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CANCELLATION OF FINITELY GENERATED MODULES OVER REGULAR RINGS

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Introduction

In [3, Theorem 2] Evans proved that an R-module M cancels from direct sums if the stable range of the endomorphism ring $\mathrm{End}_R(M)$ is 1. By using this result it follows that finitely generated projective modules over unit-regular rings cancel from direct sums [4, Corollary 4.7 and Proposition 4.12]. However, as the first author has shown [4, Example 5.13], there exists a unit-regular ring R with a cyclic module M which is directly infinite, that is, $M \cong M \oplus A$ for some nonzero R-module A, and so in particular M cannot be cancelled. Later, Menal [7, Theorem D] proved that if R is a regular ring whose primitive factor rings are all artinian, then the stable range of $\mathrm{End}_R(M)$ is 1 for every finitely generated R-module M and so again Evans' theorem implies cancellation for M. However, the converse of Menal's result is not true for arbitrary regular rings. For instance, it suffices to take any regular locally finite-dimensional algebra over a field whose primitive factor rings are not artinian. For more information on this subject we refer the reader to [8].

In this note we will show that if R is a regular ring which is N^*-complete, or right or left $\aleph_0$-continuous, or left $\aleph_0$-injective, then every finitely generated right R-module cancels from direct sums if and only if R has bounded index of nilpotence (and so all primitive factor rings of R are artinian). As an application we also obtain a characterization of those polynomial rings $R[x]$ that are semihereditary in the case that R is either N^*-complete or right or left $\aleph_0$-continuous.

Cancellation

All rings considered in this paper are associative with 1. A ring R is (*von Neumann*) *regular* if for any $x \in R$ there exists a $y \in R$ such that $x = xyx$. If we can always choose y to be a unit, then R is called *unit-regular*. If X is a subset

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of R, then we denote by $l(X)$ and $r(X)$ its left and right annihilators. By a *subring* we always mean a *unital* subring.

The example mentioned above, of a unit-regular ring which has a directly infinite cyclic module, is just $\prod_{n=1}^{\infty} M_n(F_n)$, where $F_1, F_2, \cdots$ are fields. Our method is to look for subrings of this form inside regular rings of index ∞, using N^*-completeness, or $\aleph_0$-continuity, or $\aleph_0$-injectivity to build a complete direct product.

Lemma 1. *If R is a nonzero regular ring, then there is a nonzero central idempotent $e \in R$ such that eR is an algebra over a field.*

Proof. If R is not a $\boldsymbol{Q}$-algebra, then there is a prime p such that $pR \neq R$. Since p is central, $l(pR)=eR$ for some nonzero central idempotent $e \in R$. Now eR is an algebra over $\boldsymbol{Z}/p\boldsymbol{Z}$.

Recall that the *index of nilpotence* of an ideal I in a ring R is the supremum of the indices of all nilpotent elements in I. If this supremum is finite, we say I has *bounded index* (*of nilpotence*), while if this supremum is infinite we say I has *index* ∞. The next lemma is a key result in this paper.

Lemma 2. *Let R be a regular ring and let I be a two-sided ideal of R of index ∞. Then there exist nonzero orthogonal idempotents $e_1, e_2, \cdots \in I$ such that e_nRe_n has a subring isomorphic to the $n \times n$ matrix algebra over a field.*

Proof. We first observe that it suffices to prove (1): There exist independent right ideals $A_1, A_2, \cdots$ contained in I such that A_n contains a direct sum of n nonzero pairwise isomorphic right ideals. For suppose (1) holds. Each A_n contains a right ideal B_n which is a direct sum of n nonzero pairwise isomorphic principal right ideals, and then B_n is principal. By [4, Proposition 2.13] there exist orthogonal idempotents $f_1, f_2, \cdots$ such that

$$f_1R \oplus \cdots \oplus f_nR = B_1 \oplus \cdots \oplus B_n$$

for all n. Then $f_n \in I$ and $f_nR \cong B_n$ for each n, so f_nR is a direct sum of n nonzero pairwise isomorphic right ideals, whence f_nRf_n is isomorphic to an $n \times n$ matrix ring. By Lemma 1, there is a nonzero central idempotent e_n in the ring f_nRf_n such that $e_nRe_n = e_nf_nRf_n$ is an algebra over a field. Then $e_n \in I$ and e_nRe_n has a subring isomorphic to the $n \times n$ matrix algebra over a field. Since the $f's$ are orthogonal, so are the $e's$.

For each n, let J_n be the sum of all ideals contained in I of index at most n. Then $J_1 \subseteq J_2 \subseteq \cdots$, and by [4, Corollary 7.8] each J_n has index at most n. Set $B_n = J_{n+1} \cap l(J_n)$ for each n. Observe that $(B_n \cap J_n)^2 \subseteq B_nJ_n = 0$, and so $B_n \cap J_n = 0$. Hence, the $B's$ are independent. When $B_n \neq 0$, then since $B_n \subseteq J_{n+1}$ and $B_n \cap J_n = 0$ we see that B_n has index $n+1$. In this case, [4, Theorem 7.2] implies that B_n contains a direct sum of $n+1$ nonzero pairwise isomorphic right ideals. If infinitely many B's are nonzero, say $B_{n(1)}, B_{n(2)}, \cdots$ where $n(1)<n(2)<\cdots$, then each $B_{n(k)}$ contains a direct sum of k nonzero pairwise isomorphic right ideals, and (1) is proved. Thus we may assume that only finitely many B's are not zero, and so for some $t\in\boldsymbol{N}$ we have $B_n=0$ for all $n\geq t$. Then for $n\geq t$ we have $J_{n+1}\cap l(J_n)=0$, and it follows that $J_n\leq_e J_{n+1}$ as right ideals of R. From [4, Corollary 7.5] the index of J_{n+1} is now at most n, whence $J_{n+1}=J_n$. Thus $J_n=J_t$ for $n\geq t$.

Set $A=l(J_t)\cap I$, and observe that $A\cap J_t=0$. Since I has index ∞, [4, Corollary 7.5] shows that J_t is not essential in I_R, and so $A\neq 0$. Now $A\cap J_n=0$ for all n. Thus A and all nonzero ideals contained in A have index ∞. If A contains an independent sequence of nonzero ideals $A_1, A_2, \cdots$, then by [4, Theorem 7.2] each A_n contains a direct sum of n nonzero pairwise isomorphic right ideals and (1) is proved. Thus we may assume that A does not contain an infinite sequence of independent nonzero ideals.

Now A must contain a nonzero ideal B such that any two nonzero ideals contained in B have nonzero intersection. It follows that any two nonzero right ideals $K, L\subseteq B$ must contain nonzero isomorphic right ideals. For taking nonzero elements $x\in K$ and $y\in L$, we have $RxR\cap RyR\neq 0$, whence $xay\neq 0$ for some $a\in R$, and $xayR$ is isomorphic to a right ideal contained in yR. By induction, it follows that whenever $B_1, \cdots, B_k$ are nonzero right ideals contained in B, there exist nonzero right ideals $C_i\subseteq B_i$ for $i=1, \cdots, k$ such that $C_i\cong C_j$ for all i, j.

As $B\neq 0$, it has infinite index, so it cannot be artinian. Thus B contains an infinite direct sum of nonzero right ideals. Grouping finitely many of these together at a time, we obtain nonzero independent right ideals $A_1, A_2, \cdots\subseteq B$ such that each A_n is a direct sum of n nonzero right ideals. Invoking the result of the previous paragraph, we conclude that each A_n contains a direct sum of n nonzero pairwise isomorphic right ideals. Therefore (1) holds in this case too.

Recall that a *pseudo-rank function* on a regular ring R is a map $P\colon R\to[0, 1]$ such that

(a) $P(1)=1$;

(b) $P(xy)\leq P(x), P(y)$ for all $x, y\in R$;

(c) $P(e+f)=P(e)+P(f)$ for all orthogonal idempotents $e, f\in R$.

Denote by $\boldsymbol{P}(R)$ the set of all pseudo-rank functions on R, and let

$$N^*(x)=\sup\{P(x)\colon P\in\boldsymbol{P}(R)\}$$

for all $x\in R$. Then the rule $\delta(x, y)=N^*(x-y)$ defines a pseudo-metric on R; we say R is *N^*-complete* if δ is a metric and R is complete with respect to it.

Lemma 3. *Let R be an N^*-complete regular ring of index ∞. Then R*

has a subring isomorphic to $T\times\prod_{n=1}^{\infty} M_n(F_n)$ *for some regular ring* T *and some fields* $F_1, F_2, \cdots$.

Proof. From the idempotents given in Lemma 2 we can choose non-zero orthogonal idempotents $f_1, f_2, \cdots \in R$ such that each f_nRf_n has a subring isomorphic to the $n2^n\times n2^n$ matrix algebra over a field F_n. Then f_n is a sum of orthogonal idempotents g_{ni} (for $i=1, \cdots, n2^n$) such that $g_{ni}R\cong g_{nj}R$ for all i, j and F_n is isomorphic to a subring of $g_{n1}Rg_{n1}$. Since $n2^n(g_{ni}R)\cong f_nR\subseteq R_R$, we have $N^*(g_{ni}) \leq 1/n2^n$. Set $h_n=g_{n1}+\cdots+g_{nn}$, and observe that

$$N^*(h_n)\leq N^*(g_{n1})+\cdots+N^*(g_{nn})\leq n/n2^n = 1/2^n .$$

Also $h_nRh_n\cong M_n(g_{n1}Rg_{n1})$, and so $M_n(F_n)$ is isomorphic to a subring of h_nRh_n.

Given any sequence $x=(x_n)\in\prod_{n=1}^{\infty} h_nRh_n$, we have $N^*(x_n)\leq N^*(h_n)\leq 1/2^n$ for all n, so the partial sums of $\sum x_n$ are Cauchy with respect to N^*. Hence $\sum x_n$ converges to some $\phi(x)\in R$. In particular $\sum h_n$ converges to an idempotent $h\in R$. Then ϕ gives a ring isomorphism of $\prod_{n=1}^{\infty} h_nRh_n$ onto a subring of hRh, and so $\prod_{n=1}^{\infty} M_n(F_n)$ is isomorphic to a subring of hRh.

Taking $T=(1-h)R(1-h)$, the proof is complete.

We say that a regular ring R is *right* $\aleph_0$*-continuous* if the lattice of principal right ideals $L(R_R)$ is *upper* $\aleph_0$*-continuous*, that is, every countable subset of $L(R_R)$ has a supremum in $L(R_R)$ and $A\wedge(\vee B_n)=\vee(A\wedge B_n)$ for all A and all countable linearly ordered subsets $\{B_n\}$ in $L(R_R)$. For example, any right self-injective regular ring is right $\aleph_0$-continuous [4, Corollary 13.5].

Recall that a ring R is called *right* (*left*) $\aleph_0$*-injective* provided every homomorphism from a countably generated right (left) ideal into R is given by left (right) multiplication by an element of R.

Lemma 4. *Let* R *be a regular ring of index* ∞ *which is either right or left* $\aleph_0$*-continuous. Then* R *has a subring isomorphic to* $T\times\prod_{n=1}^{\infty} M_n(F_n)$ *for some regular ring* T *and some fields* $F_1, F_2, \cdots$.

Proof. By symmetry, we may assume that R is right $\aleph_0$-continuous. By Lemma 2, there exist nonzero orthogonal idempotents $e_1, e_2, \cdots\in R$ such that each e_nRe_n has a subring isomorphic to the $n\times n$ matrix algebra over a field F_n. By [4, Corollary 14.4] there is an idempotent $e\in R$ such that $\bigoplus_{n=2}^{\infty} e_nR\leq_e eR$, and it suffices to show that $\prod_{n=2}^{\infty} M_n(F_n)$ is isomorphic to a subring of eRe (since it is clear how to find a subring of $(1-e)R(1-e)$ isomorphic to $T\times F_1$).

Let S be the maximal right $\aleph_0$-quotient ring of R (see [4, Chapter 14]), and note that $\bigoplus_{n=2}^{\infty} e_nS\leq_e eS$. Any sequence $x=(x_n)\in\prod_{n=2}^{\infty} e_nSe_n$ induces a homomorphism

$$\bigoplus_{n=2}^{\infty} e_n S \rightarrow \bigoplus_{n=2}^{\infty} e_n S \subseteq eS$$

which extends uniquely to a homomorphism $eS \rightarrow eS$ because S is right $\aleph_0$-injective [4, Theorem 14.12], and this homomorphism is left multiplication by some unique element $\phi(x) \in eSe$. We observe that ϕ is a unital ring map from $\prod_{n=2}^{\infty} e_n Se_n$ into eSe, and that ϕ is injective.

Now eSe has a subring $S' \cong \prod_{n=2}^{\infty} M_n(F_n)$. Since S' is regular and right self-injective with no nonzero abelian central idempotents, S' is generated as a ring by its idempotents [4, Theorem 13.16]. But all idempotents of S lie in R [4, Theorem 14.12]. Therefore S' is a subring of eRe.

For $\aleph_0$-injective regular rings, we have a weaker version of Lemmas 3 and 4, which is not left-right symmetric.

Lemma 5. *Let R be a left $\aleph_0$-injective regular ring of index ∞. Then R has a subring S with a two-sided ideal H such that $(S/H)_S$ is flat and $S/H \cong \prod_{n=1}^{\infty} M_n(F_n)$ for some fields $F_1, F_2, \cdots$.*

Proof. By Lemma 2, there exist nonzero orthogonal idempotents $e_1, e_2, \cdots \in R$ such that e_nRe_n has a subring isomorphic to $M_n(F_n)$ for some field F_n. Let $J = \bigoplus_{n=1}^{\infty} Re_n$ and let $\boldsymbol{I}(J)$ be the idealizer of J in R and observe that the right annihilator $r(J)$ is a two-sided ideal of $\boldsymbol{I}(J)$.

Any sequence $x = (x_n) \in \prod_{n=1}^{\infty} e_nRe_n$ induces a homomorphism $J \rightarrow J \subseteq_R R$ which must be right multiplication by some $\phi(x) \in \boldsymbol{I}(J)$, because R is left $\aleph_0$-injective. Although $\phi(x)$ is not uniquely determined by x, it is unique modulo $r(J)$. Thus the rule $x \mapsto \phi(x) + r(J)$ defines a unital ring map from $\prod_{n=1}^{\infty} e_nRe_n$ into $\boldsymbol{I}(J)/r(J)$, and this map is injective. Therefore $\boldsymbol{I}(J)$ has a subring S such that $S \supseteq r(J)$ and $S/r(J) \cong \prod_{n=1}^{\infty} M_n(F_n)$.

Since $r(J)$ is a right ideal of R, it is a directed union of right ideals eR where e is an idempotent. For any such e, observe that $S = eR \oplus (1-e)S$, so that eR is a direct summand of S_S. Therefore $S/r(J)$ is a flat right S-module.

A module M is called *directly finite* provided M is not isomorphic to any proper direct summand of itself, that is, $M \not\cong M \oplus A$ for all nonzero modules A. If M is not directly finite then M is called *directly infinite*.

Parts of the following theorem are due to Evans [3] and Menal [7].

Theorem 6. *Let R be a regular ring which is N^*-complete, or right or left $\aleph_0$-continuous, or left $\aleph_0$-injective. Then the following conditions are equivalent:*

(a) *R has bounded index of nilpotence.*

(b) *All primitive factor rings of R are artinian.*

(c) *The endomorphism ring of every finitely generated right R-module has stable range* 1.

(d) *All finitely generated right R-modules cancel from direct sums.*

(e) *All finitely generated right R-modules are directly finite.*

Proof. (a)$\Rightarrow$(b) is [4, Corollary 7.10]. (b)$\Rightarrow$(c) is by [7, Theorem D]. (c)$\Rightarrow$(d) is [3, Theorem 2]. (d)$\Rightarrow$(e) is clear.

Now we prove (e)$\Rightarrow$(a). Suppose that R has index ∞. By Lemma 3, 4, or 5, R has a subring S with a two-sided ideal H such that $(S/H)_S$ is flat and $S/H \cong \prod_{n=1}^{\infty} M_n(F_n)$ for some fields $F_1, F_2, \cdots$. By [4, Example 5.13], S/H has a cyclic right module M which is directly infinite.

Then $M \cong M \oplus A$ for some nonzero right (S/H)-module A, and consequently

$$M \otimes_S R \cong (M \otimes_S R) \oplus (A \otimes_S R) .$$

Since S/H is regular, A is flat as an (S/H)-module, and then since $(S/H)_S$ is flat, A is flat as an S-module. Hence, the natural map $A \otimes_S S \to A \otimes_S R$ is injective, and so $A \otimes_S R \neq 0$. But then $M \otimes_S R$ is a directly infinite cyclic right R-module, contradicting (e). Therefore R has bounded index.

In case the ring R in Theorem 6 is either N^*-complete or right or left $\aleph_0$-continuous, the given conditions are also equivalent to the corresponding left module versions of conditions (c), (d), (e) (because conditions (a), (b) are left-right symmetric). We do not know whether Theorem 6 holds for right $\aleph_0$-injective regular rings.

Semihereditary Polynomial Rings

By applying Lemmas 3 and 4 we will obtain a result on semihereditary polynomial rings. First we need a relatively well-known lemma.

Lemma 7. *Let S be a regular subring of a ring R. If $R[x]$ is right semihereditary, then so is $S[x]$.*

Proof. Since S is regular, ${}_S R$ is faithfully flat, and then [1, Lemma 3] shows that ${}_{S[x]}R[x]$ is faithfully flat. Then $S[x]$ is right coherent by [6, Corollary 2.1]. As S is regular, $S[x]$ has weak global dimension 1, and therefore $S[x]$ must be right semihereditary.

Recall that a ring R is *strongly π-regular* if for each element $a \in R$ there is a positive integer n such that $a^n R = a^{n+1} R$. That this condition is left-right symmetric was proved by Dischinger [2, Théorème 1].

Theorem 8. *Let R be a regular ring which is either N^*-complete or right or left $\aleph_0$-continuous. Then the following conditions are equivalent:*

(a) *$R[x]$ is right semihereditary.*
(b) *$R[x]$ is left semihereditary.*
(c) *R has bounded index of nilpotence*
(d) *R is strongly π-regular.*

Proof. (c)$\Rightarrow$(a) and (b) by [4, Corollary 7.10] and [5, Corollaire].

(a) or (b)$\Rightarrow$(c): If R has index ∞, then by Lemma 3 or 4 R has a subring $S\cong T\times\prod_{n=1}^{\infty} M_n(F_n)$ for some regular ring T and some fields $F_1, F_2, \cdots$. Then [5, Proposition 11] implies that $S[x]$ is neither right nor left semihereditary. But in view of Lemma 7 this contradicts (a) and (b). Therefore R must have bounded index.

(c)$\Rightarrow$(d) by [4, Theorem 7.15].

(d)$\Rightarrow$(c): If R has index ∞, then by Lemma 3 or 4, R has a subring $S\cong T\times\prod_{n=1}^{\infty} M_n(F_n)$ as before. Choose matrices $a_n\in M_n(F_n)$ such that a_n is nilpotent of index n. Then $(0, a_1, a_2, \cdots)$ corresponds to an element $a\in S$ such that $l_S(a^n)\neq l_S(a^{n+1})$ for all $n=1, 2, \cdots$. But then $l_R(a^n)\neq l_R(a^{n+1})$ and so $a^nR\neq a^{n+1}R$ for all n, contradicting (d). Therefore R has bounded index.

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