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WEAK CONVERGENCE OF A SEQUENCE OF STOCHASTIC PROCESSES RELATED WITH U-STATISTICS

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1. Introduction

In Hoeffding's fundamental paper [4], he proved the weak convergence of U-statistics under suitable conditions. Loynes [10] and later, Miller and Sen [12] as well as Mandelbaum and Taqqu [11] respectively considered different types of stochastic processes related with U-statistics and studied their weak convergence. In this paper, we are concerned with a sequence of stochastic processes which are similar to those developed by Mandelbaum and Taqqu [11]. We intend to show a deeper analysis of the weak convergence of the processes. This is achieved by using the martingale approach, a method used extensively by Khmaladze [8], [9]. (See Rao [13] for a survey on Martingale approach to Statistical Inference). Under this martingale approach we intend to find natural expressions for the limits of the martingale part and compensator of the processes associated with a sequence of U-statistics. These limits can be expressed by using multiple Wiener integrals.

Let F be a distribution function on $\boldsymbol{R}$ and $X_1, \cdots, X_n$, independent observations on F. Consider a parametric function $\theta=\theta(F)$, for which there exists an unbiased estimator. That is, $\theta(F)$ may be expressed as $\theta(F)=E_F(h(X_1, \cdots, X_m))$ for some function h: $\boldsymbol{R}^m\to\boldsymbol{R}$, called a "kernel", where h can be assumed to be symmetric.

Let's define:

$$h_k(x_1, \cdots, x_k):=E(h(X_1, \cdots, X_m)/X_1=x_1, \cdots, X_k=x_k) \quad \text{and}$$
$$\zeta_k:=Var(h_k(X_1, \cdots, X_k)) \quad \text{for} \quad k=1, \cdots, m\,,$$

under the assumption

$$E(h^2(X_1, \cdots, X_m))<\infty\,. \tag{1}$$

The *U-statistic for estimation of θ based on the sample $X_1, \cdots, X_n$ of size* $n\geq m$ is

$$U_n := \frac{1}{\binom{n}{m}} \sum_{C_m^n} h(X_{i_1}, \cdots, X_{i_m}),$$

where $C_m^n = \{(i_1, \cdots, i_m) \in \{1, \cdots, n\}^m / 1 \le i_1 < \cdots < i_m \le n\}$.

The study of the weak convergence of U-statistics have been extensively studied in Hoeffding [4], Gregory [3], Eagleson [6], Serfling [15], Rubin and Vitale [14], Dynkin and Mandelbaum [2], Mandelbaum and Taqqu [11] among others. Results and definitions about U-statistics can be found in Serfling [15].

2. Weak convergence of the U-processes

DEFINITION. The *stochastic process associated with a U-statistic of size n* (simply, *U-process*), is defined by:

$$U_n(t) := \frac{1}{\binom{n}{m}} \sum_{C_m^{[nt]}} h(X_{i_1}, \cdots, X_{i_m}), \quad \text{for} \quad 0 \le t \le 1$$

where [ ] is the greatest integer function, $(i_1, \cdots, i_m) \in C_m^{[nt]}$ and $U_n(t) = 0$ for $t < \frac{m}{n}$.

This process is adapted to the right continuous σ-field $\mathcal{F}_n(t) = \sigma[X_1, \cdots, X_{[nt]}]$. From now on, we will assume that $\theta(F) \equiv 0$ (if not, we can always take $h - \theta$ instead of h).

Lemma 1. *$U_n(t)$ is a semimartingale, its martingale part is given by*:

$$X_n(t) := \frac{1}{\binom{n}{m}} \sum_{C_m^{[nt]}} \{h(X_{i_1}, \cdots, X_{i_m}) - h_{m-1}(X_{i_1}, \cdots, X_{i_{m-1}})\}.$$

Following Hoeffding [4], $U_n(t)$ can also be expressed as:

$$U_n(t) = \frac{1}{\binom{n}{m}} \sum_{k=1}^{m} \binom{[nt]-k}{m-k} U_{n,k}(t), \tag{2}$$

where

$$U_{n,k}(t) = \sum_{C_k^{[nt]}} \int h_k(x_1, \cdots, x_k) \prod_{j=1}^{k} d(I_{\{X_{i_j} \le x_j\}} - F(x_j)). \tag{3}$$

Here,

$$I_{\{x \le y\}} = \begin{cases} 1 & \text{if} \quad x \le y, \\ 0 & \text{if} \quad x > y. \end{cases}$$

Similarly, we have

$$(4)\qquad X_n(t)=\frac{1}{\binom{n}{m}}\sum_{k=1}^{m}\sum_{C_k^{[nt]}}\left[\binom{i_k-k}{m-k}\int h_k(x_1,\cdots,x_k)\prod_{j=1}^{k}d(I_{\{X_{i_j}\leq x_j\}}-F(x_j))\right].$$

which can be obtained by using methods similar to the ones showed at the end of the proof of Lemma 4.

Lemma 2.

(i) *$U_{n,k}(t)$ is a martingale.*

(ii) $E(U_{n,k}^2(1))\leq\binom{n}{k}2^k\zeta_k$.

(iii) $P(\max_t|n^r U_{n,k}(t)|>\varepsilon)\to 0$ *as* $n\to\infty$ *for* $r<-\frac{k}{2}$.

Proof. (i) is easily proved by noticing that

$$E(d(I_{\{X_j\leq x\}}-F(x))/\mathcal{F}_n(s))=0\quad\text{if}\quad [ns]<j\leq[nt].$$

(ii) $$E(U_{n,k}^2(1))=E([\sum_{C_k^n}\int h_k(x_1,\cdots,x_k)\prod_{j=1}^{k}d(I_{\{X_{i_j}\leq x_j\}}-F(x_j))]^2)$$

because, if $(i_1,\cdots,i_k)\neq(l_1,\cdots,l_k)$ then

$$E(\prod_{j=1}^{k}[d(I_{\{X_{i_j}\leq x_j\}}-F(x_j))\,d(I_{[X_{i_j}\leq y_j]}-F(y_j))])=0.$$

It will be enough to prove that

$$E(\int h_k(x_1,\cdots,x_k)\,h_k(x_{k+1},\cdots,x_{2k})\prod_{j=1}^{k}d(I_{\{X_{i_j}\leq x_j\}}-F(x_j))\,d(I_{\{X_{i_j}\leq x_{j+k}\}}-F(x_{j+k})))$$

is bounded by $2^k\zeta_k$. As

$$\begin{aligned}&E(d(I_{\{X_{i_j}\leq x_j\}}-F(x_j))\,d(I_{\{X_{i_j}\leq x_{j+k}\}}-F(x_{j+k})))\\&\quad=I_{\{x_j=x_{j+k}\}}\,dF(x_j)-dF(x_j)\,dF(x_{j+k}),\end{aligned}$$

then, the above expectation becomes:

$$\begin{aligned}&\int h_k(x_1,\cdots,x_k)\,h_k(x_{k+1},\cdots,x_{2k})\prod_{j=1}^{k}\{I_{\{x_j=x_{j+k}\}}\,dF(x_j)-dF(x_j)\,dF(x_{j+k})\}\\&=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}h_k(x_1,\cdots,x_k)\,h_k(x_1,\cdots,x_i,x_{k+1},\cdots,x_{2k-i})\,dF(x_1)\cdots dF(x_{2k-i}).\end{aligned}$$

By the Cauchy-Schwartz inequality:

$$\left|\int h_k(x_1,\cdots,x_k)\,h_k(x_1,\cdots,x_i,x_{k+1},\cdots,x_{2k-i})\,dF(x_1)\cdots dF(x_{2k-i})\right|\leq\zeta_k,$$

from which we get the announced boundedness.

(iii) As $U_{n,k}(t)$ is a martingale, by Doob's inequality we have:

$$P(\max_t|n^r U_{n,k}(t)|>\varepsilon)\leq\frac{n^{2r}}{\varepsilon^2}\binom{n}{k}2^k\zeta_k\to 0\quad\text{as}\quad n\to\infty.$$

For any kernel h there exists c $(1\leq c\leq m)$ such that $\zeta_1=0,\cdots,\zeta_{c-1}=0,\zeta_c>0$. From now on, we will take c fixed.

Let's define:

$$Y_{n,c}(t):=\frac{n^{c/2}\binom{[nt]-c}{m-c}}{\binom{n}{m}}U_{n,c}(t)\,.$$

Lemma 3.

$$\sup_t|n^{c/2}U_n(t)-Y_{n,c}(t)|\xrightarrow{P}0\quad\text{as}\quad n\to\infty\,. \tag{5}$$

i.e. the process U_n can be approximated by the c-th term process appearing in the descomposition (2).

Proof. As $\zeta_1=0,\cdots,\zeta_{c-1}=0$, then $h_1=\cdots=h_{c-1}=0$. Using (2), the left term of (5) can be written as:

$$\sup_t|n^{c/2}\{\frac{1}{\binom{n}{m}}\sum_{k=c+1}^{m}\binom{[nt]-k}{m-k}U_{n,k}(t)\}|\leq\frac{n^{c/2}}{\binom{n}{m}}\sum_{k=c+1}^{m}\binom{n-k}{m-k}\sup_t|U_{n,k}(t)|\,,$$

which tends to zero by Lemma 2 (iii).

Lemma 4. *The martingale part of the process $Y_{n,c}(t)$ is given by*:

$$Z_{n,c}(t):=\frac{n^{c/2}}{\binom{n}{m}}\sum_{C_c^{[nt]}}\binom{i_c-c}{m-c}h_c(X_{i_1},\cdots,X_{i_c})\,, \tag{6}$$

and the compensator is:

$$\frac{n^{c/2}}{\binom{n}{m}}\sum_{C_c^{[nt]-1}}[h_c(X_{i_1},\cdots,X_{i_c})\{\binom{[nt]-c}{m-c}-\binom{i_c-c}{m-c}\}]\,.$$

Proof. The compensator of $Y_{n,c}(t)$ is given by:

$$\begin{aligned}&\sum_{k\leq[nt]}\{E(Y_{n,c}\left(\frac{k}{n}\right)/\mathcal{F}_n\left(\frac{k-1}{n}\right))-Y_{n,c}\left(\frac{k-1}{n}\right)\}\\&=\sum_{k=m}^{[nt]}\frac{n^{c/2}\binom{k-1-c}{m-1-c}}{\binom{n}{m}}\sum_{C_c^{k-1}}h_c(X_{i_1},\cdots,X_{i_c})\\&=\sum_{C_c^{[nt]-1}}h_c(X_{i_1},\cdots,X_{i_c})\sum_{k=i_c+1}^{[nt]}\frac{n^{c/2}\binom{k-c-1}{m-c-1}}{\binom{n}{m}}\end{aligned}$$

$$= \frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]-1}} [h_c(X_{i_1}, \cdots, X_{i_c})\{\binom{[nt]-c}{m-c} - \binom{i_c-c}{m-c}\}],$$

where we have used

$$\sum_{n=a}^{b} \binom{n}{a} = \binom{b+1}{a+1}.$$

Then, as $C_c^{[nt]} = C_c^{[nt]-1} + C_{c-1}^{[nt]-1} \times \{[nt]\}$, we have that the martingale part of $Y_{n,c}(t)$ is:

$$Z_{n,c}(t) = \frac{n^{c/2}\binom{[nt]-c}{m-c}}{\binom{n}{m}} \sum_{C_{c-1}^{[nt]-1}} h_c(X_{i_1}, \cdots, X_{i_{c-1}}, X_{[nt]})$$

$$+ \frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]-1}} \binom{i_c-c}{m-c} h_c(X_{i_1}, \cdots, X_{i_c})$$

$$= \frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]}} \binom{i_c-c}{m-c} h_c(X_{i_1}, \cdots, X_{i_c}).$$

Lemma 5. *As $n \to \infty$,*

$$\sup_t \Big| \frac{n^{c/2}}{\binom{n}{m}} \{ \sum_{C_m^{[nt]}} h_{m-1}(X_{i_1}, \cdots, X_{i_{m-1}})$$

$$- \sum_{C_c^{[nt]-1}} h_c(X_{i_1}, \cdots, X_{i_c})\{\binom{[nt]-c}{m-c} - \binom{m-c}{i_c-c}\}\} \Big| \xrightarrow[n\uparrow\infty]{P} 0.$$

i.e., the compensator of $n^{c/2} U_n$ and $Y_{n,c}(t)$ have the same limit.

Proof. Noting that $h_1 = \cdots = h_{c-1} = 0$, and using (2) with (4), the compensator of $U_n(t)$ can be written as:

$$\frac{1}{\binom{n}{m}} \sum_{k=c}^{m-1} \sum_{C_k^{[nt]-1}} \{\binom{[nt]-k}{m-k} - \binom{i_k-k}{m-k}\} \int h_k(x_1, \cdots, x_k) \prod_{j=1}^{k} d(I_{\{X_{i_j} \le x_j\}} - F(x_j)).$$

It is easy to see that the compensator of $Y_{n,c}$ is $n^{c/2}$ times the first term of the above summation. Therefore it is enough to prove that the following holds: as $n \to \infty$

$$(7) \qquad P(\sup_t \Big| \frac{n^{c/2}}{\binom{n}{m}} \sum_{k=c+1}^{m-1} \sum_{C_k^{[nt]-1}} \binom{[nt]-k}{m-k}$$

$$\times \int h_k(x_1, \cdots, x_k) \prod_{j=1}^{k} d(I_{\{X_{i_j} \le x_j\}} - F(x_j)) \Big| > \varepsilon) \to 0.$$

$$(8)\qquad P\Big(\sup_t \Big| \frac{n^{c/2}}{\binom{n}{m}} \sum_{k=c+1}^{m-1} \sum_{C_k^{[nt]-1}} \binom{i_k-k}{m-k} \times \int h_k(x_1, \cdots, x_k) \prod_{j=1}^{k} d(I_{\{X_{i_j} \le x_j\}} - F(x_j)) \Big| > \varepsilon\Big) \to 0 .$$

(7) is easily proved as in Lemma 3. (8) is proved as in Lemma 2 using the martingale structure of the process in (8) with respect to $\mathcal{G}_n(t) = \sigma[X_1, \cdots, X_{[nt]-1}]$.

A consequence of Lemmas 3 and 5 is

Lemma 6. *$n^{c/2} X_n(t)$, the martingale part of $n^{c/2} U_n(t)$, can be approximated by $Z_{n,c}(t)$, the martingale part of $Y_{n,c}(t)$.*

Now, we make an assumption on the kernel function h.

(H) There exists some $\phi \in L^2(\boldsymbol{R}, F)$ with $E(\phi(X)) = 0$ and $E(\phi^2(X)) = 1$ such that:

$$h_c(x_1, \cdots, x_c) = \prod_{i=1}^{c} \phi(x_i)$$

Dynkin and Mandelbaum [2] as well as Mandelbaum and Taqqu [11], work apparently under the same restirctions for h. From now on, we assume (H). Let

$$S_n(t) := \sum_{i=1}^{[nt]} \frac{\phi(X_i)}{\sqrt{n}},$$

$$f_n(t) := \frac{n^c \binom{[nt]-c}{m-c}}{\binom{n}{m}} .$$

Then, we have

$$(9)\qquad \begin{aligned} Z_{n,c}(t) &= \frac{n^{c/2}}{\binom{n}{m}} \sum_{C_c^{[nt]}} \binom{i_c-c}{m-c} h_c(X_{i_1}, \cdots, X_{i_c}) \\ &= \frac{n^c}{\binom{n}{m}} \sum_{i_c=c}^{[nt]} \cdots \sum_{i_1=1}^{i_2-1} \binom{i_c-c}{m-c} \prod_{j=1}^{c} \frac{\phi(X_{i_j})}{\sqrt{n}} \\ &= \int_{[0,t]} f_n(t_c) \int_{[0,t_c]} \cdots \int_{[0,t_2]} dS_n(t_1) \cdots dS_n(t) . \end{aligned}$$

Similarly,

$$(10)\qquad Y_{n,c}(t) = f_n(t) \int_{[0,t]} \cdots \int_{[0,t_2]} dS_n(t_1) \cdots dS_n(t_c) .$$

In the following we will denote by $\mathcal{L}(D[0, 1])$, the weak convergence in $D[0, 1]$ under the Skorokhod topology.

Lemma 7. *As* $n\to\infty$,

(i) $$Z_{n,c}(t)\xrightarrow{\mathcal{L}(D[0,1])}\frac{m!}{(m-c)!}\int_0^t t_c^{m-c}\int_0^{t_c}\cdots\int_0^{t_2}dW(t_1)\cdots dW(t_c)\,.$$

(ii) $$Y_{n,c}(t)\xrightarrow{\mathcal{L}(D[0,1])}\frac{m!}{(m-c)!}t^{m-c}\int_0^t\int_0^{t_c}\cdots\int_0^{t_2}dW(t_1)\cdots dW(t_c)\,.$$

where W is the Wiener process.

To prove this lemma we will make extensive use of the Skorokhod representation theorem ([16]) and the following lemma. We denote "in probability in $D[0, 1]$" by $P(D[0, 1])$.

Lemma 8. *Let A_n, A_0 be locally bounded predicatable processes in $D[0, 1]$ and M_n, M_0 be martingales in $D[0, 1]$ such that*:

(a) $M_n\to M_0$, $A_n\to A_0$ *in* $P(D[0, 1])$.

(b) $M_0, A_0\in C([0, 1])$.

(c) $\sup_n E(M_n^2(1))<\infty$.

Then $\int_0^\cdot A_n\,dM_n\to\int_0^\cdot A_0\,dM_0$ *in* $P(D[0, 1])$.

Proof. As $A_0\in C[0, 1]$ and, by tightness of $P A_n^{-1}$ for $n=0, 1, \cdots$ we have:

(α) $\lim_{k\to\infty}\sup_n P(\sup_t|A_n(t)|>k)=0$.

(β) $\lim_{\delta\downarrow 0}\overline{\lim_n} P(\sup_{|t-s|<\delta}|A_n(t)-A_n(s)|>\varepsilon)=0$ *for* $\forall\varepsilon>0$.

Let $$\varphi_k(x)=(x+k+1)\,I_{[-k-1,-k[}(x)+I_{[-k,k[}(x)+(-x+k+1)\,I_{[k,k+1[}(x)\,,$$
$$A_{n,k}=\varphi_k(A_n(t))\,A_n(t)\,,$$
$$N_{n,k}(t)=\int_0^t A_{n,k}(s)\,dM_n(s)\,,$$
$$N_n(t)=\int_0^t A_n(s)\,dM_n(s)\,.$$

As $P(d_0(N_{n,k}, N_n)>0)\le P(\sup_{0\le t\le 1}|A_n(t)|>k)$, where d_0 is the metric that defines the Skorokhod topology in $D[0, 1]$.

Then, because of (α), in order to prove $N_n\to N_0$ in $P(D[0, 1])$ it is enough to prove $N_{n,k}\to N_0$ in $P(D[0, 1])$. Therefore, we will assume from now on, that

$$|A_n(t)|\le C\quad\text{for }\forall n\quad\text{for some}\quad C>0\,.$$

Let $\tau^m=\{t_i=\frac{i}{m}/i=0, \cdots, m\}$ a partition of $[0, 1]$, and define

$$A_n^m(t)=A_n(0)\,I_{\{0\}}(t)+\sum_{i=0}^{k-1}A_n(t_i)\,I_{]t_i,t_{i+1}]}(t)\,.$$

We obtain the proof in two steps, as follows:

$$P(d_0(\int_0^\cdot A_n\,dM_n, \int_0^\cdot A_0\,dM_0)>\varepsilon)\leq P(d_0(\int_0^\cdot A_n\,dM_n, \int_0^\cdot A_n^m\,dM_n)>\frac{\varepsilon}{3})$$

$$+P(d_0(\int_0^\cdot A_n^m\,dM_n, \int_0^\cdot A_0^m\,dM_0)>\frac{\varepsilon}{3})+P(d_0(\int_0^\cdot A_0^m\,dM_0, \int_0^\cdot A_0\,dM_0)>\frac{\varepsilon}{3})$$

$$=: I_1+I_2+I_3\,.$$

We first prove that $I_1\to 0$.

$$I_1\leq P(\sup_{0\leq s\leq 1}|\int_0^t (A_n-A_n^m)\,dM_n|>\frac{\varepsilon}{3})$$

$$\leq\frac{3}{\varepsilon}E[\sup_{0\leq t\leq 1}|\int_0^t (A_n-A_n^m)\,dM_n|]\,.$$

By the Davis inequality,

$$\leq\frac{12}{\varepsilon}E(\sup_{|t-s|<\delta}|A_n(t)-A_n(s)|\langle M_n, M_n\rangle^{1/2}(1))\,.$$

By the Cauchy-Schwartz inequality,

$$\leq\frac{12}{\varepsilon^2}[E(\sup_{|t-s|<\delta}|A_n(t)-A_n(s)|^2)\,E(\langle M_n, M_n\rangle(1))]^{1/2}\,,$$

where $\delta=m^{-1}$. But, $\sup_n EM_n^2(1)<\infty$ and $E(\sup_{|t-s|<\delta}|A_n(t)-A_n(s)|^2)\to 0$ as $n\to\infty$ and $\delta\downarrow 0$, therefore, $I_1\to 0$. The proof of $I_3\to 0$ is analogous.

We next prove that $I_2\to 0$.

$$I_2\leq P(\sup_{0\leq t\leq 1}\sum_{i=0}^{m-1}C\,|M_n(t_{i+1}\wedge t)-M_0(t_{i+1}\wedge t))+(M_n(t_i\wedge t)-M_0(t_i\wedge t))|>\frac{\varepsilon}{6})$$

$$+P(\sup_{0\leq t\leq 1}|\sum_{i=0}^{m-1}(A_n(t_i)-A_0(t_i))\,(M_0(t_{i+1}\wedge t)-M_0(t_i\wedge t))|>\frac{\varepsilon}{6})\,.$$

The terms on the right-hand side tend to zero by assumptions (a) and (b).

Proof of Lemma 7.

By the Skorokhod theorem, it is possible to change the sample space and construct two processes $S_n^*(t)$ and $W^*(t)$ on the new sample space, which have the same law as $S_n(t)$ and $W(t)$, such that S_n converges a.e. to W^* in $D[0, 1]$. As $E((S_n^*(1))^2)=1$, applying Lemma 8 consecutively to obtain

$$\int_{[0,t_c)}\cdots\int_{[0,t_2)}dS_n^*(t_1)\cdots dS_n^*(t_{c-1})\xrightarrow{P(D[0,1])}\int_0^{t_c}\cdots\int_0^{t_2}dW(t_1)\cdots dW(t_{c-1})\,.$$

Noticing the uniform convergence of $f_n(t)$ to $f(t)=\frac{m!}{(m-c)!}t^{m-c}$ and the representation (9), we apply Lemma 8 again; to obtain (i). At the same time we have already proved (ii).

The next theorem is a consequence of Lemmas 3, 6 and 7.

Theorem 1. *Under (H), we have:*

$$n^{c/2}\,U_n(t)\xrightarrow{\mathcal{L}(D[0,1])}\frac{m!}{(m-c)!}\,t^{m-c}\int_0^t\int_0^{t_c}\cdots\int_0^{t_2}dW(t_1)\cdots dW(t_c)\,. \tag{11}$$

$$n^{c/2}\,X_n(t)\xrightarrow{\mathcal{L}(D[0,1])}\frac{m!}{(m-c)!}\int_0^t t_c^{m-c}\int_0^{t_c}\cdots\int_0^{t_2}dW(t_1)\cdots dW(t_c)\,. \tag{12}$$

$$\begin{aligned} &n^{c/2}(U_n(t)-X_n(t))\\ &\xrightarrow{\mathcal{L}(D[0,1])}\frac{m!}{(m-c-1)!}\int_0^t s^{m-c-1}\int_0^s\cdots\int_0^{t_2}dW(t_1)\cdots dW(t_c)\,ds\,. \end{aligned} \tag{13}$$

Proof of (13) is a direct consequence of (11), (12), Ito's formula and the fact that the limit processes are continuous.

3. Remarks

3.1. The result of Theorem 1 is different from the one obtained by Mandelbaum and Taqqu [11], Corollary 1; because we use the usual one dimensional time Wiener process, whereas they use multidimensional Wiener process. Also, the "projecting" functions h_n are easier to handle in our calculations.

3.2. Although the result of Lemma 8 is known to hold under weaker conditions than (c) (see, for example Jakubowski, et al. [7]), this version is suitable for our application and its proof is relatively easier. In fact, condition (c) can be replaced by the boundness of the jumps of the processes M_n, but the proof looses its simplicity.

3.3. The proof of Theorem 1 can also be obtained by means of Hermite polynomials, although it requires a rather complicated calculation. For example in the case of $Z_{n,c}(t)$, we have from (9) that:

$$Z_{n,c}(t)=f_n(t)\,F_c\Big(S_n(t),\sqrt{\frac{[nt]}{n}}\Big)-\int_0^t f_n'(s)\,F_c\Big(S_n(s),\sqrt{\frac{[ns]}{n}}\Big)_-d\Big(\frac{[ns]}{n}\Big)+o_P(1)\,, \tag{14}$$

where

$$f_n'(t)=\frac{n^{c-1}\binom{[nt]-c-1}{m-c-1}}{\binom{n}{m}}\,,$$

$$F_c(a,b)=\frac{b^c}{c!}H_c\Big(\frac{a}{b}\Big)\,,$$

H_c denotes the Hermite polynomial of degree c with leading coefficient 1 and the symbol $_-$ denotes limit from the left with respect of t.

Actually, the first term on the right of (14) is equal to $Y_{n,c}(t)$ (see (10)) and the second term is equal to its compensator up to a term $o_P(1)$. From here, as the Hermite polynomials are continuous, we get the limit processes in terms of H_c.

By Ito's formula it can be proved that these expressions coincide with (11), (12) and (13). In the case of $Z_{n,c}(t)$ we have:

$$\frac{m!}{m-c!}\int_0^t t_c^{m-c}\int_0^{t_c}\cdots\int_0^{t_2} dW(t_1)\cdots dW(t_c)$$
$$=f(t)\,F_c(W(t),\sqrt{t})-\int_0^t f'(s)\,F(W(t),\sqrt{t})\,ds\,.$$

3.4. Theorem 1 can be easily extended to the case

$$h_c(x_1,\cdots,x_c)=\sum_{i=1}^{\infty}\lambda_i\prod_{j=1}^{c}\phi_i(x_j)\,,$$

where ϕ_i is an orthonormal set of $L^2(\boldsymbol{R}, F)$.

This representation covers completely the case $c=2$ by taking ϕ_i as an orthonormal base of $L^2(\boldsymbol{R}, F)$.

3.5. In relation with U-processes, V-processes can be defined as (here we can't assume $\theta=0$)

$$V_n(t)=\frac{1}{n^m}\sum_{i_m=1}^{[nt]}\cdots\sum_{i_1=1}^{[nt]}h(X_{i_1},\cdots,X_{i_m})\,.$$

Under the condition

$$\zeta^*(F)=\max_{1\le i_1\le\cdots\le i_m\le m}E(h^2(X_{i_1},\cdots,X_{i_m}))<\infty\,,$$

instead of (1), analogous lemmas and theorems can be proved. A similar decomposition as in (2) can be obtained, i.e.,

$$V_n(t)=\frac{[nt]^m}{n^m}\sum_{k=0}^{m}\binom{m}{k}\int h_k(x_1,\cdots,x_k)\prod_{i=1}^{k}d(F_{[nt]}(x_i)-F(x_i))\,,$$
$$\text{where}\quad F_{[nt]}(x)=\sum_{j=1}^{[nt]}\frac{I_{\{X_j\le x\}}}{[nt]}\,.$$

In this case, $n^{c/2}(V_n(t)-\theta)$, its martingale and compensator parts have the same limits as $n^{c/2}\,U_n(t)$, $n^{c/2}\,X_n(t)$ and $n^{c/2}(U_n(t)-X_n(t))$ in Theorem 1.

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