all} \quad c\in C\,.$$

Thus we find that $\phi_1, \cdots, \phi_k$ are representatives of C-orbits on $\Lambda(C_1)$. Furthermore, since $\Omega(G_1)$ is G-stable, it is easy to see that $I_G(\theta_i)=I_G(\hat{\theta}_i)$ for all i. Also, we have $I_C(\phi_i)=I_C(\hat{\phi}_i)$ for all i, because we have assumed that $\{(\Omega_1)_A\}\Pi(G_1, A)$ has the π-lifting property. In particular, it follows that $\hat{\phi}_1, \cdots, \hat{\phi}_k$ are representatives of C-orbits on $I_\pi(C_1)$.

We divide the proof into two cases.

Case 1. G/G_1 is a π-group.

If $\chi\in\Omega_A$, then $\chi\in\mathrm{Irr}(G\,|\,\theta_i)$ for some i. So, by Theorem 2.3(1), we have $(\chi)\Pi(G, A)\in\mathrm{Irr}(C\,|\,\phi_i)$ and thus $\Lambda\subseteqq\bigcup_{i=1}^{k}\mathrm{Irr}(C\,|\,\phi_i)$. Conversely if $\psi\in\mathrm{Irr}(C\,|\,\phi_i)$, then from Theorem 2.3 (1) and Lemma 4.1 (1), $(\psi)\Pi^{-1}(G, A)\in\mathrm{Irr}(G\,|\,\theta_i)_A\subseteqq\Omega_A$. Thus $\psi\in\Lambda$ and we conclude that

$$\Lambda=\bigcup_{i=1}^{k}\mathrm{Irr}(C\,|\,\phi_i)\,.$$

On the other hand, we have clearly $I_\pi(C)=\bigcup_{i=1}^{k}I_\pi(C\,|\,\hat{\phi}_i)$. So, by Theorem 3.5, $\wedge$ defines a bijection from Λ onto $I_\pi(C)$.

Case 2. G/G_1 is a π'-group.

If $\chi\in\Omega_A$, then $\chi\in\mathrm{Irr}(G\,|\,\theta_i)\cap\Omega_A$ for some i. Thus $\Omega_A=\bigcup_{i=1}^{k}\mathrm{Irr}(G\,|\,\theta_i)\cap\Omega_A$. By Lemma 4.1 (2), each $\mathrm{Irr}(G\,|\,\theta_i)\cap\Omega$ consists of a single element. Since both $\mathrm{Irr}(G\,|\,\theta_i)$ and Ω are A-invariant, it follows that $\mathrm{Irr}(G\,|\,\theta_i)\cap\Omega=\mathrm{Irr}(G\,|\,\theta_i)\cap\Omega_A$, $1\leq i\leq k$. Set $\mathrm{Irr}(G\,|\,\theta_i)\cap\Omega_A=\{\chi_i\}$, $1\leq i\leq k$. By Lemma 4.1 (2), $\chi_i\neq\chi_j$ for $i\neq j$. Thus $\Omega_A=\{\chi_1, \cdots, \chi_k\}$ and $\Lambda=\{(\chi_i)\Pi(G, A);\ 1\leq i\leq k\}$. Furthermore, we obtain from Theorem 2.3 (1) that each $\hat{\phi}_i$ is a π-irreducible constituent of $\widehat{(\chi_i)\Pi(G, A)_{C_1}}$. Let $\hat{\psi}$ be any element of $I_\pi(C\,|\,\hat{\phi}_i)$. Let ϕ_i' be the element of $B_\pi(C_1)$ which corresponds to $\hat{\phi}_i$. Since $B_\pi(C)$ has the π-lifting property (with respect to the trivial action of A on C), we see from Lemma 4.1(2) that $B_\pi(C\,|\,\phi_i')$ consists of a single element. Set $B_\pi(C\,|\,\phi_i')=\{\psi_i\}$, $1\leq i\leq k$, so we have $I_\pi(C\,|\,\hat{\phi}_i)=\{\hat{\psi}_i\}$, $1\leq i\leq k$ and $I_\pi(C)=\{\hat{\psi}_1, \cdots, \hat{\psi}_k\}$. Set $T_i=I_G(\theta_i)$, $1\leq i\leq k$. For each χ_i, $1\leq i\leq k$, there exists a unique irreducible character $\xi_i\in\mathrm{Irr}(T_i\,|\,\theta_i)$ such that $\xi_i^G=\chi_i$. Then $\xi_i\in\mathrm{Irr}(T_i\,|\,\theta_i)\cap\Omega(T_i)$ and it follows from Lemma 4.1 (2) that ξ_i is an extension of θ_i. Since χ_i and θ_i are A-invariant, ξ_i is also A-invariant and $(\xi_i)\Pi(T_i, A)^C=(\chi_i)\Pi(G, A)$ by Theorem 2.3 (2).

If each $(\xi_i)\Pi(T_i, A)$ is π-irreducible, then $(\chi_i)\Pi(G, A)$ is also π-irreducible by Theorem 3.4 and thus $\widehat{(\chi_i)\Pi(G, A)}=\hat{\psi}_i$, $1\leq i\leq k$. This implies that $\wedge$ is a bijection from Λ onto $I_\pi(C)$.

In order to prove the π-irreducibility of $(\xi_i)\Pi(T_i, A)$, we consider two cases.

(a) $|A|$ is even.

From Hypothesis 2.1 and the Odd-Order Theorem, G is solvable. Thus G/G_1 is abelian and so $G=G_1C$. Now, we know that $\mathrm{Irr}(T_i|\theta_i)=\{\xi_i\lambda;\ \lambda\in \mathrm{Irr}(T_i/G_1)\}$ (cf. [6], Corollary 6.17). For any $\xi_i\lambda\in\mathrm{Irr}(T_i|\theta_i)$, we have

$$(\xi_i\lambda)^a(t)=\xi_i^a\lambda^a(t)=\xi_i(t)\lambda(\bar{t}^{a^{-1}})=\xi_i\lambda(t) \qquad \text{for all} \quad a\in A \text{ and } t\in T_i\,.$$

Therefore by Theorem 2.3(1), we have

$$|\mathrm{Irr}(T_i\cap C|\phi_i)|=|\mathrm{Irr}(T_i|\theta_i)|=|\mathrm{Irr}(T_i/G_1)|=|T_i\cap C\colon C_1|\,.$$

This implies that every element of $\mathrm{Irr}(T_i\cap C|\phi_i)$ is an extension of ϕ_i. In particular, we see that $(\xi_i)\Pi(T_i, A)$ is π-irreducible.

(b) $|A|$ is odd.

Then A is a solvable group. Let $A=A_0\rhd A_1\rhd\cdots\rhd A_m=\{1\}$ be a composition series of A and let $\{\bar{A}_i=A_{i-1}/A_i\}_{i=1}^m$ be composition factors of A. Let $X_j=C_{G_1}(A_j)$, $Y_j=C_{T_i}(A_j)$, $\sigma_j=(\theta_i)\Pi(G_1, A_j)$ and $\tau_j=(\xi_i)\Pi(T_i, A_j)$ for $j=0,\cdots,m$. Then

$$X_j=C_{X_{j+1}}(\bar{A}_{j+1}),\ Y_j=C_{Y_{j+1}}(\bar{A}_{j+1}),\ X_j\trianglelefteq Y_j\,,$$
$$\sigma_j\in\mathrm{Irr}(X_j)_{A_{j-1}} \text{ and } \tau_j\in\mathrm{Irr}(Y_j)_{A_{j-1}} \qquad \text{for} \quad j=1,\cdots,m\,.$$

$\tau_m=\xi_i$ is an extension of $\sigma_m=\theta_i$. Now, we assume that τ_j is an extension of σ_j for some $j\geq 1$. Let us denote by Δ (resp. Δ') the semidirect product $X_j\bar{A}_j$ (resp. $Y_j\bar{A}_j$) and let $\sigma\in\mathrm{Irr}(\Delta)$ (resp. $\tau\in\mathrm{Irr}(\Delta')$) be the canonical extension of σ_j (resp. τ_j). Then τ_Δ is irreducible and it satisfies $\ker(\det\tau_\Delta)\geqq\bar{A}_j$. So, by the uniqueness of the canonical extension of σ_j, we obtain that $\tau_\Delta=\sigma$. Since $\bar{A}_j$ is cyclic, there exist, by Theorem 2.4, $\varepsilon=\pm1$ and $\varepsilon'=\pm1$ such that

$$\tau(x\bar{a})=\varepsilon\tau_{j-1}(x),\ \sigma(x\bar{a})=\varepsilon'\sigma_{j-1}(x) \qquad \text{for all} \quad x\in X_{j-1} \text{ and } \bar{a}\in\bar{A}_j-\{\bar{1}\}\,.$$

So,

$$\tau_{j-1}(1)=\varepsilon\tau_\Delta(\bar{a})=\varepsilon\sigma(\bar{a})=\varepsilon\varepsilon'\sigma_{j-1}(1)\,.$$

This implies $\varepsilon=\varepsilon'$ and hence τ_{j-1} is an extension of σ_{j-1}. Repeating the same argument, we conclude that τ_0 is an extension of σ_0 and the proof is complete.

From Lemma 4.2, we immediately obtain the following corollary.

Corollary 4.3. *Assume Hypothesis* 2.1 *and that* G *is* π*-separable. Let* $\Omega\subseteq \mathrm{Irr}(G)$. *If* Ω *has the* π*-lifting property with respect to* A, *then the restriction map* $\wedge$ *gives a bijection from* $(\Omega_A)\Pi(G, A)$ *onto* $I_\pi(C)$.

Assume Hypothesis 2.1 and that G is π-separable. Suppose $\Omega\subseteq \mathrm{Irr}(G)$ has the π-lifting property with respect to A. Then $\wedge: \chi\to\hat{\chi}$ gives a bijection from Ω_A onto $I_\pi(G)_A$ and we have the sequence of bijections:

$$I_\pi(G)_A \xrightarrow{\wedge^{-1}} \Omega_A \xrightarrow{\Pi(G,A)} (\Omega_A)\Pi(G, A) \xrightarrow{\wedge} I_\pi(C)$$

where $\wedge^{-1}$ is the inverse map of $\wedge$. Thus the composite map $\tilde{\Pi}(G, A)=\wedge^{-1}\Pi(G, A)\wedge$ gives a bijection from $I_\pi(G)_A$ onto $I_\pi(C)$. From its construction, it seems likely that $\tilde{\Pi}(G, A)$ depends on the choice of Ω. However, as we shall see below, if A be solvable, then $\tilde{\Pi}(G, A)$ does not depend on the choice of Ω. On the other hand, if A is non-solvable, then $|G|$ is odd by the Odd-Order Theorem. When $|G|$ is odd, we shall show that $\Pi(G, A)$ actually gives a bijection from $B_\pi(G)_A$ onto $B_\pi(C)$. So the bijection map $\tilde{\Pi}(G, A)$ is naturally defined anyway.

The following Lemmas 4.4 and 4.5 correspond to Lemma 3.9 and Proposition 3.8 of Uno [10] respectively, and his proof also works in our case.

Lemma 4.4. *Assume Hypothesis* 2.1 *and that* A *is cyclic of prime order. Let,* $\chi, \psi\in \mathrm{Irr}(G)_A$ *be such that* $\hat{\chi}=\hat{\psi}\in I_\pi(G)$. *Then* $\widehat{(\chi)\Pi(G, A)}=\widehat{(\psi)\Pi(G, A)}$.

Proof. Let $|A|=q$.

Case 1. $q\in\pi$.

In this case, Γ/G is cyclic q-group and Γ is π-separable. Since χ and ψ are A-invariant, it follows from Theorem 3.5 that the restriction maps

$$\mathrm{Irr}(\Gamma|\chi)\to I_\pi(\Gamma|\hat{\chi}) \quad\text{and}\quad \mathrm{Irr}(\Gamma\psi)\to I_\pi(\Gamma|\hat{\psi})$$

are bijections.

Let χ' (resp. ψ') be the canonical extension of χ (resp. ψ) to Γ. Then $\mathrm{Irr}(\Gamma|\chi)=\{\mu\chi';\ \mu\in \mathrm{Irr}(A)\}$. So $\hat{\psi}'=\widehat{\mu\chi'}$ for some $\mu\in \mathrm{Irr}(A)$. By Theorem 2.4, there exist $\varepsilon=\pm1$ and $\varepsilon'=\pm1$ such that

$$(\chi)\Pi(G, A)(c) = \varepsilon\chi'(ca) \quad\text{and}$$
$$(\psi)\Pi(G, A)(c) = \varepsilon'\psi'(ca) \qquad \text{for all}\quad c\in C \text{ and } a\in A-\{1\}\,.$$

Thus for every π-element $c\in C$ and $a\in A-\{1\}$,

$$(\psi)\Pi(G, A)(c) = \varepsilon'\psi'(ca) = \varepsilon'\mu(a)\chi'(ca) = \varepsilon'\varepsilon\mu(a)(\chi)\Pi(G, A)(c)\,,$$

and in particular, $(\psi)\Pi(G, A)(1)=\varepsilon'\varepsilon\mu(a)(\chi)\Pi(G, A)(1)$. Since $\mu(a)$ is a root of unity, we obtain $\varepsilon\varepsilon'\mu(a)=1$. Thus $\widehat{(\chi)\Pi(G, A)}=\widehat{(\psi)\Pi(G, A)}$ as required.

Case 2. $q\in\pi'$.

Set $\pi_0=\pi\cup\{q\}$. Then G is π_0-separable. Since q does not divide $|G|$, $I_{\pi_0}(G)=I_\pi(G)$ and $\hat{\chi}=\hat{\psi}$ is a π_0-irreducible character. So the assertion is clear from the case 1.

Lemma 4.5. *Assume Hypothesis* 2.1 *and that G is π-separable and A is solvable. Let $B\trianglelefteq A$, $D=C_G(B)$ and assume that $\Omega\subseteq \mathrm{Irr}(G)$ and $\Lambda\subseteq\mathrm{Irr}(D)$ both have the π-lifting property with respect to A. Let $\chi\in\Omega_A$ and let ϕ be the unique element of $\Lambda_{A/B}$ such that $\hat{\phi}=\widehat{(\chi)\Pi(G, B)}$. (Note that $(\chi)\Pi(G, B)$ is π-irreducible by Corollary* 4.3). *Then $\widehat{(\chi)\Pi(G, A)}=\widehat{(\phi)\Pi(D, A/B)}$.*

Proof. We proceed by induction on $|A|$.

We may assume $A\neq B$. Let H be a maximal normal subgroup of A containing B. By Corollary 4.3, $\widehat{(\chi)\Pi(G, H)}$ and $\widehat{(\phi)\Pi(D, H/B)}$ are both contained in $I_\pi(C_G(H))$. From the inductive hypothesis, we may assume that $\widehat{(\chi)\Pi(G, H)}=\widehat{(\phi)\Pi(D, H/B)}$. Since A/H is a cyclic group of prime order, it follows from Lemma 4.4 that

$$\widehat{\{(\chi)\Pi(G, H)\}\Pi(C_G(H), A/H)}=\widehat{\{(\phi)\Pi(D, H/B)\}\Pi(C_G(H), A/H)}\,.$$

Thus the assertion follows immediately from Theorem 2.2 (2).

We are now ready to prove the following theorem.

Theorem 4.6. *Assume Hypothesis* 2.1 *and that G is π-separable and A is solvable. Then there exists a bijection*

$$\tilde{\Pi}(G, A)\colon I_\pi(G)_A\to I_\pi(C)\,,$$

which is independent of the choice of Ω which satisfies the π-lifting property with respect to A. And the following hold.

(1) *If $B\trianglelefteq A$, then $\tilde{\Pi}(G, A)=\tilde{\Pi}(G, B)\tilde{\Pi}(C_G(B), A/B)$.*

(2) *If A is a q-group for a prime q and $\psi\in I_\pi(G)_A$, then $(\psi)\tilde{\Pi}(G, A)$ is a unique π-irreducible constituent of ψ_C with multiplicity prime to q.*

Proof. In Lemma 4.5, let $B=\{1\}$. Then we see that $\tilde{\Pi}(G, A)$ is independent of the choice of Ω which satisfies the π-lifting property with respect to A. If $B\trianglelefteq A$, it is easily seen by Lemma 4.5 that $\tilde{\Pi}(G, A)=\tilde{\Pi}(G, B)\tilde{\Pi}(C_G(B), A/B)$. Now fix Ω which satisfies π-lifting property with respect to A. For $\psi\in I_\pi(G)_A$, there exists $\chi\in\Omega$ such that $\hat{\chi}=\psi$. If A is a q-group, then by Theorem 2.2 (3), $\chi_C=m(\chi)\Pi(G, A)+q\zeta$ where m is a positive integer prime to

q and ζ is zero or a character of C. Therefore by the definition of $\tilde{\Pi}(G, A)$, $\psi_C=m(\psi)\tilde{\Pi}(G, A)+q\hat{\zeta}$. This completes the proof.

Finally we note the following fact which asserts that $\Pi(G, A)$ induces a bijection between B_π-characters under certain circumstances.

Theorem 4.7. *Assume Hypothesis* 2.1 *and that* G *is* π*-separable. If either* $2\in\pi$ *or* $2\nmid |G|$, *then* $\Pi(G, A)$ *gives a bijection from* $B_\pi(G)_A$ *onto* $B_\pi(C)$.

Proof. It suffices to show that if $\chi\in B_\pi(G)_A$, then $\xi=(\chi)\Pi(G, A)\in B_\pi(C)$. We know that $\hat{\xi}\in I_\pi(C)$ by Corollary 4.3 (applied to $\Omega=B_\pi(G)$) and ξ has values in $\boldsymbol{Q}_\pi$ by Theorem 2.2 (5) and Theorem 3.2 (1). Thus we conclude from Theorem 3.2 (2) that $\xi\in B_\pi(C)$.

Under the assumption of the above theorem, we get the natural bijection $\tilde{\Pi}(G, A)$: $I_\pi(G)\to I_\pi(C)$. And clearly this coincides with the bijection $\tilde{\Pi}(G, A)$ given in Theorem 4.6, provided A is solvable. Thus the proof of our main theorem is completed.

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