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MODULE CORRESPONDENCE IN AUSLANDER-REITEN QUIVERS FOR FINITE GROUPS

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1. Introduction

Let G be a finite group and k be a field of characteristic $p>0$. Let Θ be a connected component of the stable Auslander-Reiten quiver $\Gamma_s(kG)$ of the group algebra kG and set $V(\Theta)=\{vx(M) \mid M$ is an indecomposable kG-module in $\Theta\}$, where $vx(M)$ denotes the vertex of M. As we shall see in Proposition 3.2 below, if Q is a minimal element in $V(\Theta)$, then $Q\leq_G H$ for all $H\in V(\Theta)$. In particular we see that Q is uniquely determined up to conjugation in G.

Let $N=N_G(Q)$ and let f be the Green correspondence with respect to (G, Q, N). Choose an indecomposable kG-module M_0 in Θ with Q its vertex. Let Δ be the connected component of $\Gamma_s(kN)$ containing fM_0. The purpose of this paper is to show that there is a subquiver Λ of Δ and a graph isomorphism $\psi\colon \Lambda\rightarrow\Theta$ such that ψ^{-1} behaves like the Green correspondence f as a bijective map between modules in Λ and those in Θ. In particular Θ is isomorphic with a subquiver of Δ. Also it will be shown that if $H\in V(\Theta)$, then $H\leq_G N_G(Q)$.

The notation is almost standard. All the modules considered here are finite dimensional over k. We write $W \mid W'$ for kG-modules W and W', if W is isomorphic to a direct summand of W'. For an indecomposable non-projective kG-module M, we write $\mathcal{A}(M)$ to denote the Auslander-Reiten sequence terminating at M. A sequence $M_0 - M_1 - \cdots - M_t$ of indecomposable kG-modules M_i $(0\leq i\leq t)$ is said to be a *walk* if there exists either an irreducible map from M_i to M_{i+1} or an irreducible map from M_{i+1} to M_i for $0\leq i\leq t-1$. Concerning some basic facts and terminologies used here, we refer to [1], [5], [6] and [8].

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2. Preliminaries

To begin with, we recall some basic facts on relative projectivity.

Let H be a subrgoup of G and $\{g_i\}_{i=1}^n$ be a right transversal of H in G. If W and W' are kG-modules, then $(W, W')^H$ denotes the k-space $\mathrm{Hom}_{kH}(W, W')$.

The trace map $t_H^G: (W, W')^H \to (W, W')^G$ is defined by $t_H^G(\phi) = \sum_{i=1}^n \phi \cdot g_i$ for $\phi \in (W, W')^H$. For a set $\mathfrak{Z}$ of subgroups of G, write $(W, W')_{\mathfrak{Z}}^G = \sum_{V \in \mathfrak{Z}} \mathrm{Im}(t_V^G)$ and $(W, W')^{\mathfrak{Z},G} = (W, W')^G/(W, W')_{\mathfrak{Z}}^G$. A kG-homomorphism φ is said to be $\mathfrak{Z}$-projective, if $\varphi \in (W, W')_{\mathfrak{Z}}^G$. A kG-module W is said to be $\mathfrak{Z}$-projective, if $W \mid \sum_{V \in \mathfrak{Z}} \oplus (W\downarrow_V)\uparrow^G$.

For a set $\mathfrak{Z}$ of subgroups of G, we set $\tilde{\mathfrak{Z}} = \mathfrak{Z} \cap_G H = \{V^g \cap H \mid V \in \mathfrak{Z}, g \in G\}$.

Lemma 2.1 ([8], Theorem 2.3). *With the notation above, let $\varphi \in (W, W')^G$.*

(1) *φ is $\mathfrak{Z}$-projective if and only if φ factors through a $\mathfrak{Z}$-projective module.*

(2) *If W or W' is $\mathfrak{Z}$-projective, then φ is $\mathfrak{Z}$-projective.*

Lemma 2.2.

(1) ([8], Cor. 5.4) *For a kG-module A and a kH-module B, the following k-isomorphisms hold:*

$$(A\downarrow_H, B)^{\tilde{\mathfrak{Z}},H} \simeq (A, B\uparrow^G)^{\mathfrak{Z},G},$$

$$(B, A\downarrow_H)^{\tilde{\mathfrak{Z}},H} \simeq (B\uparrow^G, A)^{\mathfrak{Z},G}.$$

(2) *In particular, for kH-modules A and B, the following k-isomorphism holds:*

$$((A\uparrow^G)\downarrow_H, B)^{\tilde{\mathfrak{Z}},H} \simeq (A, (B\uparrow^G)\downarrow_H)^{\tilde{\mathfrak{Z}},H}.$$

The next two results are also well-known.

Lemma 2.3 ([1], Prop. 2.17.10). *Let M be an indecomposable non-projective kG-module and H be a subgroup of G. Then the Auslander-Reiten sequence $\mathcal{A}(M)$ splits on restriction to H if and only if H does not contain $vx(M)$.*

Lemma 2.4 ([4], Lemma 1.5 and [7], Theorem 7.5). *Let H be a subgroup of G. Let M and L be indecomposable non-projective modules for G and H respectively. Assume that L is a direct summand of $(L\uparrow^G)\downarrow_H$ with multiplicity one, and that M is a direct summand of $L\uparrow^G$ such that $L \mid M\downarrow_H$. Then $\mathcal{A}(L)\uparrow^G \simeq \mathcal{A}(M) \oplus \mathcal{E}$, where $\mathcal{E}$ is a split sequence.*

Finally we note:

Lemma 2.5. *Let P be a non-trivial p-subgroup of G. Let M and L be indecomposable non-projective modules for G and $N_G(P)$ respectively. Assume that $\mathcal{A}(L)\uparrow^G \simeq \mathcal{A}(M) \oplus \mathcal{E}$, where $\mathcal{E}$ is a split sequence and that $P \leq_G vx(L)$. If M is not a direct summand of the middle term of $\mathcal{A}(L)\uparrow^G$, then $\mathcal{A}(M)\downarrow_{N_G(P)} \simeq \mathcal{A}(L) \oplus \mathcal{E}'$, where $\mathcal{E}'$ is a P-split sequence.*

Proof. Using the same argument as in the proof of [3], (2.3) Lemma (a), we have $\mathcal{A}(M)\downarrow_{N_G(P)} \simeq \mathcal{A}(L) \oplus \mathcal{E}'$, where $\mathcal{E}'$ is some exact sequence. Therefore we have only to show that $\mathcal{E}'$ is a P-split sequence. Let (,) denote the inner product on the Green ring $a(kG)$ induced by $\dim_k \mathrm{Hom}_{kG}(\ ,\ )$ [2]. For an exact sequence of kG-modules Δ: $0 \to A \to B \to C \to 0$, put $\mathcal{H}(\Delta) = B - A - C$. By [2], Theorem 3.4, it is sufficient to show that $(\mathcal{H}(\mathcal{E}')\downarrow_P, W) = 0$ for any kP-module W. Using the Frobenius reciprocity, we have

$$
\begin{aligned}
&(\mathcal{H}(\mathcal{E}')\downarrow_P, W) \\
&= (\mathcal{H}(\mathcal{A}(M))\downarrow_P, W) - (\mathcal{H}(\mathcal{A}(L))\downarrow_P, W) \\
&= (\mathcal{H}(\mathcal{A}(M)), W\uparrow^G) - (\mathcal{H}(\mathcal{A}(L)), W\uparrow^N) \\
&= (\mathcal{H}(\mathcal{A}(L)), (W\uparrow^G)\downarrow_N) - (\mathcal{H}(\mathcal{A}(L)), W\uparrow^N),
\end{aligned}
$$

where $N = N_G(P)$. By the Mackey decomposition, $(W\uparrow^G)\downarrow_N \simeq W\uparrow^N \oplus W'$, where W' is $\{P^g \cap N \mid g \in G\backslash N\}$-projective. Since L is not $\{P^g \cap N \mid g \in G\backslash N\}$-projective, we have $(\mathcal{H}(\mathcal{A}(L)), W') = 0$. Consequently we get $(\mathcal{H}(\mathcal{E}')\downarrow_P, W) = 0$ as desired.

3. Minimal element in $V(\Theta)$

Let Ξ be a subgraph of the stable Auslander-Reiten quiver $\Gamma_s(kG)$ and set $V(\Xi) = \{vx(M) \mid M \in \Xi\}$. Note that every element in $V(\Xi)$ is a non-trivial p-subgroup of G since every M is non-projective. The following Lemma 3.1 is essential in our argument.

Lemma 3.1. *Let Ξ be a subgraph of $\Gamma_s(kG)$. Assume that Ξ is connected. Take any $Q \in V(\Xi)$ with the smallest order among those p-subgroups in $V(\Xi)$. Then for any indecomposable module $M \in \Xi$, $M\downarrow_Q$ has an indecomposable direct summand whose vertex is Q.*

Proof. Let $M_0 \in \Xi$ be such that $Q = vx(M_0)$. As Ξ is connected, there is a walk $M_0 - M_1 - \cdots - M_t = M$, so that M_i is a direct summand of the middle term of the Auslander-Reiten sequence $\mathcal{A}(M_{i+1})$ or $\mathcal{A}(\Omega^{-2}M_{i+1})$. We proceed by induction on the "distance" t. Suppose that $M_{t-1}\downarrow_Q$ has an indecomposable direct summand whose vertex is Q. We may assume that $vx(M_t) \not\leq_G Q$, since otherwise $vx(M_t) =_G Q$ and Q-source of M_t is a direct summand of $M_t\downarrow_Q$. By Lemma 2.3, $\mathcal{A}(M_t)\downarrow_Q$ and $\mathcal{A}(\Omega^{-2}M_t)\downarrow_Q$ split. Since M_{t-1} is a direct summand of the middle term of $\mathcal{A}(M_t)$ or $\mathcal{A}(\Omega^{-2}M_t)$, $M_t\downarrow_Q$ has an indecomposable direct summand whose vertex is Q.

Lemma 3.1 implies that the minimal elements with respect to the partial order $\leq_G$ are those that have the smallest order. Thus the following holds.

Proposition 3.2. *Let Θ be a connected component of $\Gamma_s(kG)$. Let Q be* *an element of $V(\Theta)$ which is minimal with respect to the partial order $\leq_G$. Then for any $H\in V(\Theta)$, we have $Q\leq_G H$. In particular Q is uniquely determined up to conjugation in G.*

4. Module correspondence

Now returning to the situation of the introduction, let Q be a minimal element in $V(\Theta)$ throughout this section. Let Λ be the subquiver of Δ consisting of those kN-modules L in Δ such that there exists a walk $fM_0=L_0-L_1-\cdots-L_t=L$ with $Q\leq_G vx(L_i)$ $(i=0, 1, \cdots, t)$.

First of all we note

Lemma 4.1. *Let L be an indecomposable kN-module in Λ. Then $Q\leq vx(L)$.*

Proof. This follows immediately from Lemma 3.1.

Let $\mathfrak{X}$ be the set of all p-subgroups of N of order smaller than $|Q|$. Also let $\mathfrak{Y}=\{N\cap Q^g \mid g\in G\backslash N\}$.

Lemma 4.2. *Let W be an indecomposable kG-module in Θ. Then there exists a kN-module T satisfying the following two conditions*:

(i) $(T\uparrow^G)\downarrow_N\simeq T\oplus T'$, *where T' is $\mathfrak{Y}$-projective.*

(ii) $(W\downarrow_N, T)^{\mathfrak{X},N}\neq 0$.

Proof. By Lemma 3.1, $W\downarrow_Q$ has an indecomposable direct summand S whose vertex is Q. Let $T=S\uparrow^N$. We show that T satisfies the above two conditions. By the Mackey decomposition we have $T\downarrow_Q\simeq\sum_{g\in Q\backslash N/Q}\oplus(S\otimes g)$ and so every indecomposable direct summand of T has Q as a vertex. Hence by the Green correspondence $(T\uparrow^G)\downarrow_N\simeq T\oplus T'$, where T' is $\mathfrak{Y}$-projective. Let us show the condition (ii). Letting $\tilde{\mathfrak{X}}=\mathfrak{X}\cap_N Q$, we have by Lemma 2.2 (1)

$$\begin{aligned}(W\downarrow_N, T)^{\mathfrak{X},N} &= (W\downarrow_N, S\uparrow^N)^{\mathfrak{X},N}\\ &\simeq (W\downarrow_Q, S)^{\tilde{\mathfrak{X}},Q}\supset(S, S)^{\tilde{\mathfrak{X}},Q}\neq 0\end{aligned}$$

and the assertion follows.

Lemma 4.3. *Let T be a kN-module satisfying the condition* (i) *of Lemma* 4.2. *Let L be an indecomposable kN-module in Λ. Then the following k-isomorphisms hold*:

$$((L\uparrow^G)\downarrow_N, T)^{\mathfrak{X},N}\simeq(L, (T\uparrow^G)\downarrow_N)^{\mathfrak{X},N}\simeq(L, T)^{\mathfrak{X},N}.$$

Proof. The first k-isomorphism holds by Lemma 2.2 (2).

Let $(T\uparrow^G)\downarrow_N=T\oplus(\Sigma_i\oplus X_i)$, where X_i is an indecomposable $\mathfrak{Y}$-projective kN-module. It is enough to show that $(L, X_i)^{\mathfrak{X},N}=0$ for all X_i. So we have to show that any $\alpha\in(L, X_i)^N$ is $\mathfrak{X}$-projective. Since X_i is $\tilde{Q}=(Q^g\cap N)$-projective for some $g\in G\backslash N$, there exists $\beta\in(L\downarrow_{\tilde{Q}}, X_i)^{\tilde{Q}}$ such that $\alpha=t_{\tilde{Q}}^{N}(\beta)$. Now, there exists a walk $fM_0=L_0-L_1-\cdots-L_t=L$ such that $Q\leq vx(L_i)$ $(i=0, 1, \cdots, t)$ by Lemma 4.1. As $\tilde{Q}$ is not conjugate to Q in N, $\mathcal{A}(L_i)\downarrow_{\tilde{Q}}$ splits $(i=0, 1, \cdots, t)$ by Lemma 2.3. Since $L_0\downarrow_{\tilde{Q}}$ is $\mathfrak{X}$-projective and L_1 is a direct summand of the middle term of $\mathcal{A}(L_0)$, it follows that $L_1\downarrow_{\tilde{Q}}$ is also $\mathfrak{X}$-projective. Using this argument repeatedly, we conclude that $L\downarrow_{\tilde{Q}}$ is $\mathfrak{X}$-projective. Therefore β is $\mathfrak{X}$-projective by Lemma 2.1 and hence α is $\mathfrak{X}$-projective.

Lemma 4.4. *Let L be an indecomposable kN-module in Λ. Then $L\uparrow^G$ has a unique indecomposable direct summand M whose vertex contains Q, and we have*

(1) *L is a direct summand of $M\downarrow_N$, and*

(2) *M lies in Θ.*

Moreover letting T be a kN-module satisfying the conditions in Lemma 4.2 for M, we have:

$$((L\uparrow^G)\downarrow_N, T)^{\mathfrak{X},N}\simeq(M\downarrow_N, T)^{\mathfrak{X},N}\simeq(L, T)^{\mathfrak{X},N}\neq 0\,.$$

In particular, L is a direct summand of $(L\uparrow^G)\downarrow_N$ with multiplicity one.

Proof. Since $L\,|\,(L\uparrow^G)\downarrow_N$, $L\uparrow^G$ has an indecomposable direct summand M such that $L\,|\,M\downarrow_N$. Therefore the vertex of M contains Q and $L\uparrow^G$ has at least one indecomposable direct summand whose vertex contains Q.

Let $fM_0=L_0-L_1-\cdots-L_t=L$ be a walk. We prove the assertion by induction on the t.

If $t=0$, i.e., $L\simeq fM_0$, then the assertion follows since f is the Green correspondence.

Suppose the assertion holds for L_{t-1}. We shall derive a contradiction assuming that $L\uparrow^G$ has two indecomposable direct summands M and W whose vertices contain Q. Let $L\uparrow^G=M\oplus W\oplus W'$. We may assume that $L\,|\,M\downarrow_N$. By Lemma 2.4 $\mathcal{A}(L_{t-1})\uparrow^G\simeq\mathcal{A}(M_{t-1})\oplus\mathcal{E}$, where M_{t-1} is the unique indecomposable direct summand of $L_{t-1}\uparrow^G$ whose vertex contains Q and $\mathcal{E}$ is a split sequence. Note that the middle term of $\mathcal{E}$ does not have an indecomposable direct summand whose vertex contains Q, since M_{t-1} (resp. Ω^2M_{t-1}) is a unique indecomposable direct summand of $L_{t-1}\uparrow^G$ (resp. $(\Omega^2L_{t-1})\uparrow^G$) whose vertex contains Q. Let Y (resp. Y') be the middle term of $\mathcal{A}(M_{t-1})$ (resp. $\mathcal{A}(\Omega^{-2}M_{t-1})$). Since L is a direct summand of the middle term of $\mathcal{A}(L_{t-1})$ or $\mathcal{A}(\Omega^{-2}L_{t-1})$, it follows that $M\oplus W\,|\,Y$ or $M\oplus W\,|\,Y'$. In particular both M and W lie in Θ.

Let T and U be kN-modules satisfying the conditions (i) and (ii) for M and W respectively in Lemma 4.2 and put $T'=T\oplus U$. Then

$$
\begin{aligned}
&((L\uparrow^G)\downarrow_N, T')^{\mathfrak{X},N} \\
&\simeq (M\downarrow_N, T')^{\mathfrak{X},N}\oplus(W\downarrow_N, T')^{\mathfrak{X},N}\oplus(W'\downarrow_N, T')^{\mathfrak{X},N} \\
&\simeq (L, T')^{\mathfrak{X},N}\oplus(Z, T')^{\mathfrak{X},N}\oplus(W\downarrow_N, T')^{\mathfrak{X},N}\oplus(W'\downarrow_N, T')^{\mathfrak{X},N},
\end{aligned}
$$

where $M\downarrow_N=L\oplus Z$. But by Lemma 4.3, $((L\uparrow^G)\downarrow_N, T')^{\mathfrak{X},N}\simeq(L, T')^{\mathfrak{X},N}$. This implies that $(W\downarrow_N, U)^{\mathfrak{X},N}\subset(W\downarrow_N, T')^{\mathfrak{X},N}=0$, which is a desired contardiction. Thus $L\uparrow^G$ has a unique indecomposabie direct summand M whose vertex contains Q, and the statements (1) and (2) hold. Moreover we obtain that

$$((L\uparrow^G)\downarrow_N, T)^{\mathfrak{X},N}\simeq(M\downarrow_N, T)^{\mathfrak{X},N}\simeq(L, T)^{\mathfrak{X},N}\neq 0\,,$$

since $M\,|\,L\uparrow^G$ and $L\,|\,M\downarrow_N$. Hence L is a direct summand of $(L\uparrow^G)\downarrow_N$ with multiplicity one; for otherwise

$$(L, T)^{\mathfrak{X},N}\oplus(L, T)^{\mathfrak{X},N}\subset((L\uparrow^G)\downarrow_N, T)^{\mathfrak{X},M}\simeq(L, T)^{\mathfrak{X},N}\neq 0\,,$$

a contradiction.

For an indecomposable kN-module L in Λ, let ψL be a unique indecomposable direct summand of $L\uparrow^G$ whose vertex contains Q.

Lemma 4.5. *Let L and L' be indecomposable kN-modules in Λ. Then $\psi L\simeq\psi L'$ if and only if $L\simeq L'$.*

Proof. If $L\simeq L'$, then $\psi L\simeq\psi L'$ clearly. To show the converse, assume by way of contradiction that $\psi L\simeq\psi L'$ but $L\not\simeq L'$. Since $L\,|\,\psi L\downarrow_N$ and $L'\,|\,\psi L'\downarrow_N$, we have that $L\oplus L'\,|\,\psi L\downarrow_N\,|\,(L\uparrow^G)\downarrow_N$. Let $(L\uparrow^G)\downarrow_N\simeq L\oplus L'\oplus W$. Let T be a kN-module satisfying the conditions (i) and (ii) of Lemma 4.2 for $\psi L\,(\simeq\psi L')$. Then

$$
\begin{aligned}
&((L\uparrow^G)\downarrow_N, T)^{\mathfrak{X},N} \\
&\simeq (L, T)^{\mathfrak{X},N}\oplus(L', T)^{\mathfrak{X},N}\oplus(W, T)^{\mathfrak{X},N}.
\end{aligned}
$$

But by Lemma 4.3, $((L\uparrow^G)\downarrow_N, T)^{\mathfrak{X},N}\simeq(L, T)^{\mathfrak{X},N}$. This implies that $(L', T)^{\mathfrak{X},N}=0$, which is contrary to Lemma 4.4.

We are now ready to prove the main theorem of this paper.

Theorem 4.6. *ψ induces a graph isomorphism from Λ onto Θ which preserves edge-multiplicity and direction. Also ψ gives rise to a one-to-one correspondence between indecomposable modules in Θ and those in Λ and the following hold:*

(1) *Let M be an indecomposable kG-module in Θ. Then $M\downarrow_N\simeq\psi^{-1}M\oplus(\Sigma_i\oplus W_i)$, where $W_i\downarrow_Q$ is $\mathfrak{X}$-projective for all i.*

(2) *Let L be an indecomposable kN-module in Λ. Then $L\uparrow^G\simeq\psi L\oplus(\Sigma_i\oplus V_i)$,*

where V_i is $\mathfrak{X}$-projective for all i.

Proof. It is a direct consequence of Lemmas 4.4, 4.5 and 2.4 that ψ indeed induces a graph monomorphism. To show that ψ is an epimorphism, let M be an arbitrary element of Θ and let $M_0-M_1-\cdots-M_t=M$ be a walk in Θ. If $t=0$, i.e., $M=M_0$, then $M_0=f^{-1}(fM_0)=\psi L_0$. Now, suppose then that there exists an element L_{t-1} in Λ such that $M_{t-1}=\psi L_{t-1}$. By Lemmas 4.4 and 2.4 we have $\mathcal{A}(L_{t-1})\uparrow^G=\mathcal{A}(M_{t-1})\oplus\mathcal{E}$ and $\mathcal{A}(\Omega^{-2}L_{t-1})\uparrow^G=\mathcal{A}(\Omega^{-2}M_{t-1})\oplus\mathcal{E}'$, where $\mathcal{E}$ and $\mathcal{E}'$ are split sequences. Recall that M_t is a direct summand of the middle term of $\mathcal{A}(M_{t-1})$ or $\mathcal{A}(\Omega^{-2}M_{t-1})$. Therefore there exists some indecomposable direct summand L of the middle term of $\mathcal{A}(L_{t-1})$ or of $\mathcal{A}(\Omega^{-2}L_{t-1})$ such that $M\,|\,L\uparrow^G$. Since $Q\leq_G vx(M)\leq vx(L)$, L lies in Λ. Consequently $M=\psi L$ and ψ is an epimorphism.

Next we prove (1) by induction on the distance t from M_0 to $M=M_t$. If $t=0$, i.e., $M=M_0$, then the staement (1) follows since f is the Green correspondence. Suppose the statement (1) holds for M_{t-1}. We may assume that M_t is a direct summand of the middle term of $\mathcal{A}(M_{t-1})$ (otherwise replace M_{t-1} by $\Omega^{-2}M_{t-1}$). Let $M_t\downarrow_N\simeq\psi^{-1}M_t\oplus(\Sigma_i\oplus W_i)$ and let $M_{t-1}\downarrow_N\simeq\psi^{-1}M_{t-1}\oplus(\Sigma_i\oplus W_i')$. By Lemma 2.5, $\mathcal{A}(M_{t-1})\downarrow_N\simeq\mathcal{A}(\psi^{-1}M_{t-1})\oplus\mathcal{E}'$, where $\mathcal{E}'$ is a Q-split sequence. Note that $\mathcal{E}'$ is an exact sequence terminating at $\Sigma_i\oplus W_i'$. If W_i is a direct summand of the middle term of $\mathcal{A}(\psi^{-1}M_{t-1})$, then $Q\not\leq_G vx(W_i)$, since otherwise W_i lies in Λ but this contradicts that ψ is a graph isomorphism which preserves edge-multiplicity. Therefore $W_i\downarrow_Q$ is $\mathfrak{X}$-projective. Suppose then that W_i is a direct summand of the middle term of $\mathcal{E}'$. Then since each $W_i'\downarrow_Q$ is $\mathfrak{X}$-projective and $W_i\downarrow_Q\,|\,(\Sigma_i\oplus W_i')\downarrow_Q\oplus(\Sigma_i\oplus\Omega^2W_i')\downarrow_Q$, it follows that $W_i\downarrow_Q$ is $\mathfrak{X}$-projective.

The statement (2) follows similarly by virtue of Lemma 2.4.

As an immediate consequence of the above theorem, we have

Corollary 4.7. *Let Θ be a connected component of $\Gamma_s(kG)$ and let Q be a minimal element in $V(\Theta)$. Then for any element H of $V(\Theta)$, we have $H\leq_G N_G(Q)$.*

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