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EXAMPLES ON POLYNOMIAL INVARIANTS OF KNOTS AND LINKS II

Dedicated to Professor F. Hosokawa on his 60th birthday

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Since the discovery of the Jones polynomial in 1984, several polynomial invariants of the isotopy type of knots and links in a 3-sphere have been discovered. In general, the relationships among them, together with the classical Alexander polynomial, are as follows: the (many variable) Alexander polynomial specializes to the reduced Alexander polynomial, the 2-variable Jones polynomial, which is a skein invariant, specializes to both the reduced Alexander and the Jones polynomials, and the Kauffman polynomial specializes to both the Jones and the Q polynomials. Remember [17, Fig. 4]. For a 3-braid knot or link, the 2-variable Jones and the Q polynomials are determined by the reduced Alexander polynomial and the exponent sum [10, 22]. This is generalized to a formula for the 2-variable Jones polynomial [21]. For a 2-bridge knot or link, the Q polynomial is determined by the Jones polynomial [14]. The purpose of this paper is to consider the independency of the polynomial invariants of the 2-bridge knots and links and the closed 3-braids.

In the previous paper [13], the following examples for the 2-bridge knots and links are constructed: arbitrarily many 2-bridge knots with the same Jones polynomial, arbitrarily many skein equivalent 2-bridge links with the same 2-variable Alexander polynomial, and a pair of skein equivalent 2-bridge links with distinct 2-variable Alexander polynomials. In Sect. 3, we construct: arbitrarily many skein equivalent fibered 2-bridge knots (Theorem 1), arbitrarily many skein equivalent 2-bridge links with mutually distinct 2-variable Alexander polynomials (Theorem 2), and arbitrarily many 2-bridge links with the same 2-variable Alexander polynomial but mutually distinct Jones polynomials (Theorem 3).

In Sect. 4, we construct the following examples concerning the Kauffman polynomial of the 2-bridge knots and links: a pair of skein equivalent 2-bridge knots with the same Kauffman polynomial (Theorem 4), a pair of 2-bridge knots with the same Kauffman polynomial but distinct Alexander polynomials (Theorem 5), a pair of skein equivalent 2-bridge links with the same Kauffman and 2-variable Alexander polynomials (Theorem 6), and a pair of skein equivalent

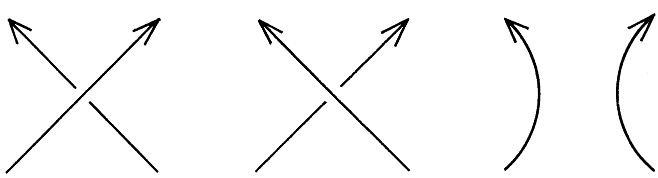
2-bridge links with the same Kauffman polynomial but distinct 2-variable Alexander polynomials (Theorem 7).

In [1], Birman constructed two examples of pairs of closed 3-braids with the same Jones polynomials; each pair of the first one has distinct signatures and the second one the same signature. In Sect. 5, the following examples are constructed: a pair of skein equivalent 3-braid knots with distinct Kauffman polynomials, in particular, nonamphicheiral 3-braid knots which are skein equivalent to their own mirror images—10_{48} [26, Appendix C] is one of them (Example 5.1), four 3-braid knots with the same Jones polynomial but have mutually distinct Kauffman polynomials (Example 5.2), a pair of skein equivalent 3-braid knots with the same Kauffman polynomial, in particular, a nonamphicheiral 3-braid knot K such that K and the mirror image of K are skein equivalent and have the same Kauffman polynomial (Example 5.3). In Examples 5.2 and 5.3, we use a computer to distinguish the knot pairs.

1. Skein equivalence

We call (L_+, L_-, L_0) a skein triple if the L_j are identical except near one point where they are as in Fig. 1. *Skein equivalence* [4, 6, 19] is the smallest equivalence relation '$\sim$' on the set of all oriented links in S^3 such that (i) if L and L' are ambient isotopic, which we denote by $L \approx L'$, then $L \sim L'$;
(ii) if (L_+, L_-, L_0) and (L'_+, L'_-, L'_0) are skein triples then

(a) $L_+ \sim L'_+$ and $L_0 \sim L'_0$ implies $L_- \sim L'_-$, and
(b) $L_- \sim L'_-$ and $L_0 \sim L'_0$ implies $L_+ \sim L'_+$.



L_+ L_- L_0

Fig. 1

The reduced Alexander polynomial $\Delta(L; t) \in \boldsymbol{Z}[t^{\pm 1/2}]$, the Jones polynomial $V(L; t) \in \boldsymbol{Z}[t^{\pm 1/2}]$ [9], and the 2-variable Jones polynomial $P(L; l, m) \in \boldsymbol{Z}[l^{\pm 1}, m^{\pm 1}]$ [5, 25] (Here we adopt the recurrence relation of Lickorish and Millett [19].) are invariants of the isotopy type of an oriented knot or link L in a 3-sphere S^3. The 2-variable Jones polynomial is defined by

$$P(U) = 1 \quad \text{for the unknot} \quad U; \tag{1.1}$$

$$lP(L_+) + l^{-1} P(L_-) + mP(L_0) = 0, \tag{1.2}$$

where (L_+, L_-, L_0) is a skein triple. The reduced Alexander polynomial and the Jones polynomial are given by the formulas:

$$P(L; -i, i(t^{1/2}-t^{-1/2})) = \Delta(L; t)\,, \tag{1.3}$$

$$P(L; it, i(t^{1/2}-t^{-1/2})) = V(L; t)\,, \tag{1.4}$$

where $i=\sqrt{-1}$. Thus the 2-variable Jones polynomial, and therefore, the Jones and the reduced Alexander polynomials are skein invariants [19, Proposition 15].

An n-braid is an element of the n-braid group generated by the elementary braids $S_1, S_2, \cdots, S_{n-1}$ as shown in Fig. 2(a). An n-braid α is called a pure n-braid, if the left j-th point and the right j-th point are connected by a string for each j. An n-knit is an element of the free semigroup with identity element 1 generated by $\{S_j, S_j^{-1}, S_j^{\infty} \mid j=1, 2, \cdots, n-1\}$, where S_j^{∞} is the braid-like element as shown in Fig. 2(b). For an n-knit κ, a closed n-knit is an unoriented diagram as shown in Fig. 3(a), which we denote by $\kappa^{\wedge}$. If β is an n-braid, then an oriented closed braid $\beta^{\wedge}$ is a diagram (or link according to the context) as shown in Fig. 3(b).

Let T be a tangle and let $T(m, n)$ and $\bar{T}(m, n)$ be oriented links with diagrams as shown in Fig. 4(a) and 4(b), respectively, where the rectangle labeled n means a 2-knit S_1^n, $n\in \boldsymbol{Z}\cup\{\infty\}$.

Lemma 1.1. *If $T_1(\varepsilon, \varepsilon')\sim T_2(\varepsilon, \varepsilon')$ for $\varepsilon, \varepsilon'=0, 1$, then $T_1(m, n)\sim T_2(m, n)$ for all $m, n\in \boldsymbol{Z}$. In particular, if $T(0, 1)\sim T(1, 0)$, then $T(m, n)\sim T(n, m)$ for all $m, n\in \boldsymbol{Z}$.*

Proof. Since $(T_i(\varepsilon, n), T_i(\varepsilon, n-2), T_i(\varepsilon, n-1))$, $i=1, 2$ and $\varepsilon=0, 1$, is a skein triple, by induction on n, $T_1(\varepsilon, n)\sim T_2(\varepsilon, n)$ for all $n\in \boldsymbol{Z}$. Similarly, since $(T_i(m, n), T_i(m-2, n), T_i(m-1, n))$ is a skein triple, by induction on m, $T_1(m, n)\sim T_2(m, n)$ for all $m, n\in \boldsymbol{Z}$.

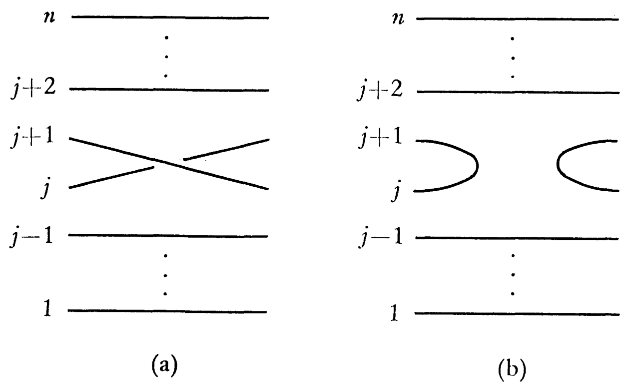
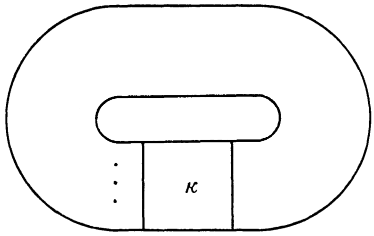
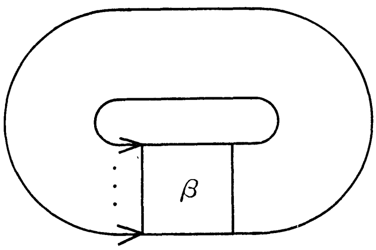


Fig. 2

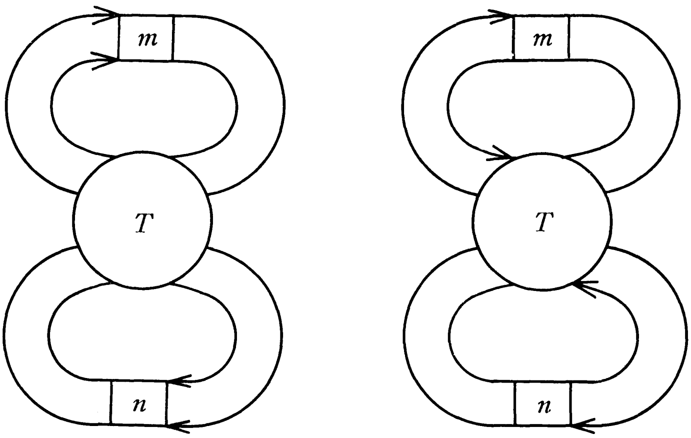


(a)



(b)

Fig. 3



(a) (b)

Fig. 4

In a similar way we can prove

Lemma 1.2. *If* $\bar{T}_1(\varepsilon, \varepsilon') \sim \bar{T}_2(\varepsilon, \varepsilon')$ *for* $\varepsilon, \varepsilon'=0, \infty$, *then* $\bar{T}_1(2m, 2n) \sim \bar{T}_2(2m, 2n)$ *for all* $m, n \in \boldsymbol{Z} \cup \{\infty\}$. *In particular, if* $\bar{T}(0, \infty) \sim \bar{T}(\infty, 0)$, *then* $\bar{T}(2m, 2n) \sim \bar{T}(2n, 2m)$ *for all* $m, n \in \boldsymbol{Z} \cup \{\infty\}$. *(We interpret* 2∞ *as* ∞.)

2. Kauffman polynomial

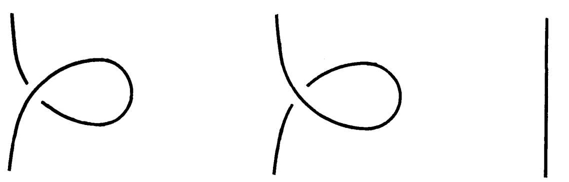
The L-polynomial $\Lambda_D(a, x) \in \boldsymbol{Z}[a^{\pm 1}, x^{\pm 1}]$ [15] is a 2-variable Laurent polynomial invariant of the regular isotopy type of a diagram D in the plane of an unoriented knot or link in S^3, that is, an invariant under Reidemeister moves II and III defined by the following formulas:

$$\Lambda_O = 1 \quad \text{for a simple closed curve} \quad O; \tag{2.1}$$

$$a^{-1}\Lambda_{C_+} = a\Lambda_{C_-} = \Lambda_{C_0}; \tag{2.2}$$

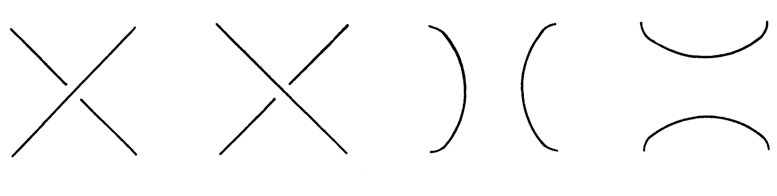
$$\Lambda_{D_+} + \Lambda_{D_-} = x(\Lambda_{D_0} + \Lambda_{D_\infty}), \tag{2.3}$$

where the C_i are curls as shown in Fig. 5, and the D_j are diagrams identical except near one point where they are as shown in Fig. 6.



C_+ C_- C_0

Fig. 5



D_+ D_- D_0 D_∞

Fig. 6

Let D be a diagram of an oriented link L, and let the writhe of D be w, which is the sum of the signs of the crossing points of D, according to the convention explained in Fig. 7. Then the Kauffman polynomial defined by $F(L; a, x) = a^{-w}\Lambda_D(a, x)$ is an isotopy type invariant of an oriented knot or link in S^3.

The Jones polynomial and the Q polynomial $Q(L;\, x)\in \boldsymbol{Z}[x^{\pm 1}]$ [2,8] are given by the following formulas [16, 17]:

$$F(L;\, -t^{3/4},\, t^{-1/4}+t^{1/4}) = V(L;\, t)\,, \tag{2.4}$$

$$F(L;\, 1,\, x) = Q(L;\, x)\,. \tag{2.5}$$

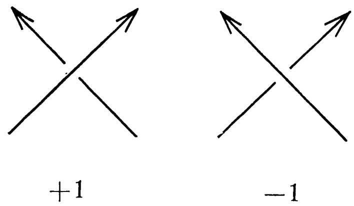


Fig. 7

The Kauffman polynomial has the following property:

Proposition 2.1.

(1) *$F(L_1 \# L_2)=F(L_1)\, F(L_2)$, where $L_1 \# L_2$ is any connected sum of L_1 and L_2.*

(2) ([15]) *$F(rL;\, a,\, x)=F(L;\, a^{-1},\, x)$, where rL is the mirror image of L.*

(3) ([18, Theorem 3]) *If L_2 is a mutant of L_1, then $F(L_1)=F(L_2)$.*

The property (1) can be proved by using the linear skein theory, see [18].

Kauffman's bracket polynomial $\langle D\rangle\in \boldsymbol{Z}[A^{\pm 1}]$ [16] is also an invariant of the regular isotopy type of an unoriented link diagram defined by the following formulas:

$$\langle O\rangle = 1 \quad \text{for a simple closed curve} \quad O; \tag{2.6}$$

$$\langle D'\rangle = (-A^2-A^{-2})\langle D\rangle; \tag{2.7}$$

$$\langle D_{\pm}\rangle = A^{\pm 1}\langle D_0\rangle+A^{\mp 1}\langle D_{\infty}\rangle\,, \tag{2.8}$$

where D' is the disjoint union of D and a simple closed curve, and the D_j are as in the above. Then the Jones polynomial $V(L;\, t)$ of an oriented link L having diagram D is given by $V(L;\, A^4)=(-A^3)^{-w}\langle D\rangle$. The bracket polynomial is a special case of the L-polynomial $\Lambda_D(a,\, x)$ as follows:

$$\langle D\rangle = \Lambda(D;\, -A^3,\, A+A^{-1}) \tag{2.9}$$

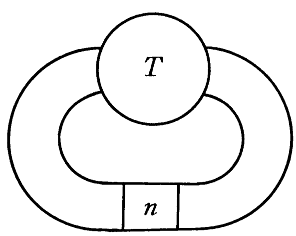


Fig. 8

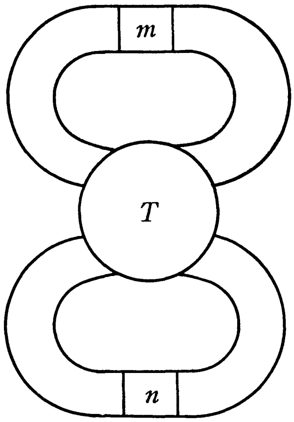


Fig. 9

Let Λ_n be the L-polynomial of the unoriented diagram $DT(n)$ as shown in Fig. 8, where T is a tangle. Then from (2.2) and (2.3),

$$\Lambda_{n-1}+\Lambda_{n+1}=x\Lambda_n+a^{-n}x\Lambda_\infty .$$

Putting $\Lambda_n=f_n\Lambda_1+g_n\Lambda_0+h_n\Lambda_\infty$, where $f_n, g_n, h_n \in \boldsymbol{Z}[a^{\pm 1}, x^{\pm 1}]$, we have

$$\begin{aligned}\Lambda_{n+1} &= x\Lambda_n-\Lambda_{n-1}+a^{-n}x\Lambda_\infty \\ &= (xf_n-f_{n-1})\Lambda_1+(xg_n-g_{n-1})\Lambda_0+(xh_n-h_{n-1}+a^{-n}x)\Lambda_\infty .\end{aligned}$$

Let $\sigma_n \in \boldsymbol{Z}[x^{\pm 1}]$ and $\tau_n \in \boldsymbol{Z}[a^{\pm 1}, x^{\pm 1}]$ be the polynomials defined by: $\sigma_{n-1}+\sigma_{n+1}=x\sigma_n$, $\tau_{n-1}+\tau_{n+1}=x\tau_n+a^{-n}x$, $\sigma_1=1$, $\sigma_0=\tau_0=\tau_1=0$. Note that σ_n is a symmetric polynomial given in [13, Sect. 2]. Then we can express Λ_n as follows:

Proposition 2.2. $\Lambda_n=\sigma_n\Lambda_1-\sigma_{n-1}\Lambda_0+\tau_n\Lambda_\infty$.

Let $\Lambda_{m,n}$ be the L-polynomial of the unoriented diagram $DT(m, n)$ as shown in Fig. 9, where T is a tangle. Then using this proposition, we readily get

Proposition 2.3.

$$\begin{aligned}\Lambda_{m,n}-\Lambda_{n,m} &= (\sigma_m\sigma_{n-1}-\sigma_n\sigma_{m-1})(\Lambda_{0,1}-\Lambda_{1,0}) \\ &+(\sigma_m\tau_n-\sigma_n\tau_m)(\Lambda_{1,\infty}-\Lambda_{\infty,1})+(\sigma_{m-1}\tau_n-\sigma_{n-1}\tau_m)(\Lambda_{\infty,0}-\Lambda_{0,\infty}) .\end{aligned}$$

Concerning the coefficients, the following holds.

Lemma 2.1. (1) *$\sigma_m\sigma_{n-1}=\sigma_n\sigma_{m-1}$ iff $m=n$.*
(2) *$\sigma_m\tau_n=\sigma_n\tau_m$ iff either $m=n$, $m=0$, or $n=0$.*
(3) *$\sigma_{m-1}\tau_n=\sigma_{n-1}\tau_m$ iff either $m=n$, $m=1$, or $n=1$.*

Proof. We shall prove the "only if" part of (2). Let r_n^+ (resp. r_n^-) be the maximum (resp. minimum) a-power of any term of τ_n. By induction on n, if $n\geqq 2$, then $r_n^-=1-n$ and $r_n^+\leqq -1$, and if $n\leqq -2$, then $r_n^-\geqq 0$ and $r_n^+=-n-1$. So if $|m|$, $|n|\geqq 2$, then $\sigma_m\tau_n=\sigma_n\tau_m$ implies $m=n$. Since $\tau_0=\tau_1=0$, $\tau_{-1}=x$, $\sigma_0=0$, $\sigma_1=1$, $\sigma_{-1}=-1$, for other cases, $\sigma_m\tau_n=\sigma_n\tau_m$ implies $m=n$ or $mn=0$. This completes the proof.

For the bracket polynomial, we have

Proposition 2.4. (*cf.* [13, Lemma 1.1 and Theorem 1]).

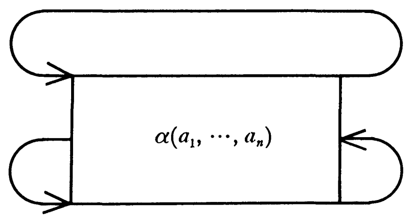
$$\langle DT(n)\rangle = A^n\langle DT(0)\rangle+\frac{A^n(1-(-A^{-4})^n)}{A^2+A^{-2}}\langle DT(\infty)\rangle,$$

$$\langle DT(m,n)\rangle-\langle DT(n,m)\rangle =$$

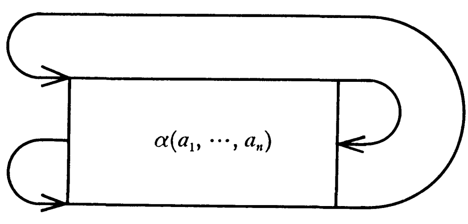
$$\frac{A^{m+n}}{A^2+A^{-2}}((-A^{-4})^m-(-A^{-4})^n)(\langle DT(0,\infty)\rangle-\langle DT(\infty,0)\rangle).$$

3. Skein equivalent 2-bridge knots and links

Let α be a 3-braid. We denote a 3-knit $\alpha S_2^{a_1}\alpha^{-1}S_1^{a_2}\alpha\cdots\alpha^{-1}S_1^{a_n}\alpha$ with n even and $\alpha S_2^{a_1}\alpha^{-1}S_1^{a_2}\alpha\cdots\alpha S_2^{a_n}\alpha^{-1}$ with n odd by $\alpha(a_1, a_2, \cdots, a_n)$, $a_i\in \boldsymbol{Z}\cup\{\infty\}$. For $n=0$, we interpret $\alpha(a_1, a_2, \cdots, a_n)$ as α. If α is a pure 3-braid and a_i is even, then we denote an oriented 2-bridge knot and an oriented 2-bridge link as shown in Fig. 10 by $K_{\alpha(a_1,a_2,\cdots,a_n)}$. If α is a trivial 3-braid, then we denote the



n is odd



n is even

Fig. 10

2-bridge knot or link $K_{\alpha(2c_1,-2c_2,\cdots,(-1)^{n-1}2c_n)}$ by $D(c_1, c_2, \cdots, c_n)$ as in [11, 13]. Any 2-bridge knot or link can be put in this form. Its reduced Alexander polynomial has degree n, and so the genus is $n/2$ or $(n-1)/2$ according as if n is even or odd. See [28].

Proposition 3.1. *Let α_1 and α_2 be pure 3-braids. If the 2-bridge knots K_{α_1} and K_{α_2} are skein equivalent, then so are $K_{\alpha_1(2b_1,2b_2,\cdots,2b_n)}$ and $K_{\alpha_2(2b_1,2b_2,\cdots,2b_n)}$, $b_i \in \boldsymbol{Z}$.*

Proof. We prove by induction on n. Suppose that $K_{\alpha_1(2b_1,2b_2,\cdots,2b_p)}$ and $K_{\alpha_2(2b_1,2b_2,\cdots,2b_p)}$ are skein equivalent for $p<n$. We have a skein triple $(K_{\alpha_i(2b_1,2b_2,\cdots,2b_{n-1},2b_n-2)}, K_{\alpha_i(2b_1,2b_2,\cdots,2b_{n-1},2b_n)}, K_{\alpha_i(2b_1,2b_2,\cdots,2b_{n-1},\infty)})$, $i=1, 2$. Since $K_{\alpha_i(2b_1,2b_2,\cdots,2b_{n-1},\infty)})$ $=K_{\alpha_i(2b_1,2b_2,\cdots,2b_{n-1})} \# K_i$, where K_i is either K_{α_i} or rK_{α_i} with its orientation reversed, according as if n is even or odd, and so K_1 and K_2 are skein equivalent. Now $K_{\alpha_i(2c_1,2c_2,\cdots,2c_{n-1},0)}$ is $K_{\alpha_i(2c_1,2c_2,\cdots,2c_{n-2})}$ or a trivial 2-component link according as if $n \geqq 2$ or $n=1$, and so $K_{\alpha_1(2b_1,2b_2,\cdots,2b_n)}$ and $K_{\alpha_2(2b_1,2b_2,\cdots,2b_n)}$ are skein equivalent by induction on b_n. This completes the proof.

From Lemma 1.2, we have

Proposition 3.2. *If α is a pure 3-braid, then the 2-bridge knots $K_{\alpha(2c_1,2c_2)}$ and $K_{\alpha(2c_2,2c_1)}$ are skein equivalent.*

Using Schubert's classification theorem of 2-bridge knots and links [27], we can prove the following in the same way as in [3, Proposition 12. 13].

Lemma 3.1. *The oriented 2-bridge knots or links $D(a_1, \cdots, a_m)$ and $D(b_1, \cdots, b_n)$ are ambient isotopic iff $m=n$, and $a_i=b_i$ or $a_i=b_{m-i+1}$, $1 \leqq i \leqq m$.*

We give other well-known properties of 2-bridge knots and links.

Lemma 3.2. (*cf.* [27, Satz 5]). *A 2-bridge knot is amphicheiral iff it has a presentation of the form $D(a_1, a_2, \cdots, a_m, a_m, \cdots, a_2, a_1)$.*

Lemma 3.3. (*cf.* [13, Lemma 6.3]). *A 2-bridge knot or link $D(a_1, a_2, \cdots, a_m)$ is fibered iff $a_i=\pm 1$ for all i.*

In [13, Theorem 6], arbitrarily many 2-bridge knots with the same Jones polynomial are constructed. But they may not be skein equivalent to one another. In fact, 10_{22} and 10_{35} in the table of Rolfsen [26], the simplest example of a pair of 2-bridge knots constructed there, have distinct Alexander polynomials. Here we have

Theorem 1. *There exist arbitrarily many, skein equivalent, amphicheiral, fibered, 2-bridge knots.*

Proof. Let $D(a_1, a_2, \cdots, a_n, a_n, \cdots, a_2, a_1)$ and $D(b_1, b_2, \cdots, b_n, b_n, \cdots, b_2, b_1)$, $|a_i|=|b_i|=1$ for all i, be distinct, skein equivalent, amphicheiral, fibered, 2-

bridge knots. Then by Propositions 3.1 and 3.2, the following four 2-bridge knots are distinct, skein equivalent, amphicheiral, and fibered:

$$D(a_1, \cdots, a_{2n}, 1, -a_{2n}, \cdots, -a_1, 1, a_1, \cdots, a_{2n}),$$
$$D(a_1, \cdots, a_{2n}, -1, -a_{2n}, \cdots, -a_1, -1, a_1, \cdots, a_{2n}),$$
$$D(b_1, \cdots, b_{2n}, 1, -b_{2n}, \cdots, -b_1, 1, b_1, \cdots, b_{2n}),$$
$$D(b_1, \cdots, b_{2n}, -1, -b_{2n}, \cdots, -b_1, -1, b_1, \cdots, b_{2n}),$$

where $a_1=a_{2n-i+1}$ and $b_i=b_{2n-i+1}$, $1\leqq i\leqq n$. Beginning with $D(1, 1)$, we get a desired set of skein equivalent 2-bridge knots. Note that the second constructed knots are $D(1, 1, 1, -1, -1, 1, 1, 1)$ and $D(1, 1, -1, -1, -1, -1, 1, 1)$.

For 2-bridge links, although Theorem 7 of [13] claims the existence of arbitrarily many skein equivalent fibered 2-bridge links with the same 2-variable Alexander polynomial, we have the following, which is a stronger version of Theorem 8 in [13].

Theorem 2. *There exist arbitrarily many skein equivalent 2-bridge links which have mutually distinct 2-variable Alexander polynomials.*

Proof. Let $\{D(a_{i1}, a_{i2}, \cdots, a_{i,2n}) \mid 1\leqq i\leqq k\}$ be the set of skein equivalent 2-bridge knots such that the $2k$ integers $\lambda_i=|a_{i1}|+|a_{i3}|+\cdots+|a_{i,2n-1}|$, $\mu_i=|a_{i2}|+|a_{i4}|+\cdots+|a_{i,2n}|$, $1\leqq i\leqq k$, are mutually distinct. From this set, we get $2k$ skein equivalent 2-bridge links $\{D(a_{i1}, a_{i2}, \cdots, a_{i,2n}, d, -a_{i,2n}, \cdots, -a_{i2}, -a_{i1})$, $D(a_{i,2n}, \cdots, a_{i2}, a_{i1}, d, -a_{i1}, -a_{i2}, \cdots, -a_{i,2n}) \mid 1\leqq i\leqq k\}$ by Proposition 3.1 or [13, Lemma 6.5]. Their 2-variable Alexander polynomials have t_1-degrees $2\lambda_i+|d|-1$, $2\mu_i+|d|-1$, $1\leqq i\leqq k$ [11, Theorem 3], which are mutually distinct. Also from the above set of 2-bridge knots, we get the set of $2k$ skein equivalent 2-bridge knots

$$\{D(a_{i1}, \cdots, a_{i,2n}, p, -a_{i,2n}, \cdots, -a_{i1}, q, a_{i1}, \cdots, a_{i,2n}),$$
$$D(a_{i,2n}, \cdots, a_{i1}, p, -a_{i1}, \cdots, -a_{i,2n}, q, a_{i,2n}, \cdots, a_{i1}) \mid 1\leqq i\leqq k\}$$

by Proposition 3.1, where p and q are integers such that the $4k$ integers $3\lambda_i+|p|$, $3\mu_i+|q|$, $3\mu_i+|p|$, $3\lambda_i+|q|$, $1\leqq i\leqq k$, are mutually distinct. So beginning with $D(1, 2)$, we can obtain the desired set of 2-bridge links.

Conversely we have

Theorem 3. *There exist arbitrarily many 2-bridge links with the same 2-variable Alexander polynomial, but have mutually distinct Jones polynomials.*

Proof. For a pure 3-braid α, we denote a 2-bridge link as shown in Fig. 11 by L_α. Let us consider the 2-bridge links $L_{\alpha_{m,n}}$, where $\alpha_{m,n}=\alpha S_1^{2m}\alpha^{-1}S_1^{2n}\alpha$ with α a pure 3-braid. Using Propositions 3.1 and 3.2 in [13], we can readily compute the 2-variable Alexander polynomial of this link to obtain:

$$\Delta(L_{\alpha_{m,n}}) = \Delta(L_\alpha)\ \{mn(t_1-1)^2(t_2-1)^2\ \Delta(L_\alpha)\ \Delta(L_{\alpha^{-1}})+1\}$$

Using Lemma 1.1 in [13], we obtain its Jones polynomial as follows:

$$V(L_{\alpha_{m,u}}) = V(L_\alpha)\ \{(t^{2m}-1)(t^{2n}-1)\ \mu_V^{-2}\ V(L_\alpha)\ V(L_{\alpha^{-1}})+t^{2m}+t^{2n}-t^{2m+2n}\}\ ,$$

where $\mu_V=-t^{1/2}-t^{-1/2}$ is the Jones polynomial of the trivial 2-component link. Thus choosing a pure 3-braid α and a set $\{(m_1, n_1), \cdots, (m_k, n_k)\,|\,m_i\,n_i=c\}$ appropriately, we get a desired set of 2-bridge links.

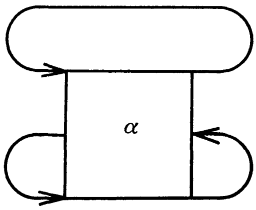
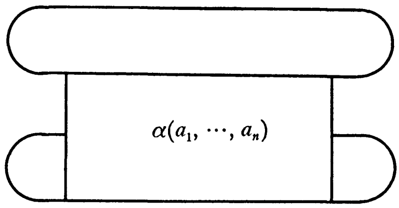


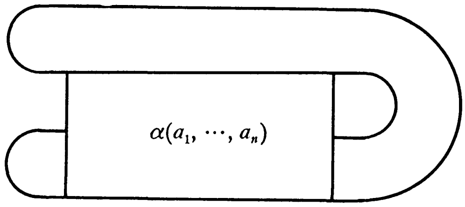
Fig. 11

4. Kauffman polynomials of 2-bridge knots and links

For a 3-braid α, let $G_{\alpha(a_1,a_2,\cdots,a_n)}$ be the unoriented diagram as shown in Fig. 12 and $\Lambda_{\alpha(a_1,a_2,\cdots,a_n)}$ be its L polynomial.



n is odd



n is even

Fig. 12

Proposition 4.1. *If β is a 3-braid of the form $(S_2^{b_1} S_1^{b_2}\cdots)(\cdots S_2^{-b_2} S_1^{-b_1})$ and $p, q, r \in \boldsymbol{Z}$, then*

$$\Lambda_{\beta(p,\underbrace{r,\cdots,r}_{i},q)} = \Lambda_{\beta(q,\underbrace{r,\cdots,r}_{i},p)},$$

$$\Lambda_{\beta(\underbrace{r,\cdots,r}_{j},p,q,\underbrace{r,\cdots,r}_{j})} = \Lambda_{\beta(\underbrace{r,\cdots,r}_{j},q,p,\underbrace{r,\cdots,r}_{j})},$$

where $i, j \geqq 0$.

Proof. We prove only the first equation, since the proof of the second one is similar. For $i=0$, the following hold:

$$\begin{aligned}\Lambda_{\beta(0,1)} &= \Lambda_{\beta(1,0)} = a^{-1}\Lambda_\beta,\\ \Lambda_{\beta(1,\infty)} &= \Lambda_{\beta(\infty,1)} = \Lambda_{\beta(1)}\Lambda_\beta,\\ \Lambda_{\beta(\infty,0)} &= \Lambda_{\beta(0,\infty)} = \delta\Lambda_\beta,\end{aligned}$$

where $\delta=x^{-1}(a+a^{-1})-1$. Thus by Proposition 2.3, $\Lambda_{\beta(p,q)}=\Lambda_{\beta(q,p)}$. For $i\geqq 1$, the following hold:

$$\Lambda_{\beta(1,\underbrace{r,\cdots,r}_{i},0)} = a^{-r}\Lambda_{\beta(1,\underbrace{r,\cdots,r}_{i-1})},$$

$$\Lambda_{\beta(0,\underbrace{r,\cdots,r}_{i},1)} = a^{-r}\Lambda_{\beta(\underbrace{r,\cdots,r}_{i-1},1)},$$

$$\Lambda_{\beta(1,\underbrace{r,\cdots,r}_{i},\infty)} = \Lambda_{\beta(1,\underbrace{r,\cdots,r}_{i})}\Lambda_\beta,$$

$$\Lambda_{\beta(\infty,\underbrace{r,\cdots,r}_{i},1)} = \Lambda_{\beta(\underbrace{r,\cdots,r}_{i},1)}\Lambda_\beta,$$

$$\Lambda_{\beta(\infty,\underbrace{r,\cdots,r}_{i},0)} = \Lambda_{\beta(0,\underbrace{r,\cdots,r}_{i},\infty)} = a^{-r}\Lambda_{\beta(\underbrace{r,\cdots,r}_{i-1})}.$$

Thus by induction on i and Proposition 2.3, the first equation follows.

Theorem 4. *There exist a pair of fibered, amphicheiral, skein equivalent, 2-bridge knots with the same Kauffman polynomial.*

Proof. Using Proposition 3.2, Lemmas 3.1–3.3, and Proposition 4.1, we are convinced that $D(1, 1, 1, -1, -1, 1, 1, 1)$ and $D(1, 1, -1, -1, -1, -1, 1, 1)$ are such a pair. See the proof of Theorem 1.

REMARK. Although $D(1, 2, p, -2, -1, q, 1, 2)$ and $D(1, 2, -q, -2, -1, -p, 1, 2)$ are skein equivalent by Proposition 3.2, they have distinct Kauffman polynomials. In fact, let $\alpha=S_2^2 S_1^{-4}$. Then $\Lambda_{\alpha(0,1)}=\Lambda_{\alpha(1,0)}$, $\Lambda_{\alpha(1,\infty)}=\Lambda_\alpha\Lambda_{\alpha(1)}$, $\Lambda_{\alpha(\infty,1)}=\Lambda_\alpha\Lambda_{\alpha'(1)}$, and $\Lambda_{\alpha(\infty,0)}=\Lambda_{\alpha(0,\infty)}$, where $\alpha'=S_2^{-4} S_1^2$ Since $\Lambda_{\alpha[0;1,1]}+\Lambda_{\alpha[0;1,-1]}=x(a\delta+\Lambda_{\alpha(1)})$ and $\Lambda_{\alpha[0;1,1]}+\Lambda_{\alpha[0;-1,1]}=x(a\delta+\Lambda_{\alpha'(1)})$, where $\alpha[0;p,q]$

$=(\alpha S_2^p \alpha^{-1} S_1^q)^\wedge$ (see Sect. 5), and $\Lambda_{\alpha[0;-1,1]} \neq \Lambda_{\alpha[0;1,-1]}$ applying [22, Theorem 12.2] by a calculation of computer, $\Lambda_{\alpha(p,q)} \neq \Lambda_{\alpha(q,p)}$.

Theorem 5. *There exist a pair of 2-bridge knots with the same Kauffman polynomial but distinct Alexander polynomials.*

Proof. The 2-bridge knots with the diagrams $G_{\beta(1,-2)}$ and $G_{\beta(-2,1)}$, where $\beta=S_2^2 S_1^{-2}$ are such a pair. In fact the former is $D(1, 2, -1, -1, 2, -1)$, which has genus 3, and the latter is $D(-1, 1, 1, -2, 1, -1, 1, 1)$, which has genus 4.

REMARK. At the Santa Cruz conference on Artin's braid group in July 1986, W.B.R. Lickorish mentioned the example of the two 11 crossing knots with the same Kauffman polynomial but distinct Alexander polynomials.

In order to prove Theorem 6 below, we consider the Conway potential function instead of the Alexander polynomial, see [4, 7]. The Conway potential function $\nabla(L; t_1, t_2, \cdots, t_\nu) \in \boldsymbol{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_\nu^{\pm 1}]$ is a uniquely determined invariant of the isotopy type of an oriented link with ν components $L=L_1 \cup L_2 \cup \cdots \cup L_\nu$, where L_i corresponds to the label t_i. This is related to the Alexander polynomial up to multiplication by a unit $\pm t_1^{n_1} t_2^{n_2} \cdots t_\nu^{n_\nu}$ by the following formula:

$$\Delta(L; t_1^2, t_2^2, \cdots, t_\nu^2) = \begin{cases} (t_1 - t_1^{-1}) \nabla(L; t_1) & \text{if } \nu = 1, \\ \nabla(L; t_1, t_2, \cdots, t_\nu) & \text{if } \nu > 1. \end{cases}$$

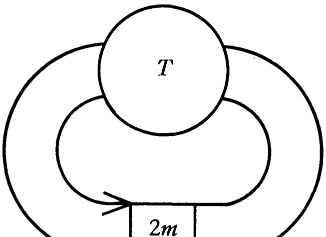


Fig. 13

Let $T(2m)$ be the oriented link as shown in Fig. 13, where the 2-braid consists of the different components L_i and L_j. Let ∇_{2m} be the potential function of $T(2m)$. Then

Lemma 4.1. $\nabla_2 + \nabla_{-2} = (t_i t_j + t_i^{-1} t_j^{-1}) \nabla_0$.

Lemma 4.2. *If L^* is obtained from L by reversing the orientation of the i-th component, then* $\nabla(L^*; t_1, \cdots, t_i, \cdots, t_\nu) = -\nabla(L; t_1, \cdots, t_i^{-1}, \cdots, t_\nu)$.

From Lemma 4.1, we have

Proposition 4.2. $\nabla_{2n}=\sigma_n \nabla_2+\sigma_{n-1}\nabla_0$, *where* $\sigma_p=\sigma_q(t_i\, t_j)$.

Let $\nabla_{2m,2n}$ be the potential function of $T(2m, 2n)$ (Fig. 4(a)), where each 2-braid consists of the different components L_i and L_j. Then we have

$$\nabla_{2m,2n} = \sigma_m \sigma_n \nabla_{2,2}-\sigma_m \sigma_{n-1} \nabla_{2,0}-\sigma_{m-1} \sigma_n \nabla_{0,2}+\sigma_{m-1} \sigma_{n-1} \nabla_{0,0}\,,$$

and so we obtain

Proposition 4.3. $\nabla_{2m,2n}-\nabla_{2n,2m}=(\sigma_m \sigma_{n-1}-\sigma_{m-1}\sigma_n)(\nabla_{0,2}-\nabla_{2,0})\,.$

Theorem 6. *There exist a pair of fibered, skein equivalent, 2-bridge links with the same 2-variable Alexander and Kauffman polynomials.*

Proof. Using Propositions 3.1, 4.1 and 4.3 and Lemmas 3.1–3.3 and 4.2, we are convinced that $D(1, 1, 1, -1, -1, 1, 1, 1, -1, -1, -1)$ and $D(1, 1, -1, -1, -1, 1, 1, 1, 1, -1, -1)$ are such a pair.

Theorem 7. *There exist a pair of skein equivalent 2-bridge links with the same Kauffman polynomial but distinct 2-variable Alexander polynomials.*

Proof. Let β be a 3-braid of the form $(S_2^{2b_1} S_1^{2b_2}\cdots)(\cdots S_2^{-2b_2} S_1^{-2b_1})$ and k, l, $m\in \boldsymbol{Z}$. For a 3-braid $\alpha=\beta(k, 2l, m)$, let $\boldsymbol{L}_\alpha$ be the oriented 2-bridge knot or link as shown in Fig. 14. Consider the pair of 2-bridge knots or links $\boldsymbol{L}_{\beta(p,2r,q)}$ and $\boldsymbol{L}_{\beta(q,2r,p)}$, where p, q, $r\in \boldsymbol{Z}$. From Proposition 4.1, they have the same Kauffman polynomial. Since $\boldsymbol{L}_{\beta(1,2r,0)}\approx\boldsymbol{L}_{\beta(0,2r,1)}$, by Lemma 1.1, $\boldsymbol{L}_{\beta(p,2r,q)}$ and $\boldsymbol{L}_{\beta(q,2r,p)}$ are skein equivalent.

$\boldsymbol{L}_{\beta(p,2r,q)}$ is a 2-bridge link, iff both p and q are odd or even. If both p and q are even, then $\boldsymbol{L}_{\beta(p,2r,q)}$ and $\boldsymbol{L}_{\beta(q,2r,p)}$ have the same 2-variable Alexander polynomial by Proposition 4.3. If $\beta=S_2^2 S_1^{-2}$, then $\boldsymbol{L}_{\beta(3,2,-1)}\approx D(-1, 2, -1, 1, 1, -1, 1, 1, -1, -2, 1)$ and $\boldsymbol{L}_{\beta(-1,2,3)}\approx D(-1, 1, 2, -1, 1, 1, -2, 1, -1, -1, 1)$. The former has t_1-degree 5 and the latter 7 by [11, Theorem 3], and so they are the desired pair of 2-bridge links.

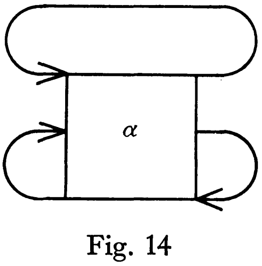


Fig. 14

5. Closed 3-braids

Let α be a 3-braid and let $\alpha[2n; p, q]$ be the closed 3-knit $(\Delta^{2n}\, \alpha S_i^p\, \alpha^{-1}\, S_j^q)^\wedge$, where $\Delta^2=(S_1 S_2 S_1)^2$ is a generator of the center of B_3, $n\in \boldsymbol{Z}$, $p, q\in \boldsymbol{Z}\cup\{\infty\}$, and $i, j=1, 2$. Then from Lemma 1.1, we have

Proposition 5.1. *If α is a 3-braid and $n, p, q\in \boldsymbol{Z}$, then $\alpha[2n; p, q]$ and $\alpha[2n; q, p]$ are skein equivalent.*

Proposition 5.2. *If α is a 3-braid and $p, q\in \boldsymbol{Z}$, then the closed 3-braids $\alpha[0; -q, -p]$ and $r\alpha[0; p, q]$, the mirror image of $\alpha[0; p, q]$, are skein equivalent. In particular, $\alpha[0; p, -p]$ is skein equivalent to its own mirror image.*

Proof. $r\alpha[0; p, q]$ is ambient isotopic to $\alpha[0; -p, -q]$ with its orientation of every component reversed, which is skein equivalent to $\alpha[0; -p, -q]$ [19, p. 129]. Thus the proposition follows from Proposition 5.1.

Since $\alpha[2n; 1, 0]$ and $\alpha[2n; 0, 1]$ are regular isotopic, $\Lambda_{\alpha[2n\,;\,\infty,0]}=\Lambda_{\alpha[2n\,;\,0,\infty]}=a^{-2n}\,\delta$, $\Lambda_{\alpha[2n\,;\,1,\infty]}=a^{-2n}\,\Lambda_{\alpha[0\,;\,1,\infty]}$, and $\Lambda_{\alpha[2n\,;\,\infty,1]}=a^{-2n}\,\Lambda_{\alpha[0\,;\,\infty,1]}$, from Proposition 2.3 we have

Proposition 5.3. *If $m, n\in \boldsymbol{Z}$, then $\Lambda_{\alpha[2p\,;\,m,n]}-\Lambda_{\alpha[2p\,;\,n,m]}=a^{-2p}(\sigma_m\tau_n-\sigma_n\tau_m)$ $(\Lambda_{\alpha[0\,;\,1,\infty]}-\Lambda_{\alpha[0\,;\,\infty,1]})$. (Note that $F(\alpha[2n; p, q])=a^{-(6n+p+q)}\,\Lambda_{\alpha[2n\,;\,p,q]}$.)*

EXAMPLE 5.1. Let us consider the case $\alpha=S_1^2\,S_2^{-3}$, $i=1, j=2$. By Proposition 5.1, the closed 3-braids $\alpha[2n; p, q]$ and $\alpha[2n; q, p]$ are skein equivalent. Since $\alpha[0; \infty, 1]$ is 10_{35} and $\alpha[0; 1, \infty]$ is 10_{22} in the table of Rolfsen [26], by Proposition 5.3, $\Lambda_{\alpha[2n\,;\,p,q]}-\Lambda_{\alpha[2n\,;\,q,p]}=a^{1-2n}(\sigma_p\,\tau_q-\sigma_q\,\tau_p)\,(F(10_{35})-F(10_{22}))$. Lickorish [18] calculated that $F(10_{22})\neq F(10_{35})$, and so by Lemma 2.1, $F(\alpha[2n; p, q])=F(\alpha[2n; q, p])$ iff $p=q$ or $pq=0$. Thus $\alpha[2n; p, q]\approx\alpha[2n; q, p]$ iff $p=q$ or $pq=0$. In particular, $\alpha[0; p, -p]$ ($\alpha[0; 1, -1]=10_{48}$, see [10, Table]) is skein equivalent to its own mirror image $\alpha[0; -p, p]$, but nonamphicheiral. Now we combine this with Birman's construction [1]. Let $\alpha[0; p, q]=\beta_1^{\wedge}$ and $\alpha[0; q, p]=\beta_2^{\wedge}$, and $p>0>q$. Then $\beta_1=S_1^2\,S_2^{-3}\,S_1^{p-1}\,S_2^{-1}\,S_1^2\,S_2^{-1}\,S_1\,S_2^{q+1}$ and $\beta_2=S_1\,S_2^{-2}\,S_1\,S_2^{q+1}\,S_1^3\,S_2^{-2}\,S_1^{p-1}\,S_2^{-1}$ are alternating principal braids, each of which has the exponent sum $p+q$. If $p+q=6r\neq 0$, $r\in \boldsymbol{Z}$, then by [1, Proposition 2], $\beta_i^{\wedge}$ and $(\Delta^{4r}\,\beta_i^{-1})^{\wedge}=\alpha[4r; -p, -q]$, $i=1, 2$, have the same Jones polynomial but have distinct signatures: the former has $6r$, and the latter $2r$. Therefore the four 3-braid knots $\alpha[0; p, q]$, $\alpha[0, q, p]$, $\alpha[4r; -p, -q]$, $\alpha[4r; -q, -p]$ have the same Jones polynomial, and so the same 2-variable Jones and Q polynomials [10, 21, 22], but they have mutually distinct Kauffman polynomials [22, Sect. 14].

Proposition 5.4. *If α is a 3-braid and $p, q\in \boldsymbol{Z}$, then*

$$\begin{aligned}\langle\alpha[0; p, q]\rangle = &A^{p+q}\{(-A^{-4})^p+(-A^{-4})^q+A^4+A^{-4}\\ &+(1-(-A^{-4})^p)(1-(-A^{-4})^q)(-A^2-A^{-2})^{-2}\langle\alpha[0; \infty, \infty]\rangle\}\,.\end{aligned}$$

In particular, $\langle\alpha[0; 1, -1]\rangle=\langle\alpha[0; \infty, \infty]\rangle$.

Since $\alpha[0; \infty, \infty]$ is a connected sum of two 2-bridge knots or links, combining this proposition with Theorem 6 or 7 in [13], or Theorem 1, we might construct arbitrarily many closed 3-braids with the same Jones polynomials. The following example shows the possibility.

EXAMPLE 5.2. Let $\alpha_1=\alpha(1, 1, -1)$, $\alpha_2=\alpha(-1, 1, 1)$, $\alpha_3=\beta(1, 1, -1)$, and $\alpha_4=\beta(-1, 1, 1)$, where $\alpha=S_2^2 S_1^{-3}$ and $\beta=S_2^3 S_1^{-2}$. Then the four diagrams of 2-bridge knots G_{α_i}, $i=1, 2, 3, 4$, have the same bracket polynomial, see [13, Lemma 6.2]. Thus the four 3-braid knots $(\alpha_i S_2^p \alpha_i^{-1} S_2^q)^\wedge$ with p, q odd have the same bracket polynomial and the same exponent sum by Proposition 5.4. However we can see that the four knots $(\alpha_i S_2^3 \alpha_i^{-1} S_2^{-3})^\wedge$ have distinct Kauffman polynomials applying [22, Theorem 12.2] by a calculation of computer.

REMARK. Let β be a pure 3-braid and let $\beta[1/m, 1/n]$, $m, n \in \boldsymbol{Z} \cup \{\infty\}$, be the knot or link as shown in Fig. 15. For $m=0$ and ∞, we interpret $1/m$ as ∞ and 0, respectively. The family of knots $K_{p,q}$ in [12] is $\beta[1/2p, 1/2q]$ and $K(a, b)$ in [13, Sect. 4] is $\beta[-1/a, -1/b]$ with $\beta=S_2^2 S_1^{-2}$. If either $m=\infty$ or $n=\infty$, then $\beta[1/m, 1/n]$ is the trivial 2-component link, so we can show that $\beta[1/m, 1/n] \sim \beta[1/m', 1/n']$ if $m+n=m'+n'$ and $m \equiv m'$, $n \equiv n'$ (mod 2), see [13, Proposition 4.3]. Using Proposition 2.4, we have $\langle \beta[1/m, 1/n] \rangle = A^{-m-n}(\langle \beta[\infty, \infty] \rangle - 1 + (-A^4)^{m+n})$, in particular, $\langle \beta[1/m, -1/m] \rangle = \langle \beta[\infty, \infty] \rangle$, see [13, Proposition 4.2].

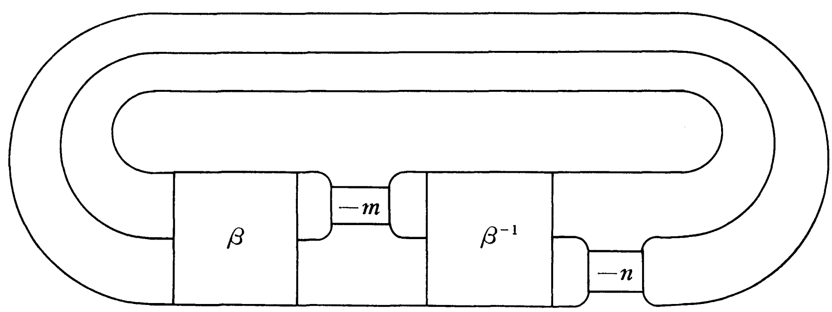


Fig. 15

For a 3-braid α, let $\alpha[2n; p, q, r, s]$ be the closed 3-knit $(\Delta^{2n} \alpha S_2^p \alpha^{-1} S_1^q \alpha S_2^r \alpha^{-1} S_1^s)^\wedge$.

Proposition 5.5. *If* α *is a 3-braid of the form* $(S_2^{b_1} S_1^{b_2} \cdots)(\cdots S_2^{-b_2} S_1^{-b_1})$ *and* $n, p, q, r, s \in \boldsymbol{Z}$, *then*

$$\Lambda_{\alpha[2n\,;\,p,q,r,s]} = \Lambda_{\alpha[2n\,;\,p,s,r,q]}$$

Proof. Let $\alpha[2n; p, q] = (\alpha S_2^p \alpha^{-1} S_1^q)^\wedge$. Since $\Lambda_{\alpha[2n\,;\,p,1,r,0]} = \Lambda_{\alpha[2n\,;\,p,0,r,1]} = \Lambda_{\alpha[2n\,;\,p+r,1]}$, $\Lambda_{\alpha[2n\,;\,p,1,r,\infty]} = a^{-2n} \Lambda_{\alpha(p,1,r)} = a^{-2n} \Lambda_{\alpha(r,1,p)} = \Lambda_{\alpha[2n\,;\,p,\infty,r,1]}$ by Proposition

4.1, and $\Lambda_{\alpha[2n\,;\,p,\infty,r,0]}=\Lambda_{\alpha[2n\,;\,p,0,r,\infty]}=a^{-2n}\Lambda_{\alpha(p+r)}$, the equation follows from Proposition 2.3.

If $\alpha=(S_2^{b_1}S_1^{b_2}\cdots)(\cdots S_2^{-b_2}S_1^{-b_1})$, then the mirror image of the closed 3-braid $\alpha[0; p, q, r, s]$ is ambient isotopic to $\alpha[0; -s, -r, -q, -p]$ with the orientation of every component reversed. Thus from Propositions 5.1 and 5.5, we have

Proposition 5.6. *If $\alpha=(S_2^{b_1}S_1^{b_2}\cdots)(\cdots S_2^{-b_2}S_1^{-b_1})$, then the closed 3-braids $\alpha[0; p, q, -p, -q]$ and its mirror image are skein equivalent and have the same Kauffman polynomial.*

EXAMPLE 5.3. Let us consider the case $\alpha=S_2^2S_1^{-2}$. J. Murakami [23, Proposition (2.4.7)] shows that $\alpha[0; 1, 3, -1, 1]$ and $\alpha[0; 1, 1, -1, 3]$ are not ambient isotopic considering the Jones polynomials of their 3-parallel links with the help of computer. By the above proposition, $\alpha[0; p, q, -p, -q]$ and its mirror image are skein equivalent and have the same Kauffman polynomial. But $\alpha[0; p, q, -p, -q]\approx(S_2^2S_1^{-2}S_2^{p-1}S_1^{-1}S_2S_1^{-2}S_2^{q-1}S_1^{-1}S_2S_1^{-1}S_2S_1^{1-p}S_2^2S_1^{-1}S_2S_1^{1-q})^{\wedge}$, so $\alpha[0; p, q, -p, -q]$ and its mirror image have distinct normal forms, if $p>q>0$, see [24]. In fact we can show that $\alpha[0; 2, 1, -2, -1]$ is nonamphicheiral using the above method of J. Murakami.

REMARK. Birman [1, Lemma 4] discovered a family of pairs of closed 3-braids with the same Jones polynomial and the exponent sum. Morton and Short [20] calculate the Jones polynomials of the 2-cable knots of some of them and J. Murakami [22, Sect. 14] calculates the Kauffman polynomials to distinguish them. Note that the Jones polynomial of the 2-parallel link of a knot K is produced from the Kauffman polynomial of K [29], so the Jones polynomial of the 2-cable knot of K is also produced from the Kauffman polynomial of K.

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