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A FUNDAMENTAL DOMAIN FOR SOME QUASI-FUCHSIAN GROUPS

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1. Introduction

Let A and B be two loxodromic elements of $\mathrm{SL}(2, \boldsymbol{C})$ having no common fixed point and subjected to the condition that $ABA^{-1}B^{-1}$ is parabolic. Writing $tr\ A=x$, $tr\ B=y$ and $tr\ AB=z$, one sees that the condition on the commutator of A and B reduces to $x^2+y^2+z^2=xyz$. In [1] Keen observed the case $z>2$, which is a special case considered there, and obtained a sufficient condition on traces x, y, z for which the group $\langle A, B\rangle$ generated by A and B is Kleinian. Her idea is first to take a symmetric normalization of the fixed points of A and B and then to construct a symmetric fundamental polygon for $\langle A, B\rangle$. In this article we shall take another normalization and construct a fundamental domain to show the following.

Theorem. *Let* $A=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ *and* $B=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $bc\neq 0$, *be loxodromic elements of* $\mathrm{SL}(2, \boldsymbol{C})$ *such that* $ABA^{-1}B^{-1}$ *is parabolic and let* $G=\langle A, B\rangle$. *If, for each integer* n, *the inequality*

$$\frac{|\alpha^n a|+|\beta^n d|}{|\alpha^n a+\beta^n d|}<\frac{|\alpha|+|\beta|}{|\alpha-\beta|} \tag{1}$$

holds, then G *is quasi-Fuchsian and represents a pair of once punctured tori.*

Restricting to the case $z>2$, one might wish to compare (1) with the Keen condition (**) of Theorem 5.1 in [1]. This will be discussed elsewhere.

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REMARK 1. Since A is an element of $\mathrm{SL}(2, \boldsymbol{C})$, $\beta=1/\alpha$. Also since A is loxodromic, $|\alpha|\neq 1$ and we may assume $|\alpha|>1>|\beta|$. Hence the left hand side of (1) tends to 1 as the absolute value of n tends to ∞. If one examines (1) for $n=0$ and if it holds, then the right hand side of (1) must be greater than 1. Therefore, for given A and B, we may examine (1) only for a finite number of integres n.

REMARK 2. Geometric meaning of (1) appears in the course of the proof of Theorem. It is given stepwise in Propositions 2 and 4 and in Section 3. In Propositions 2 and 4 we examine some restricted cases and use some simple conditions instead of (1).

2. Preliminary and special cases

Let $A=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ and $B=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $bc\neq 0$, be loxodromic elements of $\mathrm{SL}(2, \boldsymbol{C})$ such that $ABA^{-1}B^{-1}$ is parabolic. We first note that the parabolicity of $ABA^{-1}B^{-1}$ and the inequalities (1) are invariant under the conjugations by $\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$ and $\begin{pmatrix}k & 0\\ 0 & 1/k\end{pmatrix}$, where $i=\sqrt{-1}$ and k is a non zero complex number. They are also invariant under the change of generators A and B to A and BA^n. Now, conjugating A and B by $\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$, if necessary, and by $\begin{pmatrix}k & 0\\ 0 & 1/k\end{pmatrix}$ with a suitable k, we may assume and do hereafter that

$$|\alpha|>1 \quad \text{and} \quad c=\frac{\alpha+\beta}{\alpha-\beta}. \tag{2}$$

The explicit form of the matrix $ABA^{-1}B^{-1}$ is

$$\begin{pmatrix}ad-\alpha^2 bc & (\alpha^2-1)\, ab\\ (\beta^2-1)\, cd & ad-\beta^2 bc\end{pmatrix}, \tag{3}$$

so from the condition that (3) is parabolic (or from the equality $x^2+y^2+z^2=xyz$ with $x=\alpha+\beta$, $y=a+d$, $z=\alpha a+\beta d$) we obtain easily

$$ad=\left(\frac{\alpha+\beta}{\alpha-\beta}\right)^2. \tag{4}$$

Using (2) and (4) we have the following equalities:

$$\begin{aligned} &ad=c^2,\ bc=c^2-1,\ c/d=a/c\,,\\ &\alpha^2=(c+1)/(c-1),\ \alpha^2-\beta^2=4c/(c^2-1) \quad \text{and}\\ &\beta^2-1=-2/(c+1)\,. \end{aligned} \tag{5}$$

Using these equalities, we see from (3) that the fixed point of $ABA^{-1}B^{-1}$ is

$$-(\alpha^2-\beta^2)\, bc/2(\beta^2-1)\, cd=(c+1)/d=a/c+1/d\,.$$

We denote it by p_1. We put $p_2=A^{-1}(p_1)$, $p_3=B^{-1}A^{-1}(p_1)$ and $p_4=AB^{-1}A^{-1}(p_1)$. By the aid of (5) we calculate them and obtain

$$\begin{aligned} p_1&=a/c+1/d\,, & p_2&=a/c-1/d\,,\\ p_3&=-d/c+1/a\,, & p_4&=-d/c-1/a\,. \end{aligned} \tag{6}$$

Let $C_1=\{z\,|\,|z-a/c|=1/|d|\}$ and $C_2=\{z\,|\,|z+d/c|=1/|a|\}$. Note that the center of C_1 (resp. C_2) coinsides with that of the isometric circle of B^{-1} (resp. B) and that (6) implies that the segment connecting p_1 with p_2 (resp. p_3 with p_4) is a diameter of C_1 (resp. C_2). Denoting by D_i the inside of C_i ($i=1, 2$), we have the following.

Proposition 1. *B maps C_2 to C_1 such that B maps the exterior of D_2 to D_1.*

Proof. Let l be the line passing through p_3 and p_4. Since B maps p_3, p_4 and ∞ to p_2, p_1 and a/c, respectively, $B(l)$ is a line passing through p_2, p_1 and the center of C_1. Since C_2 and C_1 are orthogonal to l and $B(l)$, respectively, B maps C_2 to C_1. Since the center of C_1 is $B(\infty)$, B maps the exterior of D_2 onto D_1.

In case A is hyperbolic, a suitable position of C_1 and C_2 gives us some fundamental domain for G, the picture of which is easy to draw. As a fundamental idea of this aritcle we show the following.

Proposition 2. *Let $A=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ and $B=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $bc\neq 0$, be hyperbolic and loxodromic elements of $\mathrm{SL}(2, \boldsymbol{C})$, respectively, such that $ABA^{-1}B^{-1}$ is parabolic and let $G=\langle A, B\rangle$. If C_1 and C_2 are separated by the imaginary axis, then G is quasi-Fuchsian and represents a pair of once punctured tori.*

Proof. By Proposition 1 we see that the assumption on C_1 and C_2 implies that the exterior of $D_1\cup D_2$ is a fundamental domian for $\langle B\rangle$. We shall construct a fundamental domain R for $\langle A\rangle$ such that $R\supset D_1\cup D_2$. Since A is hyperbolic, the action of A is a streching from the origin so that p_1 and p_2 (resp. p_3 and p_4) lie on a ray emanating from the origin. Let H_i be one of the half planes $\{z\,|\,\mathrm{Re}\, z>0\}$ and $\{z\,|\,\mathrm{Re}\, z<0\}$ containing D_i ($i=1, 2$). We put

$$R_1 = \{z\,|\,|p_2|<|z|<|p_1|\}\cap H_1 \quad\text{and}$$
$$R_2 = \{z\,|\,|p_3|<|z|<|p_4|\}\cap H_2\,.$$

Then $R_i\supset D_i$ ($i=1, 2$). If the set $\bar{R}_1\cap\bar{R}_2$ consists of two closed intervals on the imaginary axis, then we put

$$R = R_1\cup R_2\cup \mathrm{Int}(\bar{R}_1\cap\bar{R}_2)\,,$$

where $\bar{R}_i$ denotes the closure of R_i and Int $(\bar{R}_1\cap\bar{R}_2)$ denotes two open intervals lying on the imaginary axis. If $\bar{R}_1\cap\bar{R}_2$ consists of at most two points, then we change the generators for G so that we are in the same situation just above. This is done simply by changing B to BA^n with a suitable integer n. Since the ring $\{z\,|\,|p_2|\leqq|z|<|p_1|\}$ is a fundamental set for $\langle A\rangle$, there is an integer n such that $|p_2|\leqq|A^{-n}(p_3)|<|p_1|$. For this n we see that $\bar{R}_1\cap A^{-n}(\bar{R}_2)$ containes more than two points. Changing B by BA^n and conjugating it with a suitable matrix of the form $\begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}$, we are in the same situation cited above. In any case we have a fundamental domain R for $\langle A \rangle$ such that $R \supset D_1 \cup D_2$. Let J_i be the closed boundary curve of R passing through p_i $(i=3, 4)$. Then putting $J_i = C_i$ $(i=1, 2)$, the proof of the proposition is completed by the following.

Proposition 3. *Let J_1 and J_2 be disjoint circles passing through p_1, p_2 and p_3, p_4, respectively, such that $B(J_2)=J_1$ and that J_1 and J_2 bound a fundamental domain for $\langle B \rangle$, where p_i are the points in* (6). *Assume that there are disjoint Jordan curves J_3 and J_4 passing through p_2, p_3 and p_1, p_4, respectively, such that $A(J_3)=J_4$, $J_1 \cap J_3 = \{p_2\}$, $J_1 \cap J_4 = \{p_1\}$, $J_2 \cap J_3 = \{p_3\}$, $J_2 \cap J_4 = \{p_4\}$ and that J_3 and J_4 bound a fundamental domain for $\langle A \rangle$. Then G is quasi-Fuchsian and represents a pair of once punctured tori.*

Our proof of Proposition 3 is somewhat long, so it is given in Section 4.

Next, assuming A is hyperbolic, we treat the case where C_1 and C_2 are not necessarily separated by the imaginary axis. As is shown in the proof of Proposition 2, we may assume and do hereafter that the center of the isometric circle of B lies in a ring $\{z \mid |p_2| \leqq |z| < |A(p_2)|\}$.

Proposition 4. *Let $A=\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $bc \neq 0$, be hyperbolic and loxodromic elements of* $\mathrm{SL}(2, \boldsymbol{C})$, *respectively, such that $ABA^{-1}B^{-1}$ is parabolic and let $G=\langle A, B \rangle$. Assume that $|p_2| \leqq |-d/c| < |p_1|$, where p_1 and p_2 are in* (6). *If the inequality*

$$\frac{|a|+|d|}{|a+d|} < \frac{\alpha+\beta}{\alpha-\beta} \tag{7}$$

holds, then G is quasi-Fuchsian and represents a pair of once punctured tori.

Proof. First we show that (7) means that two discs D_1 and D_2 lie in a positive distance. The centers and the radii of D_1 and D_2 are a/c, $-d/c$ and $1/|d|$, $1/|a|$, respectively. Hence the condition that D_1 and D_2 lie in a positive distance is equivalent to

$$1/|a| + 1/|d| < |(a+d)/c| \,.$$

By (2) and (4) we see easily that this inequality is equivalent to (7). If $|p_3| = |p_2|$, then by (5) and (6) we have $|a| = |d| = c$. Hence we see that the circles C_1 and C_2 are symmetric with respect to the imaginary axis so that we are in a situation of Proposition 2. So assume $|p_3| < |p_2|$. Draw a circle $C = \{z \mid |z| = |p_2|\}$. Then C intersects C_2 at two points. Denoting by D the inside of C, we put

$$J = (C \cap D_2^c) \cup (C_2 \cap D) \,.$$

Then the ring domain bounded by J and $A(J)$ is a fundamental domain for $\langle A\rangle$ and contains D_1 and D_2 in it. Since $J\cap C_2\neq\{p_3\}$, we need a slight deformation of a part of J lying on C_2 to apply Proposition 3. We put $C_2^*=\{z\mid |z+(1+\varepsilon/c)\,d/c|=(1+\varepsilon)/|a|\}$ with a so small positive number ε that $C_2^*\cap C_1=\emptyset$. This is possible, for we have shown that the condition (7) is equivalent to $C_1\cap C_2=\emptyset$. We put

$$J_3=(C\cap(D_2^*)^c)\cup(C_2^*\cap D)\,,$$

where D_2^* denotes the inside of C_2^*. Now it is easy to see that the curves $J_1=C_1$, $J_2=C_2$, J_3 and $J_4=A(J_3)$ satisfy the conditions of Proposition 3. Thus we have Proposition 4.

REMARK 3. Let $A=\begin{pmatrix}2 & 0\\ 0 & 1/2\end{pmatrix}$ and $B=\begin{pmatrix}(5/6)\,e^{i\theta} & 16/15\\ 5/3 & (10/3)\,e^{-i\theta}\end{pmatrix}$, $0<|\theta|<\pi/2$. Since the square of the absolute value of the trace of B is $225/36+(100/9)\cos^2\theta$, B is loxodromic. We see by (2) and (4) that $ABA^{-1}B^{-1}$ is parabolic. Since $\cos\theta\neq 0$, the square of the left hand side of (7) is smaller than 25/9. The right hand side of (7) is 5/3 so that (7) holds. Thus we have seen that A and B satisfy the assumptions of Proposition 4 except for $|p_2|\leqq|-d/c|<|p_1|$. Since $AB=\begin{pmatrix}(5/3)\,e^{i\theta} & 32/15\\ 5/6 & (5/3)\,e^{-i\theta}\end{pmatrix}$, the trace of AB is $(10/3)\cos\theta$. Hence we can find a θ such that AB is elliptic of the infinite order. This implies that $\langle A, B\rangle$ is not necessarily Kleinian. Hence the assumption on the absolute value of $-d/c$ can not be dropped.

3. Proof of Theorem

Finally we Shall go into the case where A and B are not necessarily hyperbolic. Recall that when A is hyperbolic we are in a good situation in which the line tangent to C_1 at p_2 (resp. C_2 at p_3) is mapped by A to the line tangent to C_1 at p_1 (resp. C_2 at p_4), which enables us easily to draw fundamental domains for $\langle A\rangle$ and $\langle B\rangle$ by which the condition of the Klein combination theorem is satisfied. We are now not in the same situation. That is, for a non-hyperbolic A, the line tangent to C_1 at p_2 is mapped by A to a line not tangent to C_1 at p_1. So we shall introduce new circles C_1' and C_2' such that the exterior of both D_1' and D_2' is a fundamental domain for $\langle B\rangle$ and that the line tangent to C_1' at p_2 (resp. C_2' at p_3) is mapped by A to the line tangent to C_1' at p_1 (resp. C_2' at p_4), where D_i' denotes the disc bounded by C_i' $(i=1, 2)$. Then we will be in a similar situation as before.

Let θ be the argument of α lying in the interval $(-\pi/2, \pi/2]$. The left hand side of (1) is not smaller than 1, so the right hand side of (1) is greater than 1. This implies that $\theta\in(-\pi/2, \pi/2)$. The action of A is a stretching from the origin of magnitude $|\alpha|^2$ followed by a rotation about the origin of the angle 2θ. We define the new circles as follows:

$$(8)\qquad \begin{aligned} C_1' &= \{z \mid |z-a/c+i(\tan\theta)/d| = 1/|d|\cos\theta\} \quad \text{and} \\ C_2' &= \{z \mid |z+d/c-i(\tan\theta)/a| = 1/|a|\cos\theta\}\,, \end{aligned}$$

where $i=\sqrt{-1}$. Note that by (2) and (4) we see that the centers of C_1' and C_2' are respectively

$$(9)\qquad \frac{|\alpha|+|\beta|}{d(\alpha-\beta)\cos\theta} \quad \text{and} \quad -\frac{|\alpha|+|\beta|}{a(\alpha-\beta)\cos\theta}\,.$$

Since the vector from p_1 to the center of C_1' is $-(1+i\tan\theta)/d=-\alpha/d|\alpha|\cos\theta$, the line l_1 tangent to C_1' at p_1 has the form with a real parameter s

$$l_1\colon p_1+is\,\alpha/d|\alpha|\cos\theta\,.$$

Likewise we see that the line l_2 tangent to C_1' at p_2 has the form with a real parameter t

$$l_2\colon p_2+it\,\beta/d|\beta|\cos\theta\,.$$

Since $A(z)=\alpha^2 z$ and $A(p_2)=p_1$, we have $A(l_2)=l_1$. We can also see that the line tangent to C_2' at p_3 is mapped by A to the line tangent to C_2' at p_4. Hence we have our first requirement. Another requirement is that the exterior of both D_1' and D_2' is a fundamental domain for $\langle B\rangle$. To see this we may only to show that $B(C_2')=C_1'$, because the center of the isometric circle of B is the mid point of p_3 and p_4 so that it lies in D_2' and is mapped by B to ∞. Since $B(p_3)=p_2$ and $B(p_4)=p_1$, it suffices to show that a point on $C_2'\setminus\{p_3, p_4\}$ is mapped by B to a point on C_1'. From (8) we see that the points $q=-d/c+i(\tan\theta+1/\cos\theta)/a$ and $q'=a/c-i(\tan\theta-1/\cos\theta)/d$ lie on C_2' and on C_1', respectively. A direct calculation shows $B(q)=q'$ and we have our another requirement.

Now we have two chains of circles $\{A^n(C_1')\}$ and $\{A^n(C_2')\}$ such that, for each integer n, $A^n(C_i')$ and $A^{n+1}(C_i')$ are tangent at $A^n(p_1)$ if $i=1$ and at $A^n(p_4)$ if $i=2$ and that they tend to 0 and ∞ as n tends to $-\infty$ and ∞, respectively. We show that the inequalities (1) imply that two chains of circles $\{A^n(C_i')\}$ are disjoint. Since $\{A^n(C_i')\}$ are invariant under A, it suffices to show that C_1' and $\{A^n(C_2')\}$ are disjoint. From (8) and (9) we see that the center and the radius of $A^n(C_2')$ are respectively

$$(10)\qquad -\alpha^{2n}(|\alpha|+|\beta|)/a(\alpha-\beta)\cos\theta \quad \text{and} \quad |\alpha|^{2n}/|a|\cos\theta\,.$$

By (9) and (10), the distance between the centers of C_1' and $A^n(C_2')$ is

$$((|\alpha|+|\beta|)/|\alpha-\beta|\cos\theta)\,|1/d+\alpha^{2n}/a|$$

and by (8) and (10) the sum of the radii of C_1' and $A^n(C_2')$ is

$$(1/\cos\theta)(1/|d|+|\alpha|^{2n}/|a|).$$

Hence the condition that C_1' and $A^n(C_2')$ are disjoint becomes

$$1/|d|+|\alpha|^{2n}/|a|<((|\alpha|+|\beta|)/|\alpha-\beta|)|1/d+\alpha^{2n}/a|,$$

which is clearly equivalent to (1).

Taking $J_i=C_i'$ $(i=1, 2)$, we shall draw a simple closed curve J_3 satisfying the condition of Proposition 3. As we noted after Proposition 3, we may assume that $|p_2|\leqq|d/c|<|p_1|$. Let $c=\{z\mid |z|=|p_2|\}$ and denote by D the disc enclosed by C. We put

$$\begin{aligned} J &= (C\cap(\cup A^n(D_1'\cup D_2'))^c)\cup(*)\cup(**), \\ (*) &= \bigcup_{n\geqq 0} A^n(C_1'\cup C_2')\cap D \quad \text{and} \\ (**) &= \bigcup_{n<0} A^n(C_1'\cup C_2')\cap D^c. \end{aligned} \tag{11}$$

That is, J is a Jordan curve separating $\{A^n(D_i')\}_{n=0}^{\infty}$ from $\{A^n(D_i')\}_{n=-1}^{-\infty}$. A sligh deformation of parts of J which lie on $\{A^n(C_i')\}$ will give us J_3. We need a finite number of replacements as follows:

Let ε be a positive number. Let m be the maximum number of n such that $C\cap A^n(C_1'\cup C_2')\neq\emptyset$. If $m>0$, then we put

$$C_{i,m,n}'' = \{z\mid |z-q_i| = r_i+\varepsilon n/m\}$$

and replace

$$A^n(C_i') \text{ by } A^n(C_{i,m,n}'') \quad (n=1, \cdots, m), \tag{12}$$

where q_i and r_i denote the center and the radius of C_i', respectively. Replace

$$C_i' \text{ by } C_i^* = \{z\mid |z-(q_i+\delta(q_i-p_{i+1}))| = (1+\delta)r_i\}, \tag{13}$$

where $\delta=\varepsilon/2mr_i$. Let m_i be the minimum number of n such that $C\cap A^n(C_i')\neq\emptyset$. We put

$$C_{i,m_i,n}'' = \{z\mid |z-q_i| = r_i+\varepsilon n/m_i\}.$$

If $m_1<-1$, then replace

$$A^n(C_1') \text{ by } A^n(C_{1,m_1,n}'') \quad (n=-2, \cdots, m_1). \tag{14}$$

If $m_2<0$, then replace

$$A^n(C_2') \text{ by } A^n(C_{2,m_2,n}'') \quad (n=-1, \cdots, m_2). \tag{15}$$

We put

$$C_1^{**} = \{z\mid |z-(q_1+\delta_1(q_1-p_1))| = (1+\delta_1)r_1\}$$

and replace

(16) $$A^{-1}(C_1') \text{ by } A^{-1}(C_1^{**}),$$

where $\delta_1=\varepsilon/2|m_1|r_1$. In the case where $|p_3|=|p_2|$, we need a modification. If so, then we put

$$C_2^{**}=\{z\,|\,|z-(q_2+\delta_2(q_2-p_4))|=(1+\delta_2)r_2\}$$

and replace

(17) $$A^{-1}(C_2') \text{ by } A^{-1}(C_2^{**}),$$

where $\delta_2=\varepsilon/2|m_2|r_2$. The result of these replacements of $A^n(C_i')$ and the corresponding $A^n(D_i')$ in (11) denote we by J_3.

Now, we show that, for a small ε, J_1, J_2, J_3 and $J_4=A(J_3)$ satisfy the conditions of Proposition 3. It is already shown that J_1 and J_2 are disjoint circles passing through p_1, p_2 and p_3, p_4, respectively, such that $B(J_2)=J_1$ and that J_1 and J_2 bound a fundamental domain for $\langle B\rangle$. Since the replacements above are the deformations of a finite number of circular arcs on J in small neighborhoods of them and do not move p_2 and p_3, we see that, for a small ε, J_3 is a Jordan curve passing through p_2 and p_3. By (11), (13), (16) and (17) we see that, except for one point p_{i+1}, C_i' and $A^{-1}(C_i')$ lie outside and inside of J_3, respectively, $(i=1, 2)$ so that $J_1\cap J_3=\{p_2\}$, $J_1\cap J_4=\{p_1\}$, $J_2\cap J_3=\{p_3\}$ and $J_2\cap J_4=\{p_4\}$. To see that J_3 and J_4 are disjoint and that they bound a fundamental domain for $\langle A\rangle$ it suffices to show the former, because this and two facts that J_3 separates the fixed points of A and $J_4=A(J_3)$ imply the latter. Assume that J_3 and J_4 are not disjoint and let p be a point lying on both J_3 and J_4. This implies that both p and $A^{-1}(p)$ lie on J_3.

Assume that $|p|>|p_2|$. Then by (11), (14), (15), (16) and (17) we see that p lies on one of the circles $A^{-1}(C_1^{**})$, $A^n(C_{i,m_i,n}'')$ and $A^{-1}(C_2^{**})$. Assume that p lies on $A^{-1}(C_1^{**})$. There are two cases. The first case is that $m_1<-1$. It is easy to see that C_1^{**} is contained in the inside of $C_{1,m_1,-2}''$ so that $A^{-2}(C_1^{**})$ is contained in the inside of $A^{-2}(C_{1,m_1,-2}'')$. Since J_3 does not pass through the inside of $A^{-2}(C_{1,m_1,-2}'')$, we see that this case does not occur. The second case is that $m_1=-1$. Then $A^{-2}(C_1')$ is disjoint to C. Hence the distance between J_3 and $A^{-2}(C_1')$ is positive. Since $A^{-2}(C_1^{**})$ lies in a small neighborhood of $A^{-2}(C_1')$, we see that, for a small ε, $A^{-2}(C_1^{**})$ does not meet J_3. Hence this case does not occur, too. Assume that p lies on $A^n(C_{1,m_1,n}'')$. There are two cases. The first case is that $m_1<n$. Then $A^{-1}(p)$ lies on $A^{n-1}(C_{1,m_1,n}'')$. Since $C_{1,m_1,n}''$ is a circle concentric with $C_{1,m_1,n-1}''$ and the radius of the former is smaller than that of the latter, we see that $A^{n-1}(C_{1,m_1,n}'')$ is contained in the inside of $A^{n-1}(C_{1,m_1,n-1}'')$. Since J_3 does not pass through the inside of $A^{n-1}(C_{1,m_1,n-1}'')$, we see that this case does not occur. The second case is that $n=m_1$. Then $A^{n-1}(C_1')$ is disjoint to

C and $A^{n-1}(C''_{1,m_1,n})$ lies near $A^{n-1}(C'_1)$. By the same argument in the case where p lies on $A^{-1}(C_1^{**})$ and $m_1=-1$ we see that this case does not occur, too. The rested two cases where p lies on $A^n(C''_{2,m_2,n})$ and on $A^{-1}(C_2^{**})$ are treated similarly with a remark on the minor change (17) and we see that those two cases do not occur. Hence we conclude that $|p|\leqq|p_2|$.

Assume that $|p|\leqq|p_2|$. By (11), (12) and (13) we see that $A^{-1}(p)$ lies on one of the circles C_i^* and $A^n(C''_{i,m,n})$. Assume that $A^{-1}(p)$ lies on C_i^*. Then p lies on $A(C_i^*)$. There are two cases. The first case is that $m>1$. It is easy to see that C_i^* is contained in the inside of $C''_{i,m,1}$ so that $A(C_i^*)$ is contained in the inside of $A(C''_{i,m,1})$. Since J_3 does not pass through the inside of $A(C''_{i,m,1})$, we see that this case does not occur. The second case is that $m=1$. Then $A(C'_i)$ is disjoint to C. Hence the distance between J_3 and $A(C'_i)$ is positive. Since $A(C_i^*)$ lies in a small neighborhood of $A(C'_i)$, we see that, for a small ε, $A(C_i^*)$ does not meet J_3. Hence this case does not occur, too. Assume that $A^{-1}(p)$ lies on $A^n(C''_{i,m,n})$. There are two cases. The first case is that $n<m$. Then p lies on $A^{n+1}(C''_{i,m,n})$. Since $C''_{i,m,n}$ is a circle concentric with $C''_{i,m,n+1}$ and the radius of the former is smaller than that of the latter, we see that $A^{n+1}(C''_{i,m,n})$ is contained in the inside of $A^{n+1}(C''_{i,m,n+1})$. Since J_3 does not pass through the inside of $A^{n+1}(C''_{i,m,n+1})$, we see that this case does not occur. The second case is that $n=m$. That this case does not occur is shown similarly by the method used above.

Thus we have shown that J_3 and J_4 are disjoint. Therefore we have checked all the conditions of Proposition 3. Hence Proposition 3 completes the proof of Theorem.

4. Proof of Proposition 3

Let D_1 and D_2 be the fundamental domains for $\langle A\rangle$ and $\langle B\rangle$ bounded by J_3, J_4 and by J_1, J_2, respectively. Let Q_1 (esp. Q_2) be the component of $D_1\cap D_2$ on whose boundary four points p_1, p_2, p_3 and p_4 lie clockwise (resp. counterclockwise). The Klein combination theorem (see, for example, page 139 of [3]) tells us that Q_1 and Q_2 are contained in the region of discontinuity $\Omega(G)$ of G and no two points of $Q_1\cup Q_2$ are equivalent under G. To see G is quasi-Fuchsian it suffices to show that the component Ω_i of G which contains Q_i is an invariant one ($i=1, 2$) and that Ω_1 and Ω_2 are disjoint (for a proof of this fact see [2]). First we show that Ω_1 is an invariant component of G. Denote by a the open arc $J_3\cap\partial Q_1\setminus\{p_2, p_3\}$ and let $a'=A(a)$, where ∂Q_1 denotes the boundary of Q_1. Then we see that $a'=J_4\cap\partial Q_1\setminus\{p_1, p_4\}$. Also denote by b the open arc $J_2\cap\partial Q_1\setminus\{p_3, p_4\}$ and we see that $b'=J_1\cap\partial Q_1\setminus\{p_1, p_2\}$, where $b'=B(b)$.

Lemma 1. *Four arcs a, a', b and b' on ∂Q_1 are contained in $\Omega(G)$.*

Proof. If there is a limit point p on one of the four arcs, say a, then there is a fixed point of a loxodromic element of G in any neighborhood of p. Since p is an interior point of $Q_1 \cup a \cup A^{-1}(Q_1)$, this implies that there is a loxodromic element g of G with the attractive fixed point on a. Since $g(Q_1) \cap Q_1 = \emptyset$, we see that $g(Q_1) \cap A^{-1}(Q_1) \neq \emptyset$. Hence $Ag(Q_1) \cap Q_1 \neq \emptyset$. Since Ag is not the identity, this contradicts the fact that no two points of Q_1 are equivalent. Thus we have Lemma 1.

By Lemma 1 we see that $A(Q_1)$, $A^{-1}(Q_1)$, $B(Q_1)$ and $B^{-1}(Q_1)$ which are adjacent Q_1 along a', a, b' and b, respectively, are contained in Ω_1. Writing each element of G by a word in generators A and B and appealing to the induction for lengths of words, we see easily that, for each g of G, $g(Q_1)$ is contained in Ω_1 so that Ω_1 is an invariant component of G. Likewise we see Ω_2 is so, too.

It remains to show that Ω_1 and Ω_2 are disjoint. In order to show this we need several lemmas. We shall consider the G-orbit of the circle J_1. It is easy to show the following lemma.

Lemma 2. *The set of circles $\{A^n(J_i) \mid i=1, 2; n \in \boldsymbol{Z}\}$ has the following properties:*

i) $A^n(J_1)$ *and* $A^{n+1}(J_1)$ *are tangent at* $A^n(p_1)$,

ii) $A^n(J_2)$ *and* $A^{n+1}(J_2)$ *are tangent at* $A^n(p_4)$,

iii) $A^n(J_i)$ *and* $A^m(J_i)$ *are disjoint whenever* $|n-m|>1$,

iv) $\lim_{n\to\infty} A^n(J_i)=\infty$.

v) $\lim_{n\to-\infty} A^n(J_i)=0$ *and*

vi) *for any n and m,* $A^n(J_1)$ *and* $A^m(J_2)$ *are disjoint.*

We denote by K_1 the set of circles $\{A^n(J_i) \mid i=1, 2; n \in \boldsymbol{Z}\}$.

Lemma 3. *The set $K_1 \cup \{0, \infty\}$ separates Q_1 from Q_2.*

Proof. By Lemma 2 we see that $\{A^n(J_1)\}$ constitutes a chain of circles which are tangent successively in the sence of i) and iii) of Lemma 2 and that its ends tend toward 0 and ∞. This is also true for $\{A^n(J_2)\}$. These and vi) of Lemma 2 imply the lemma.

Now consider the sst of circles $\{B(c) \mid c \in K_1 \setminus \{J_2\}\}$ (resp. $\{B^{-1}(c) \mid c \in K_1 \setminus \{J_1\}\}$. There are two circles in this set one of which is tangent to J_1 at p_1 (resp. J_2 at p_3) and the other at p_2 (resp. p_4). Except for these two, each circle of the set is contained in the inside of J_1 (resp. J_2). Let K_2 be the set of circles

$$\{A^n B(c) \mid c \in K_1 \setminus \{J_2\}; n \in \boldsymbol{Z}\} \cup \{A^n B^{-1}(c) \mid c \in K_1 \setminus \{J_1\}; n \in \boldsymbol{Z}\} .$$

It is easy to see that

$$K_2 = (\{A^n B(c) \mid c \in K_1; n \in \boldsymbol{Z}\} \cup \{A^n B^{-1}(c) \mid c \in K_1; n \in \boldsymbol{Z}\}) \setminus K_1 .$$

It is easy to see that Lemma 3 holds when we change K_1 by K_2. Inductively we put

$$K_i = ((*) \cup (**)) \setminus (K_1 \cup \cdots \cup K_{i-1}) ,$$
$$(*) = \{A^n B(c) \mid c \in K_{i-1};\, n \in \boldsymbol{Z}\} \quad \text{and}$$
$$(**) = \{A^n B^{-1}(c) \mid c \in K_{i-1};\, n \in \boldsymbol{Z}\}$$

and have the following.

Lemma 4. *The set $K_i \cup \{0, \infty\}$ separates Q_1 from Q_2.*

Lemma 5. *Let $\{k_i\}$ be a sequence of circles such that $k_i \in K_i$ and that, for each i, k_i is contained in the closed disc bounded by k_{i-1}. Then the sequence of the diameters of k_i's converges to* 0.

The proof of Lemma 5 is given later. Now we show that Ω_1 and Ω_2 are disjoint. Assume contrarily that $\Omega_1 = \Omega_2$. Then there is an arc J lying in $\Omega(G)$ and connecting a point of Q_1 to a point of Q_2. By Lemmas 3 and 4 we see that, for each opsitive integer i, there is a circle k_i of K_i which intersects J. Since each circle of K_i is contained in the closed disc bounded by a circle of K_{i-1}, we may assume that $\{k_i\}$ satisfies the condition of Lemma 5. On the other hand, there is a positive number r such that each open disc centered at a point of J with radius r is contained in $\Omega(G)$. By Lemma 5 we see that there is circle k_i which is contained entirely in $\Omega(G)$. But k_i is an element of G-orbit of J_1 and there is a fixed point p_1 of a parabolic element $ABA^{-1}B^{-1}$ on J_1, we have a contradiction. Therefore Ω_1 and Ω_2 are distinct.

Before going to prove Lemma 5 we need two more lemmas.

Lemma 6. *For each positive integer n, $(ABA^{-1}B^{-1})^n(J_1)$, $(A^{-1}BAB^{-1})^n(J_1)$, $(A^{-1}B^{-1}AB)^n(J_2)$ and $(AB^{-1}A^{-1}B)^n(J_2)$ are circles of K_{2n} passing through p_1, p_2, p_3 and p_4, respectively. Moreover, they are disjoint to both insides of J_1 and J_2.*

Proof. We show the lemma for $ABA^{-1}B^{-1}$. The other cases are shown similarly. Since $ABA^{-1}B^{-1}(J_1) = ABA^{-1}(J_2)$ and $A^{-1}(J_2)$ is an element of K_1 different from J_2, we see that $ABA^{-1}B^{-1}(J_1)$ is an element of K_2 lying in the closed disc bounded by $A(J_1)$. The fixed point of $ABA^{-1}B^{-1}$ is p_1 and is a point on J_1, so $ABA^{-1}B^{-1}(J_1)$ passes through p_1. This describes completely the action of the parabolic transformation $ABA^{-1}B^{-1}$, so we have Lemma 6.

Lemma 7. *There exists a positive constant M such that, for any two disjoint circles k and k' of $K = \cup K_i$, the module of the ring domain bounded by k and k' is not smaller than M.*

Proof. We denote by $R(k, k')$ the ring domain bounded by k and k'. We recall that the modules are conformally invariant and satisfy the comparison principle as follows: if $R(k, k')$ is contained in $R(k'', k''')$ and if k separates the complementary components of $R(k'', k''')$, then the module of $R(k, k')$ is not greater than that of $R(k'', k''')$. Since each element of K is equivalent to J_1, we may assume that $k'=J_1$. There are three cases.

Case 1. k is an element of $K_1\setminus\{J_1, A(J_1), A^{-1}(J_1)\}$ or lies in the closed disc bounded by it.

Case 2. k is contained in one of the closed discs bounded by $A(J_1)$ and $A^{-1}(J_1)$.

Case 3. k is contained in the inside of J_1.

In Case 1, by the comparison principle of the modules we may assume that k is an element of $K_1\setminus\{J_1, A(J_1), A^{-1}(J_1)\}$. By iii)$\sim$vi) of Lemma 2, we see that there is a positive number d such that no element of $K_1\setminus\{J_1, A(J_1), A^{-1}(J_1)\}$ meets the open disc concentric with J_1 and having the radius $r+d$, where r is the radius of J_1. Also by the comparison principle we see that the module of $R(k, J_1)$ is not smaller than $\log(r+d)/r$. We denote this number by M_1.

Case 2. Let k be contained in the closed disc bounded by $A(J_1)$. Another case with respect to $A^{-1}(J_1)$ is treated similarly. It is easy to see that $ABA(J_2)$ and $ABA^{-1}(J_2)$ are the elements of K_2 which are contained in the closed disc bounded by $A(J_1)$ and tangent to $A(J_1)$ at $A(p_1)$ and p_1, respectively, and that all other elements of K_2 lying inside of $A(J_1)$ are $\{AB(c)\,|\,c\in K_1\}\setminus\{ABA(J_2), ABA^{-1}(J_2), A(J_1)\}$. If k lies in the closed disc bounded by an element of the last set, then the images of k and J_1 under $B^{-1}A^{-1}$ are contained in the closed discs bounded by an element of $K_1\setminus\{A(J_2), J_2, A^{-1}(J_2)\}$ and by J_2, respectively. The comparison principle implies that this case is reduced to Case 1 with a change J_1 by J_2. We denote by M_2 the lower bound of modules of the present case. If k lies in the closed disc bounded by $ABA(J_2)$, then the module of $R(k, J_1)$ is not smaller than that of $R(ABA(J_2), J_1)$. We denote this module by M_3. If k lies in the closed disc bounded by $ABA^{-1}(J_2)$, then there is a positive integer n such that k lies in the closed domain bounded by $(ABA^{-1}B^{-1})^n(J_1)$ and $(ABA^{-1}B^{-1})^{n+1}(J_1)$. Operating $(ABA^{-1}B^{-1})^{-n}$, we see that k is transformed in the closed disc bounded by an element of either $K_1\setminus\{A(J_1), J_1\}$ or $\{AB(c)\,|\,c\in K_1\}\setminus\{ABA^{-1}(J_2), A(J_1)\}$ and J_1 is transformed to a circle lying in the closed disc bounded by J_1 and being tangent to J_1 at p_1. If the transformed k lies in the closed disc bounded by $A^{-1}(J_1)$, then operating A once more, we can reduce this case to the previously considered case with respect to $R(ABA(J_2), J_1)$. If k is transformed in the closed disc bounded by an element of $K_1\setminus\{A(J_1), J_1, A^{-1}(J_1)\}$, then by the comparison principle we reduce this case to Case 1. If k is transformed in the closed disc bounded by an element of $\{AB(c)\,|\,c\in K_1\}\setminus\{ABA^{-1}(J_2), A(J_1)\}$, then replacing J_1 by the transformed J_1 and using the comparison principle, we reduce this case to a case already considered in this paragraph.

Case 3. By the action of B we see that $B^{-1}(J_1)=J_2$ and $B^{-1}(k)$ lies in the closed disc bounded by an element of $K_1\setminus\{J_2, A(J_2), A^{-1}(J_2)\}$. By the same reasonning in Case 1 we see that the module of $R(B^{-1}(J_1), B^{-1}(k))$ is not smaller than a positive constant. We denote it by M_4.

Now, we conclude that $M=\min(M_1, M_2, M_3, M_4)$ is the desired constant.

We return to Lemma 5.

Proof of Lemma 5. If there is a subsequence of $\{k_i\}$ such that any two of which are disjoint, then Lemma 7 implies that the diameters of elements of the subsequence must converge to 0, so that that of $\{k_i\}$ must do, too. Hence we consider the case where there is no subsequence of $\{k_i\}$ such that any two of which are disjoint. We assert that there are a positive integer i_0 and a point p equivalent to p_1 such that, for each $i>i_0$, k_i passes through p. Assuming the contrary of the assertion, we shall construct a subsequence having the property that any two of which are disjoint. We first note that only two points p_1 and p_2 on J_1 are the points at which elements of $K\setminus\{J_1\}$ can meet J_1 and that each k_i is equivalent to J_1. Put $k_{i_1}=k_1$. If k_1 and k_2 are disjoint, then put $k_{i_2}=k_2$. If k_1 and k_2 are not disjoint, then denote by $p_{1,2}$ the tangent point of them. Since $p_{1,2}$ is equivalent to p_1, there is a number $i_2>2$ such that k_{i_2} does not pass through $p_{1,2}$ so that k_1 and k_{i_2} are disjoint. If k_{i_2} and k_{i_2+1} are disjoint, then put $k_{i_3}=k_{i_2+1}$. If k_{i_2} and k_{i_2+1} are not disjoint, then denote by p_{i_2,i_2+1} the tangent point of them. Since p_{i_2,i_2+1} is equivalent to p_1, there is a number $i_3>i_2+1$ such that k_{i_3} does not pass through p_{i_2,i_2+1}, and so on. Then the subsequence $\{k_{i_j}\}$ has the property stated above, a contradiction. Thus we have our assertion. Let g be an element of G such that $g(p)=p_1$. Then $\{g(k_i)\}_{i>i_0}$ is an infinite subset of K passing through p_1. Lemma 6 tells us that the diameter of $g(k_i)$ tends to 0. Hence that of k_i tends to 0, too.

Now, we have shown that Ω_1 and Ω_2 are disjoint so that G is quasi-Fuchsian. To complete the proof of Proposition 3 we must show that G represents a pair of once punctured tori. Let $G'=\langle A', B'\rangle$ be a Fuchsian group keeping the real axis invariant and f be a quasiconformal mapping such that $A=fA'f^{-1}$ and $B=fB'f^{-1}$. Then we see that the commutator of A' and B' is parabolic. Writing $A'=\begin{pmatrix}\alpha' & 0\\ 0 & \beta'\end{pmatrix}$ and $B'=\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}$, we may assume that the equalities (2)~(6) hold for the letters with a prime α', $\cdots$, d' and p_i'. Then we have that

$$p_1'p_2'=((c')^2-1)/(d')^2>0\,,\quad p_3'p_4'=((c')^2-1)/(a')^2>0 \quad \text{and}$$
$$p_1'p_3'=-((c')^2-1)/(c')^2<0\,.$$

Hence we see that two circles orthogonal to the real axis and passing through p_1', p_2' and p_3', p_4' are separated by the imaginary axis. Now it is clear that two quadrilateral domains bounded by four circles which are orthogonal to the real axis and pass through p_1', p_2'; p_3', p_4'; p_1', p_4' and p_2', p_3', respectively, form a fundamental domain for G' so that G' represents a pair of once punctured tori. As G is a quasiconformal deformation of G', G also represents a pair of once punctured tori. Thus we have completed the proof of Proposition 3.

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