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# AN EXTENSION OF A RESULT OF KOMATSU AND TAKASHIMA

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## Introduction

It has been shown by Komatsu and Takashima (1984) [7] that in dimensions higher than one the Brownian path has Hausdorff dimension 2 for quasi-all paths. This result extends the result of Taylor (1953) [10] who proved it held a.s. with respect to Wiener measure. In this paper we prove:

**Theorem.**

*Let $A$ be any time set of dimension $\alpha$, and let $\{B(t)\colon t\geq 0\}$ be a $d$-dimensional Brownian motion. The image of $A$ by $\{B(t)\colon t\geq 0\}$ has Hausdorff dimension equal to the minimum of $2\alpha$ and $d$ quasi-everywhere.*

This result extends the following result found in Kahane (1968) [5]:

*The image of a time set of Hausdorff dimension $\alpha$ by $d$-dimensional Brownian motion has dimension equal to the minimum of $2\alpha$ and $d$ a.s..*

Fukushima (1984) [2] proved that Levy's modulus of continuity held quasi-everywhere rather than just a.s.. Consequently it is easy to see that for a fixed set $A$ of Hausdorff dimension $\alpha$, the dimension of the image of $A$ is bounded by $2\alpha$ quasi-everywhere. Hence, the Theorem will be established once we have shown that for fixed $A$, the dimension of the image of $A$ by $B$ has dimension greater than the minimum of $2\alpha$ and $d$ quasi-everywhere. We shall in fact explicitly perform the calculations only for the case $d=1$ and $\alpha<d/2$, but the method we exhibit will also work for higher dimensions.

In Section 1 we give a general plan of the method; in the Section 2 we collect together the building blocks for the final proof which appears in Section 3.

## 1. Pre-requisites

For $\alpha>0$ one defines the function $\phi_\alpha^\delta$ on a set $A$ by

$$\phi_\alpha^\delta(A)=\inf\sum_{i=1}^{\infty}|A_i|^\alpha,$$

where the infimum is taken over all sequences of intervals $A_i$ such that

$$A \subset \bigcup_{i=1}^{\infty} A_i \text{ and } |A_i| < \delta \quad (|A_i| \text{ denotes the length of the interval } A_i).$$

We define the set function $\phi_\alpha$ by taking the (increasing) limit of $\phi_\alpha^\delta$ as $\delta$ tends to zero:

$$\phi_\alpha(A) = \operatorname*{limit}_{\delta \to 0} \phi_\alpha^\delta(A).$$

The Hausdorff dimension of a set A is defined to be $\inf_\alpha : \phi_\alpha(A) = 0$. With these definitions, it is easy to see that the Hausdorff dimension could also have been defined as $\sup_\alpha : \phi_\alpha(A) = \infty$.

The Hausdorff dimension of a set A is equal to its *capacitory dimension*, which is defined as the supremum over $\beta$ such that

$\exists$ a probability measure $\mu$ supported on A such that $\sup_y \int \frac{\mu(dx)}{|x-y|^\beta} < \infty$.

This condition is equivalent to

$\exists$ a probability measure $\mu$ supported on A such that $\iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^\beta} < \infty$.

The equality of these two dimensions (Hausdorff and capacitory) implies that if there exists a measure $\mu$ supported on a set A such that for each $n$

$$\iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^{\alpha-1/n}} < \infty,$$

then the Hausdorff dimension of A is greater than or equal to $\alpha$. This is how Kahane proved his result: it was shown that for a positive measure $\mu$ supported on A (which can be assumed to be closed) and a standard $d$-dimensional Brownian motion $\{B(t): t \geq 0\}$, then the (random) measure $\nu$ defined by

$$\nu(E) = \mu\{t: B(t) \in E\}$$

is supported by the Brownian image of A and that (for $2\beta < d$)

$$\iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^\beta} < \infty$$

implies that a.s.

$$\iint \frac{\nu(dx)\,\nu(dy)}{|x-y|^{2\beta}} < \infty. \tag{1}$$

Our method of proving the Theorem is showing that a.s. the local time integral (1) is finite for all paths visited by the Ornstein-Uhlenbeck process. Let us fix the set A. Without loss of generality, we may assume that A is a closed subset of the interval [0, 1] and that $\exists$ a probability measure $\mu$ supported by A such that $\exists$ C such that $\forall\, y, r,\ \mu\{(y-r, y+r)\} \leq Cr^\alpha$. This last condition of course implies

$$\forall\ \beta<\alpha,\ \sup_y \int \frac{\mu(dx)}{|x-y|^\beta} < M_\beta < \infty\ .$$

The existence of such a measure $\mu$ does not follow merely from the fact that the Hausdorff dimension of A is equal to $\alpha$ (see Taylor (1961) [11] for details). Let $\{O_s(\cdot): s\geq 0\}$ be the Ornstein-Uhlenbeck process on Wiener space. Define the measures $\nu_s$ to be

$$\nu_s(E) = \mu\{t: O_s(t)\in E\}.$$

Our result will be established if we can show for each $s$ and every $n$,

$$F_n(s) = \iint \frac{\nu_s(dx)\,\nu_s(dy)}{|x-y|^{2(\alpha-1/n)}} < \infty$$

for every $s$ and every $n$ a.s., which will follow provided we show $\{F_n(s): s\geq 0\}$ is a.s. continuous.

To show $\{F_n(s): s\geq 0\}$ is a.s. continuous we use the following criterion of Kolmogorov:

*A process $\{Z(t): t\geq 0\}$ has a continuous version if $\exists\ \beta>0$ and an integer $k$ such that for all $s$ and $t$,*

$$E[|X(t)-X(s)|^k] < K|t-s|^{1+\beta} \tag{2}$$

(see e.g. Berman (1970) [1]).
If a process has a version with continuous paths, then any separable version must also have continuous paths (see Jain and Marcus (1978) [4]). We show conditon (2) holds for $F_n(\cdot)$ (Section 3) and that $\{F_n(s): s\geq 0\}$ is separable (Lemma 1 below), thereby establishing the Theorem.

**Lemma 1.** *For each $n$, the process $\{F_n(s): s\geq 0\}$ is separable.*

Proof. Define $F_n^M(s)$ as follows:

$$F_n^M(s) = \iint \frac{\nu_s(dx)\,\nu_s(dy)}{\max(|x-y|^{2(\alpha-1/n)},\ 1/M)}\ .$$

Then by the continuity of the Ornstein-Uhlenbeck process, $F_n^M(\cdot)$ is a.s. continuous. Letting $M$ go to infinity, we see that the process $\{F_n(s): s\geq 0\}$ must be separable.

It only remains to show that condition (2) holds for each $F_n(\cdot)$. Meyer (1982) [8] noted that a realization of the Ornstein-Uhlenbeck process is

$$\{O_s(\cdot): s\geq 0\} = \{e^{-s/2}\,W(e^s,\ \cdot): s\geq 0\}\ ,$$

where $W$ denotes the Brownian sheet. Therefore $F_n$ has continuous sample paths if and only if

$$V_n(s)=\iint \frac{\mu(dx)\,\mu(dy)}{|W(s,x)-W(s,y)|^{2(\alpha-1/n)}}\,.$$

has continuous sample paths, where $s\geq 1$. Purely for greater notational simplicity we will work with $V_n(\cdot)$ rather than $F_n(\cdot)$.

## 2. Details

The following lemmas will be used in the proof of the Theorem given in Section 3. The proofs rest on tiresome calculations and are therefore omitted.

**Lemma 2.** *Let $X$ and $Y$ be Gaussian random variables with respective means $m_x$ and $m_y$ and variances $\tau_x^2$ and $\tau_y^2$. Then for $\alpha<1$ and some $C$,*

$$E\left[\left|\frac{1}{|X|^\alpha}-\frac{1}{|X-Y|^\alpha}\right|\right]<C(|m_y|^{1-\alpha}+|\tau_y|^{1-\alpha})/\tau_x\,.$$

Unfortunately, Lemma 2 gives a poor bound when either $|m_y|$ or $|\tau_y|$ is greater than $|\tau_x|$. For these cases we use the following:

**Lemma 3.** *Let $X$ and $Y$ be as in Lemma 2. Then for some finite $C$,*

$$E\left[\left|\frac{1}{|X|^\alpha}-\frac{1}{|X-Y|^\alpha}\right|\right]<\frac{C}{\tau_x^\alpha}\,.$$

**Lemma 4.** *Let $\{B(t)\colon t\geq 0\}$ be a Brownian motion and $T=\{0, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\}$ be a set of time points. Then given the values of $B$ in $T$, the conditional distribution of $B(x)-B(y)$ is Gaussian with variance greater than*

$$\min\{(\max\{d(\{x\},T)/2,\ d(\{y\},T)/2\}),\ |x-y|/2\}\,.$$

*Here $d(E,F)$ is defined to be equal to the infimum of the distance between a point in the set $E$ and a point in the set $F$. The conditional mean of $B(x)-B(y)$ is a Gaussian variable with mean zero and variance bounded by $|x-y|$.*

In the succeeding section we consider terms of the form

$$\prod_{i=1}^{n}\left|\frac{1}{|W(s,u_i)-W(s,v_i)|^{2\alpha}}-\frac{1}{|W(s+h,u_i)-W(s+h,v_i)|^{2\alpha}}\right|\,.$$

Let $\mathcal{F}_r$ denote the $\sigma$-field generated by $W(s,u_i)$, $W(s,v_i)$, $W(s+h,u_i)$, $W(s+h,v_i)$, $i=1,2,\cdots r$. Given $\mathcal{F}_r$, the conditional distribution of $(W(s,u_{r+1})-W(s,v_{r+1}),\ W(s+h,u_{r+1})-W(s+h,v_{r+1}))$ is that of $(Y, Y+Z)$, where $Y$ and $Z$ are independently distributed as $N(c_Y, s\cdot k_r)$ and $N(c_Z, h\cdot k_r)$ respectively. The conditional means $c_Y$ and $c_Z$ are $\mathcal{F}_r$-measurable random variable with means zero and variances bounded by $s\cdot(v_{r+1}-u_{r+1})$ and $h\cdot(v_{r+1}-u_{r+1})$ respectively. The term $k_r$ is bounded below by

$$d_{r+1}=\min\{\max\{d(u_{r+1},\{0,u_i,v_i\})/2, d(v_{r+1},\{0,u_i,v_i\})/2\},\ |u_{r+1}-v_{r+1}|/2\}, i\leq r. \tag{3}$$

By applying Lemmas 2 and 3 and iterating Lemma 4, we obtain the following:

**Lemma 5.** *Given numbers $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$, $u_i<v_i$ for each $i$, and numbers $d_i$, $i=1, \cdots, n$, defined as in* (3). *Then for $s\geq 1$,*

$$E\left[\prod_{i=1}^{n}\left|\frac{1}{|W(s,u_i)-W(s,v_i)|^{2(\alpha-1/n)}}-\frac{1}{|W(s+h,u_i)-W(s+h,v_i)|^{2(\alpha-1/n)}}\right|\right]$$
$$\leq\prod_{i=1}^{n}\left\{\frac{C}{d_{i+1}^{\alpha-1/n}},\frac{C_n h^{1/2-\alpha+1/n}}{d_{i+1}^{1/2}}\right\}$$
$$\leq\prod_{i=1}^{n}\left\{\min\left\{\frac{C}{d_{i+1}^{\alpha-1/n}},\frac{C_n h^{1/2n}}{d_{i+1}^{\alpha-1/2n}}\right\}\right\}.$$

Once it is noted that the variance of $c_Z$ is bounded by $h$, the proof of Lemma 5 is easy.

The following Lemma is well known, but I have been unable to find a reference.

**Lemma 6.** *Let $\mu$ be a measure on $[0, 1]$ such that $\forall\, y, r$, $\mu\{(y-r, y+r)\}\leq Cr^{\alpha}$ for some constant $C$ not depending on $y$ or $r$. Then for some $K$ not depending on $h$ or $y$,*

$$\forall\, y, h, \quad \int_{|x-z|<h}\frac{\mu(dx)}{|x-y|^{\alpha-\varepsilon}}<Kh^{\varepsilon}.$$

This lemma gives the following corollary which will be used in Section 3.

**Corollary.** *Given any fixed points $\{u_1, v_1, u_2, v_2, \cdots, u_r, v_r\}$ and a measure $\mu$ as in Lemma 6, the integral*

$$\int_{u_{r+1}<v_{r+1}}\mu(du_{r+1})\,\mu(dv_{r+1})\min\left\{\frac{C}{d_{r+1}^{\alpha-1/n}},\frac{C_n h^{1/2n}}{d_{r+1}^{\alpha-1/2n}}\right\}$$

*is majorized by $K_{r,n}h^{1/2n}$, where $K_{r,n}$ depends on $r$ and $n$ only and is otherwise independent of the particular $\{u_i\}$ and $\{v_i\}$.*

We are now prepared to assemble the proof of the Theorem.

## 3. Proof of the Theorem

As was noted in Section 1, the final result will be established if we can show $\forall\, n$ and $s\geq 1$

$$V_n(s)=\int\frac{\mu(dx)\,\mu(dy)}{|W(s,x)-W(s,y)|^{2(\alpha-1/n)}}$$

is continuous. Let us now consider $n$ to be fixed. By Lemma 1 and Kolmogorov's criterion, it is sufficient to find a $k$ such that

$$E[|V_n(s+h)-V_n(s)|^{2k}|\leq Ch^{1+\beta}$$

for some fixed $C$ and $\beta>0$ and all $h$. Now $E[(V_n(s+h)-V_n(s))^{2k}]=$

$$E\Big[\sum_{A\subset\{1,2,\cdots,2k\}}(-1)^{|A|}\int\cdots\int\prod_{i\in A}\frac{\mu(du_i)\,\mu(dv_i)}{|W(s,u_i)-W(s,v_i)|^{2(\alpha-1/n)}}$$

$$\int\cdots\int\prod_{i\in A^c}\frac{\mu(du_i)\,\mu(dv_i)}{|W(s+h,v_i)-W(s+h,v_i)|^{2(\alpha-1/n)}}$$

$$\leq\int\cdots\int\prod_{i=1}^{2k}\mu(du_i)\,\mu(dv_i)\,E\Big[\prod_{i=1}^{2k}\Big|\frac{1}{|W(s,u_i)-W(s,v_i)|^{2(\alpha-1/n)}}$$

$$-\frac{1}{|W(s+h,u_i)-W(s+h,v_i)|^{2(\alpha-1/n)}}\Big|\Big].$$

By Lemma 5 this last quantity is majorized by

$$\int\cdots\int\prod_{i=1}^{2k}\mu(du_i)\,\mu(dv_i)\min\left\{\frac{C}{d_{r+1}^{\alpha-1/n}},\frac{C_nh^{1/2n}}{d_{r+1}^{\alpha-1/2n}}\right\},$$

which by Lemma 6 is less than or equal to

$$\prod_{i=1}^{2k}K_{i,n}h^{1/2n}\leq Q_{2k,n}h^{k/n}$$

for some constant $Q_{2k,n}$ not depending on h. Taking $k$ greater than $n$, we complete the proof.

## 4. Related Questions

Given this Theorem, it is of interest to further investigate the case $2\alpha\geq d$. We know that if the Hausdorff dimension of a set A is greater than 1/2, then the image of A by one-dimensional Brownian motion a.s. has interior points (see Kaufmann (1975) [6] and Pitt (1978) [9]). By creating an appropriate continuous local time for the Ornstein-Uhlenbeck process, we can easily extend this result from a.s. to q.e.. Kahane (1968) [5] also showed that if a set has positive capacity w.r.t. $t^{-1/2}$, then a.s. its image by Brownian motion has positive Lebesgue measure. A simple martingale argument show s that this result also holds q.e.. Unfortunately we cannot find an extension of the converse (see Hawkes (1977) [3]) from a.s. to q.e..

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