ON DIRECTLY FINITE REGULAR RINGS

Dedicated to Professor Manabu Harada on his sixtieth birthday

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This paper is concerned with the following open problem for directly finite, von Neuman regular rings. The problem was given by Goodearl and Handelman [3]: what conditions on a regular ring \( R \) induce that the maximal right quotient ring of \( R \) is right and left self-injective. In [4], the author showed an example of directly finite, right self-injective regular ring which is not left self-injective. So we have an interest in this problem. In Theorem 17 in §3, we give necessary and sufficient conditions for this problem. In §2, we consider the maximal left quotient ring \( Q \) of a directly finite, right self-injective regular ring. We show that \( Q \) is directly finite (Theorem 7) and the factor ring \( Q/\mathcal{M} \) is the maximal left quotient ring of the factor ring \( R/#m \) for every maximal ideal \( \mathcal{M} \) (resp. \( D*n \)) of \( Q \) (resp. \( R \)) (Theorem 9). In §3, we give one generalization of a result in [5]: the maximal left quotient ring of a directly finite, right self-injective regular ring is left and right self-injective. Further we obtain necessary and sufficient conditions for the maximal right quotient ring of a regular ring to be directly finite (Theorem 16).

1. Preliminaries

All rings in this paper are associative with unit and ring homomorphisms are assumed to preserve the unit. A ring \( R \) is said to be directly finite if \( xy=1 \) implies \( yx=1 \) for all \( x, y \in R \). A ring is said to be directly infinite if \( R \) is not directly finite. A regular ring means von Neumann regular ring.

A rank function on a regular ring \( R \) is a map \( N: R \to [0, 1] \) satisfying the following conditions:

(a) \( N(1)=1 \),
(b) \( N(xy) \leq N(x) \) and \( N(xy) \leq N(y) \) for all \( x, y \in R \),
(c) \( N(e+f)=N(e)+N(f) \) for all orthogonal idempotents \( e, f \in R \),
(d) \( N(x)>0 \) for all non-zero \( x \in R \).

If \( R \) is a regular ring with a rank function \( N \), then \( \delta(x, y)=N(x-y) \) defines a metric on \( R \), this metric \( \delta \) is called \( N \)-metric or rank metric and the (Hausdorff)
completion of $R$ with respect to $\delta$ is a ring $\bar{R}$ which we call the $N$-completion of $R$.

An idempotent $e$ of a regular ring $R$ is said to be abelian if $eRe$ is strongly regular i.e., all idempotents of $eRe$ are central idempotents of $eRe$. An idempotent $e$ of $R$ is said to be directly finite if $eRe$ is directly finite as a ring. For a ring $R$, we use $B(R)$ to denote the central idempotents in $R$. We note that $B(R)$ is a Boolean algebra in which $e \lor f = e + f - ef$ and $e \land f = ef$, while $e' = 1 - e$. If $R$ is regular and right self-injective, then $B(R)$ is complete [2].

Let $R$ be a regular, right self-injective ring. For a given element $x$ in $R$, put $H = \{ g \in B(R) | xg = 0 \}$ and $1 - h = \lor_{g \in H} g$ in $B(R)$. The idempotent $h$ is called the central cover of $x$, denote $c.c(x)$.

Let $R$ be a directly finite, right self-injective regular ring. Then $R$ is said to be Type II$_f$ if $R$ contains no abelian idempotents. And $R$ is said to be Type I$_f$ if $R$ contains an abelian idempotent $f$ with $c.c(f) = 1$. Note that $R$ is uniquely a direct product of rings of Type I$_f$, II$_f$ ([2] Theorem 10.13).

For a regular ring $R$ and elements $a, b \in R$, we use $aR \leq bR$ to mean that $aR$ is isomorphic to a direct summand of $bR$. A regular ring satisfies general comparability provided that for any $x, y \in R$, there exists $g \in B(R)$ such that $gxR \preceq_R gyR$ and $(1 - g)xR \succeq_R (1 - g)yR$. Note that every regular right self-injective ring satisfies general comparability ([2] Corollary 9.15).

Let $R$ be a subring of a ring $Q$. For every element $x$ of $Q$ and right ideal $I$ of $Q$, we use $(x^Q, I)_R$, $(J, x)_{Q'}$ to denote the right ideal $\{a \in R | xa \in I\}$, the left ideal $\{a \in R | ax \in J\}$, respectively.

Lemma A. For two idempotents $e, f$ of a ring $R$, the following conditions are equivalent.
1). $eR = fR$.
2). There exist elements $x \in eR, y \in fRe$ such that $yx = f, xy = e$.
3). $Re = Rf$.

Proof. It is trivial.

Lemma B. Let $R$ be a subring of a ring $Q$ and $\bar{R}$ be a factor ring of $R$. For two idempotents $e, f$ with $eR = fR$, the followings hold.
1). $\bar{e}R = \bar{f}R$ and $\bar{R}e = \bar{R}f$.
2). $eQ = fQ$ and $Qe = Qf$.

Proof. By Lemma A, it is easy.

Lemma C. Let $R$ be a ring. For two idempotents $e, f$ and an integer $n$, the followings are equivalent.
1) $n(eR) = fR$.
2) $n(Re) = Rf$. 
where \( n(eR), n(Re) \) are direct sums of \( n \)-copies of \( eR, Re \), respectively.

Proof. It is easy.

**Lemma D.** Let \( R \) be a regular ring. For a right ideal \( \sum_{i=1}^{\infty} a_i R \), there exist pairwise orthogonal idempotents \( \{ e_i \}_{i=1}^{\infty} \) which satisfy the following:

1. \( \sum_{i=1}^{m} \oplus e_i R = \sum_{i=1}^{m} a_i R \) for all \( m \).
2. \( e_i R \leq a_i R \) for all \( i \).

If \( \sum_{i=1}^{\infty} \oplus a_i R \) is directsum, then \( \{ e_i \} \) satisfy (1), (3).

3. \( e_i R \leq a_i R \) for all \( i \).

Proof. We prove Lemma by induction on \( m \). For \( m = 1 \) it is trivial.

Assume that \( \{ e_i \}_{i=1}^{n} \) satisfy (1), (2) or (3). Now \( \sum_{i=1}^{m+1} a_i R = \sum_{i=1}^{m} \oplus e_i R + a_{m+1} R = \sum_{i=1}^{m} \oplus e_i R \oplus (1 - \sum_{i=1}^{m} e_i) a_{m+1} R \).

Let \( e'_{m+1} \) be an idempotent with \( e'_{m+1} R = (1 - \sum_{i=1}^{m} e_i) a_{m+1} R \). Put \( e_{m+1} = e'_{m+1} (1 - \sum_{i=1}^{m} e_i) \). Then \( e_{m+1}^2 = e_{m+1} \).

And \( (1 - \sum_{i=1}^{m} e_i) a_{m+1} R = e'_{m+1} R \supset e'_{m+1} (1 - \sum_{i=1}^{m} e_i) R \supset e'_{m+1} (1 - \sum_{i=1}^{m} e_i) e'_m R = e'_{m+1} R \), i.e., \( e_{m+1} R = (1 - \sum_{i=1}^{m} e_i) a_{m+1} R \).

Thus \( \{ e_i \}_{i=1}^{m+1} \) are orthogonal and satisfy (1). Since \( (1 - \sum_{i=1}^{m} e_i) a_{m+1} R \) is projective, we have \( (1 - \sum_{i=1}^{m} e_i) a_{m+1} R = e_{m+1} R \leq a_{m+1} R \). For (3), we have \( A = \sum_{i=1}^{m} \oplus e_i R \oplus a_{m+1} R = \sum_{i=1}^{m} \oplus e_i R \oplus e_{m+1} R \). We denote by \( p \) the projection from \( A \) to \( e_{m+1} R \) induced by the decomposition \( A = \sum_{i=1}^{m} e_i R \oplus e_{m+1} R \). Then \( p \) induce an isomorphism of \( a_{m+1} R \) to \( e_{m+1} R \).

2. **Directly finite maximal quotient ring**

We consider the necessary and sufficient condition for a regular ring \( R \) to have the directly finite maximal right quotient ring of \( R \). For a prime regular ring with a rank function, the following theorem is known [2], [3].

**Theorem 1.** ([2] Theorem 21.18 and 19) Let \( R \) be a prime regular ring with a rank function \( N \). Then \( Q(R) \) is directly finite if and only if \( Q(R) \leq \widehat{R} \) as a subring if and only if \( \sup \{ N(x) \mid x \in I \} = 1 \) for all essential right ideals \( I \) of \( R \) where \( Q(R) \) is the maximal right quotient ring of \( R \) and \( \widehat{R} \) is the completion of \( R \) in the \( N \)-metric.

In general case, we have the following Proposition and we consider again this property in Theorem 16.

**Proposition 2.** For a regular ring \( R \), the following conditions are equivalent.
(1) The maximal right quotient ring $Q$ of $R$ is directly finite.

(2) Every right ideal isomorphic to some essential right ideal is an essential right ideal of $R$.

Proof. (1)$\Rightarrow$(2): Let $Q$ be directly finite. Suppose that $R$ does not satisfy (2). Let $I, J$ be isomorphic right ideals such that $J$ is essential in $R$ but $I$ is not essential in $R$. There exists an element $q \in Q$ such that $q: J \rightarrow I$ ($x \mapsto qx$) is a given isomorphism. Since the right $R$-module $J$ is essential in right $R$-module $Q$, the homomorphism $q: Q \rightarrow Q$ is a monomorphism. Since $I$ is not essential in right $R$-module $Q$, $qQ$ is a proper direct summand of right $R$-module $Q$. This contradicts that $Q$ is directly finite.

(2)$\Rightarrow$(1): Assume that $Q$ is directly infinite. Then there exists an element $q$ of such that $r_0(q) = \{x \in Q | qx = 0\} = 0, Q \neq qQ$. Then $(q' R) = \{x \in R | qx \in R\}$ is an essential right ideal. By $Q \neq qQ$ and $qQ \cap R \supseteq q(q' R)$, $q(q' R)$ is not essential in $R_R$. On the other hand, by $r_0(q) = 0$, the homomorphism $q: (q' R) \rightarrow q(q' R)$ is an isomorphism between two right ideals of $R$. So (2) does not hold.

We consider the maximal left quotient ring of $R$ which is directly finite, right self-injective regular ring. For the end, we start with the following proposition.

**Proposition 3.** Let $R$ be a directly finite, right self-injective regular ring with no abelian idempotents. Then there exists a set $\{e^*\}_{n=1}^\infty$ of orthogonal idempotents such that $\sum_{n=1}^\infty e^*_R$ is an essential right ideal and $\sum_{n=1}^m e^*_R (1 - \sum_{i=1}^n e^*_R) R = e^*_R, 2^m (e^*_R) R = R$ for all $m = 1, 2, \ldots$.

Proof. By [2] Theorem 10.28, there exists an idempotent $e^* \in R$ such that $2e^*_R R = R, (1 - e^*_R) R = e^*_R R$. For $R_1 = (1 - e^*_R) R (1 - e^*_R)$, there exists an idempotent $e^*_R \in R_1$ such that $2(e^*_R R_1) R = R$, i.e., $2^2 (e^*_R R) R = R$ and $(1 - e^*_R - e^*_R) R = e^*_R R$. We obtain inductively a set $\{e^*_R\}_{n=1}^\infty$ of orthogonal idempotents such that $2^n (e^*_R) R = R, (1 - \sum_{i=1}^n e^*_R) R = e^*_R R$ for all $n = 1, 2, \ldots$.

For a given nonzero idempotent $f \in R$, suppose that $f R \leq e^*_R R$ for all $n$. Then $f R \subseteq \sum_{n=1}^\infty e^*_R R \subseteq R$. Since $R$ is directly finite, we have $f = 0$ from [2] Corollary 9.23. This is a contradiction. So, for $f \in R$, we obtain from general comparability on $R$ that $f R \geq e^*_R R$ for some integer $m$ and some nonzero idempotent $g \in B(R)$.

Suppose that $\sum_{n=1}^\infty e^*_R R$ is not essential in $R_R$, i.e., there exists a nonzero idempotent $e$ with $(\sum_{n=1}^\infty e^*_R R) \cap e R = 0$. By the above argument, $e R \geq e^*_R R$ for some integer $m$ and some nonzero idempotent $g \in B(R)$. Then $\sum_{n=1}^\infty e^*_R R \oplus$
geR ⊕ gR, i.e., there exist exists a nonzero idempotent f' such that ∑\( n=1 \) \( ge_{n}^{*}R \oplus geR \oplus f' \)R = gR. While \( gR \lesssim \sum_{n=1}^{m} \oplus ge_{n}^{*}R \oplus geR \) from (1− ∑ \( n=1 \) \( e_{n}^{*} \))R \( \simeq e_{n}^{*}R \) and \( eR \succeq ge_{m}^{*}R \), that is, gR is isomorphic to a proper direct summand of gR.

This contradicts that R is directly finite. So \( \sum_{n=1}^{m} \oplus e_{n}^{*}R \) is essential right ideal of R.

Now we have (1− \( n=1 \) \( e_{n}^{*} \))R ⊃ \( \sum_{n=m+1}^{\infty} \oplus e_{n}^{*}R \). By the same argument, we obtain that \( \sum_{n=m+1}^{\infty} \oplus e_{n}^{*}R \) is essential in (1− \( n=1 \) \( e_{n}^{*} \))R.

Throughout this paper, we use \( \{e_{n}^{*}\} \) to denote the orthogonal idempotents as above.

From Proof of Proposition 3, we have the following proposition.

**Proposition 4.** Let R be a directly finite right self-injective regular ring with no abelian idempotents. For every idempotent e, there exist an integer m and a nonzero central idempotent g \( \in B(R) \) and an idempotent f of eRe such that \( ge_{m}^{*}R = fR \).

**Corollary of Proposition 3.** Let R be as above. Then, for every n, \( \sum_{i=n+1}^{\infty} \oplus Re_{i}^{*} \) is essential in R(1− \( \sum_{i=1}^{n} e_{i}^{*} \)).

Proof. Since \( \{e_{i}^{*}\}_{i=1}^{\infty} \) are pairwise orthogonal, \( R(1− \sum_{i=1}^{n} e_{i}^{*}) \) contains \( \sum_{i=n+1}^{\infty} Re_{i}^{*} \).

Suppose that there exists a nonzero idempotent e \( \in R(1− \sum_{i=1}^{n} e_{i}^{*}) \) with \( Re \cap \sum_{i=n+1}^{\infty} Re_{i}^{*} = 0 \). By Proposition 4, we may assume that eRe = c·c(e)e_{m}^{*}R for some m. Put \( g = c·c(e) \in B(R) \).

(i) A case of \( m < n \). Then \( Re_{m}^{*}g \simeq Re \subset R(1− \sum_{i=1}^{n} e_{i}^{*})g \simeq Re_{m}^{*}g \) implies \( ge_{m}^{*}R \lesssim ge_{m}^{*}R \). So the following holds:

\[
2^{m}(ge_{m}^{*}R) \succeq 2^{m}(ge_{m}^{*}R) = gR.
\]

On the other hand we obtain from \( m < n \) that \( 2^{m}(ge_{m}^{*}R) \) is isomorphic to a proper direct summand of \( gR = 2^{m}(ge_{m}^{*}R) \). By (1), we obtain that gR is isomorphic to a proper direct summand of gR. This contradicts that R is directly finite.

(ii) A case of \( m \geq n \). Considering \( \sum_{i=1}^{n} \oplus Re_{i}^{*} \oplus \sum_{i=n+1}^{\infty} Re_{i}^{*}g \oplus Re \) in a regular ring, we may assume that \( \{ge_{i}^{*}\}_{i=1}^{\infty} \cup \{e\} \) are pairwise orthogonal. Then \( \sum_{i=1}^{n} \oplus ge_{i}^{*}R \oplus eR \) is a proper direct summand of \( \sum_{i=1}^{n} \oplus ge_{i}^{*}R \oplus eR \), i.e., of gR. On the other hand \( Re \simeq Re_{m}^{*}g \simeq R(1− \sum_{i=1}^{n} e_{i}^{*})g \) implies \( \sum_{i=1}^{n} \oplus ge_{i}^{*}R \oplus eR \simeq \sum_{i=1}^{n} \oplus ge_{i}^{*}R \oplus g(1− \sum_{i=1}^{n} e_{i}^{*})R = gR \). This contradicts that R is directly finite. Consequently we
obtain that \( \sum_{i=\sigma+1}^{\sigma} \bigoplus Re_i^k \) is essential in \( R(1-\sum_{i=1}^{n} e_i^k) \).

In a right self-injective regular ring \( R \), \( B(R) \) is complete Boolean algebra ([2] Proposition 9.9). For any subset \( \{g_i\}_{i \in I} \subset B(R) \) of pairwise orthogonal idempotents, \( (\bigvee g_i)R \) is the injective hull of a right ideal \( \sum_i \bigoplus g_iR \), and the natural ring homomorphism \( (\bigvee g_i)R \to \prod g_iR \) is an isomorphism ([2] Proposition 9.9 and 9.10), where \( \prod g_iR \) is the ring of direct product of rings \( \{g_i\}_{i \in I} \).

So we regard as \( (\bigvee g_i)R = \prod g_iR \) and we denote by \( \prod a_i(a_i \in g_iR) \) the element \( a \in (\bigvee g_i)R \) such that \( a g_i = a_i \) for all \( i \in I \). A subset \( I \) of \( R \) is said to be centrally closed if, for every subset \( A = \{a_i \in I\} \) satisfying the following condition (*), \( I \) contains \( a = \prod a_i \in \prod \{c \cdot c(a_i)R = (\bigvee c \cdot c(a_i))R. \)

(*): \( \{c \cdot c(a) | a \in A \} \) are pairwise orthogoanal idempotents in \( B(R) \).

Note that every essentially closed left ideal is centrally closed in right self-injective regular rings.

**Proposition 5.** Let \( R \) be a directly finite, right self-injective regular ring with no abelian idempotents and let \( I \) be a centrally closed non-zero left ideal of \( R \). Then there exist orthogonal idempotents \( \{e_i\}_{i=1}^{\tau} \) in \( I \) and an central idempotent \( 1 - g = B(R) \cap I \) which satisfy the following conditions:

1) \( e_i R = c \cdot c(e_i) e_i^k R \) for all \( i \), where \( \{n(i)\}_{i=1}^{\tau} \) is a strictly increasing sequence of integers.

2) \( \sum_{i=1}^{\tau} \bigoplus Re_i \) is essential in \( Ig \) and \( 0 = Ig \cap B(R) \).

Proof. Let \( \{g_a\} \) be a maximal subset of orthogonal idempotents in \( I \cap B(R) \). Since \( I \) is centrally closed, \( I \) contains \( 1 - g = \prod h \in B(R), n \not \in I \). We may assume that \( I \cap B(R) = 0 \). We may assume that \( I \cap B(R) = 0 \) and \( I \) is centrally closed.

By induction on \( m \), we will show that there exist orthogonal idempotents \( \{e_i\}_{i=1}^{\tau} \) of \( I \) and integers \( n(1) < n(2) < \cdots < n(m) \) satisfying 1) and the following condition:

2') For any idempotent \( f \) in \( I(1-\sum_{i=1}^{n} e_i) \), \( fR = c \cdot c(f) e_i^k R \) implies \( i > n(m) \).

For \( m=1 \), let \( n(1) \) be the smallest integer of \( \{n | c \cdot c(f) e_i^k R = fR \) for some nonzero idempotent \( f \) in \( I \) which is not empty by Proposition 4. Let \( \{g_a\}_{a \in A} \) be a maximal subset of family of orthogonal idempotents in \( \{g \in B(R) | eR = ge_i^k \} \) for some idempotent \( e \) in \( I \). Let \( \{e_a\}_{a \in A} \) be a set of idempotents of \( I \) such that \( c \cdot c(e_a) = g_a, e_a R = ge_a^k R \) for all \( a \in A \). Put \( e_i = \prod e_a \), \( g_i = \prod g_a \) in \( \prod g_a R = (\bigvee g_a)R \). Since \( I \) is centrally closed, it follows that \( I \) contains \( e_i \) and...
\( I = R_1 \oplus I(1-e_1) \). Let \( t \) be an integer such that \( fR = c \cdot c(f)e_i^*R \) for some non-zero idempotent \( f \) in \( I(1-e_1) \). Since \( n(1) \) is minimal and \( \{g_\alpha\}_\Lambda \) is maximal and \( I \cap B(R) = 0 \), we have \( n(1) < t \).

Assume that \( \{e_{1}^\tau\}_{\tau=1} \) satisfy (1), (2'). Now we consider a case of \( I(1 - \sum_{i=1}^m e_i) = 0 \), i.e., \( I = \sum_{i=1}^m \oplus R e_i \). By Corollary of Proposition 3, we have
\[
R(1 - \sum_{i=1}^m e_i^\tau) \supset \sum_{i=n(m)+1}^m \oplus Re_i^\tau.
\]
Since \( R_{e_i} = Re_{i(m)}^\tau c \cdot c(e_{m}) = R(1 - \sum_{i=1}^m \oplus e_i^\tau) c \cdot c(e_{m}) \), it follows that \( R_{e_i} \) has an essential submodule isomorphic to \( \sum_{i=n(m)+1}^m \oplus Re_i^\tau c \cdot c(e_{m}) \).

We can see from Lemma D that there exist orthogonal idempotents \( \{e_i'\}_{i=m} \) in \( e_m R_{e_i} \) and a sequence \( \{n(i) = n(m) + 1 + (i-m)\}_{i=m} \) such that \( e_i'R = c \cdot c(e_{m}) e_i^\tau R \) for all \( m, m+1, \cdots, \sum_{i=1}^m \oplus e_i \) is essential in \( R \) and \( \{e_i'\}_{i=m} \cup \{e_i\}_{i=1}^{n-1} \) are orthogonal idempotents. Thus we have orthogonal idempotents \( \{e_i\}_{i=1}^{n} \) and a sequence \( \{n(i)\}_{i=1}^{n} \) satisfying (1), (2).

Next, we consider a case of \( I(1 - \sum_{i=1}^m e_i) \neq 0 \). Using \( I(1 - \sum_{i=1}^m e_i) \) in place of \( I \), the same argument for \( m=1 \) implies that there exist an idempotent \( e_{m+1} \) in \( I(1 - \sum_{i=1}^m e_i) \) and an integer \( n(m+1) \) satisfying the same conditions's above. We can see that the idempotent \( e_{m+1} = (1 - \sum_{i=1}^m e_i) e_{m+1}' \) satisfy 1) and \( \{e_i\}_{i=1}^{n+1} \) are orthogonal and satisfy 2'). Thus we obtain orthogonal idempotents \( \{e_i\}_{i=1}^{n+1} \) satisfying 1).

We will show that \( \sum_{i=1}^m \oplus Re_i \) is essential in \( R \). Suppose that \( \sum_{i=1}^m \oplus Re_i \cap Rf = 0 \) for some non zero idempotent \( f \) of \( R \). By Proposition 4, we may assume that \( fR = c \cdot c(f)e_i^*R \) for some integer \( t \). Let \( m \) be an integer with \( n(m) > t \). Since \( I = \sum_{i=1}^m Re_i \oplus I(1 - \sum_{i=1}^m e_i) \supset \sum_{i=1}^m Re_i \oplus Rf \) implies \( I(1 - \sum_{i=1}^m e_i) \supset Rf \), there exists an idempotent \( f' \) of \( I(1 - \sum_{i=1}^m e_i) \) such that \( Rf' \subset Rf = Re_i^* \cdot c \cdot c(f) \). This contradicts 2'), So \( \sum_{i=1}^m \oplus Re_i \) is essential in \( R \).

**Corollary.** Let \( R, I \) be as above. Then there exist pairwise orthogonal idempotents \( \{f_i\}_{i=1}^{n-1} \) satisfying the following conditions:

(a) \( f_i R = c \cdot c(f_i)e_i^*R \) for all \( i \), where \( \{n(i)\}_{i=1}^{n-1} \) is a strictly increasing sequence of integers.

(b) \( \sum_{i=1}^{n} \oplus Rf_i \) is essential in \( R \).

**Proof.** Let \( g, \{e_i\}, \{n(i)\} \) be as in Proposition 5. If \( 1 - g \neq 0 \), the assertion is trivial. Suppose that \( 1 - g = 0 \). Put \( f_i = (1 - g) e_i^* + e_j \) if \( i = n(j) \) for some \( j \), put \( f_i = (1 - g) e_i^* \) if \( i \neq n(j) \) for all \( j \). Put \( n(i) = i \) for all \( i \). By Proposition 5, the
assertion is clear.

**Proposition 6.** Let $R$ be a directly finite, right self-injective regular ring, $Q$ be the maximal left quotient ring of $R$. Then, for a given element $q \in Q$, there exists a central idempotent $g \in \mathcal{B}(R)$ and orthogonal idempotents $\{e_n\}_{n=1}^\infty$ which satisfy the following

1. $R(1-g) \bigoplus \bigoplus_{n=1}^\infty \oplus Re_n$ is essential in $(R^*, q)$, $\sum_{n=1}^\infty \oplus Re_n$ is essential in $Rg$.

2. $e_n R = ge_n^* R$ for all $n$.

**Proof.** By [2] Theorem 10.13, there is a central idempotent $g^* \in \mathcal{B}(R)$ such that $(1-g^*)R$ is a ring of type $I_f$, $g^* R$ is a ring of type $II_f$. Then by [2] Corollary 10.25, we have $(1-g^*)R = (1-g^*)Q$, so we have $(1-g^*) \in (R^*, q)$. It is sufficient to show that $g^* R$ satisfy the assertion. So we may assume that $R$ is type $II_f$.

Let $\{a_\alpha\}_A$ be a subset of $(R^*, q)$ such that $\{c \cdot c(a_\alpha)\}_A$ are pairwise orthogonal idempotents in $\mathcal{B}(R)$. Since $a_\alpha R \subseteq R$ for all $\alpha \in A$, it follows that $(\prod_\alpha a_\alpha)q = \prod_\alpha (a_\alpha q) \subseteq \prod_\alpha (c \cdot c(a_\alpha)R) = (\bigvee_\alpha (c \cdot c(a_\alpha)))R$, i.e., $\prod_\alpha a_\alpha \in (R^*, q)$. So $(R^*, q)$ is centrally closed. By Proposition 5, there exist orthogonal idempotents $\{e_t\}_{t=1}^\infty$ and a central idempotent $1-g \in \mathcal{B}(R) \cap (R^*, q)$ satisfying 1), 2) of Proposition 5. Then we may assume for the sake of simplicity that $R = Rg$, $B(R) \cap (R^*, q) = 0$. Since $(R^*, q)$ is an essential left ideal, $\bigoplus_{i=1}^\infty \oplus Re_t$ is essential left ideal from 2) of Proposition 5.

We will show that $c \cdot c(e_t) = 1$ and $n(i) = i$ for all $i \in N$. Suppose that $c \cdot c(e_t) \neq 1$ or $n(i) \neq i$ for some $t$. Then we have $K_2 = \bigoplus_{i=1}^\infty \bigoplus (1-c \cdot c(e_t))e_i^* R = \bigoplus_{i=1}^\infty e_i^* R \neq 0$. While we obtain $\bigoplus_{i=1}^\infty \oplus e_i R \cong \bigoplus_{i=1}^\infty \bigoplus c \cdot c(e_t)e_i^* R = K_2$ from 1) of Proposition 5 and and Lemma A. So $\bigoplus_{i=1}^\infty \oplus e_i R$ is isomorphic to a proper direct summand $K_2$ of an essential right ideal $\bigoplus_{i=1}^\infty \bigoplus e_i^* R = K_1 \oplus K_2$. By Proposition 2, $\bigoplus_{i=1}^\infty \oplus e_i R$ is not essential in $R$. Since $R$ is right self-injective, there exists an idempotent $e$ in $R$ such that $eR \cap \bigoplus_{i=1}^\infty \oplus e_i R = 0$ and $e(\bigoplus_{i=1}^\infty \oplus e_i R) = 0$. Put $x = xe = \bigoplus_{i=1}^\infty x_i e_i \in R \cap \bigoplus_{i=1}^\infty \oplus Re_i$. Then we obtain that $0 = xee = x_i e_i$ for all $i \in N$, so $Re \cap \bigoplus_{i=1}^\infty \oplus Re_i = 0$. This contradicts that $\bigoplus_{i=1}^\infty \oplus Re_i$ is essential in $R$. We obtain that $c \cdot c(e_t) = 1$, $n(i) = i$ for all $i \in N$.

**Theorem 7.** Let $R$ be a directly finite, right self-injective regular ring. Then the maximal left quotient ring $Q$ of $R$ is directly finite.

**Proof.** We may assume that $R$ is Type $II_f$. Suppose that $Q$ is directly
infinite. Then there is an element $q$ of $Q$ such that $1_q(q)=0$ and $Q_q\neq Q$. There exist a central idempotent $g$ and orthogonal idempotents $\{e_i\}_{i=1}^\infty$ of $R$ which satisfy the following conditions:

1. $(R, R) \supseteq R(1-g) \oplus \sum_{i=1}^\infty \oplus Re_i$

2. $Rg \supseteq \sum_{i=1}^\infty \oplus Re_i$

3. $e_iR \cong ge_i^*R$ for all $i$.

We will show that (i) $Q(1-g)q=Q(1-g)$ and (ii) $Qgq=Qg$.

(i): Since $1-g \in (R, R)$ implies $(1-g)q \in R$, we have $R(1-g)q \oplus A = R(1-g)q$. While we have $R(1-g) = R(1-g)q$ from $1_R/(q) \subset 1_q(q)=0$. Since $R$ is directly finite, we obtain $A=0$, i.e., $Q(1-g) = Q(1-g)q$.

(ii): Here we claim that $\sum_{i=1}^\infty \oplus Re_i$ is essential in $Rg$. Suppose that there exists a nonzero idempotent $e$ in $Rg$ such that $Re \cap \sum_{i=1}^\infty \oplus Re_i = 0$. Further we may assume from Proposition 4 that $eR \cong ge_i^*R$ for some integer $m$ and some nonzero central idempotent $g'$ in $Rg$. Considering the proper direct summand $\sum_{i=1}^\infty \oplus Re_i g' \oplus Re_i$ of $Rg'$, there exist orthogonal idempotents $\{e_i\} \cup \{e'_i\}_{i=1}^{m+1}$ such that $Re = Re', Re = Re_i g' = Re_i g'$ for all $n=1, 2, \ldots, m+1$. Then $(e'+\sum_{i=1}^{m+1} e_i)R$ is a proper direct summand of $g'R$. On the other hand it easily follows from (3) and Proposition 3 that $g'R \cong \sum_{i=1}^\infty \oplus e_i R \oplus e'R$. This contradicts that $R$ is directly finite. So we obtain that $\sum_{i=1}^\infty \oplus Re_i$ is essential in $Rg$.

Since $Rg$ is essential in $kQg$, we have $\sum_{i=1}^\infty \oplus Re_i \subset Qg$. While we obtain from (2) and non-singularity of $kQ$ that $\sum_{i=1}^\infty \oplus Re_i$ is essential in $Qg$. Thus we obtain $Qgq = Qg$.

From (i) and (ii), we obtain $Qq = Q$. This is a contradiction. Thus $Q$ is directly finite.

**Proposition 8.** Let $R$ be a directly finite, right self-injective regular ring which contains no nonzero abelian idempotents and $\mathfrak{M}$ a maximal ideal of $B(R)$. Let $m$ be the maximal ideal of $R$ such that $m \supset \mathfrak{M}R$ are essential right ideal of $\bar{R}$. We denote by $\bar{R}$ the factor ring $R/\mathfrak{M}R$.

1. For a given idempotent $e$ of $R$, the following conditions are equivalent.
   a. $m$ contains $e$ but $\mathfrak{M}R$ does not contain $e$.
   b. $\mathfrak{M}$ does not contain the central cover $c \cdot c(e)$ of $e$ and $\mathfrak{M}$ contains all central idempotents $g \in B(R)$ satisfying $ge_n^*R \leq eR$ for some integer $n$.
   c. There exist orthogonal central idempotents $\{g_i\}_{i=1}^\infty$ and idempotents $e_i, e_j$ and integers $\{n(t)\}_{t=1}^\infty$ which satisfy the following conditions;
(i) \[ ge_i \cdot g = g \cdot ge_{\alpha(i) - 1} \] and \[ ge_i = g \cdot ge_{\alpha(i)} \] for all \( t \).

(ii) \( \{n(t)\}_{t=1}^\infty \) is a strictly increasing sequence of integers.

(iii) \( \alpha_t R \subseteq e R \subseteq \alpha_t^* R \).

(iv) \( \bigwedge_{t=1}^\infty g_t = c \cdot c(e) \in \mathcal{M}, \) \( g_t \in \mathcal{M} \) for all \( t \).

(d) \( \mathfrak{K}_0(\bar{R}) \subseteq \bar{R} \) and \( \bar{e} \neq 0 \).

(II) For an idempotent \( e \) in \( R \), \( m \) does not contain \( e \) if and only if there exist a nonzero central idempotent \( g \) and an integer \( n \) such that \( ge_n^* R \subseteq e R \) and \( g \in \mathcal{M} \).

Proof. (a) \( \Rightarrow \) (b): Let \( e \) be an idempotent in \( m \backslash \mathcal{M} R \). Then \( e \in \mathcal{M} R \) implies \( c \cdot c(e) \in \mathcal{M} \). Suppose that there exist a central idempotent \( g \) and an integer \( n \) satisfying the following condition:

(1) \[ ge_n^* R \subseteq e R \] and \( g \in \mathcal{M} \).

From \( 2^*(e_n^* R) = R \), there exist orthogonal idempotents \( \{e_i\}_{i=1}^\infty \) such that \[ \sum_{i=1}^\infty e_i = 1 \], \( e_i R = e_n^* R \) for all \( i = 1, 2, \ldots, 2^n \) and \( e_i = e_n^* \). So we obtain that \( \sum_{i=1}^\infty \bar{e}_i = \bar{1} \), \( \bar{e}_i R \subseteq \bar{e}_n^* R \), \( \bar{e}_n^* \neq 0 \) in \( \bar{R} \). So we have the following:

(2) \[ \bar{R} = \bar{e}_n^* R \].

On the other hand, we obtain from (1) and Lemma B that \( g = \bar{1} \) and \( g_n^* R = g \bar{e}_n^* R \). Then by (2), we obtain \( \bar{R} = \bar{e}_n^* R \subseteq \bar{R} \). This contradicts \( e \in m \).

We obtain the last part of (b).

(b) \( \Rightarrow \) (c): Assume that (b) for an idempotent \( e \) holds. Let \( n(1) \) be a minimal integer of \( \{n \geq 0 | e R \supseteq g e_n^* R \text{ for some } 0 \neq g \in B(R) \} = N \) where \( e_n^* = 1 \). Let \( \{g_i\}_I \) be a maximal subset of orthogonal idempotents in \( \{g \in B(R) | e R \supseteq ge_n^* R \} = J \). Since \( R \) is injective and \( \sum_{i=1}^\infty \oplus g_i e_n^* R \subseteq e R \), it follows that \( g_i e_n^* R \subseteq e R \) for \( g_i = \bigvee_{i=1}^\infty g_i \). Since \( \{g_i\}_I \) is maximal and \( R \) satisfies general comparability, we obtain \( (1 - g_i) e R \subseteq e_{n(i)}^* R \). Since \( n(1) \) is minimal, we have \( g_i e R \subseteq g_i e_{n(i)}^* R \) when \( n(1) > 0 \). From \( c \cdot c(e) \in \mathcal{M} \) and \( g_i \in \mathcal{M} \) we see that \( (1 - g_i)e \neq 0 \) and \( (1 - g_i) e \) holds (b). By the same argument as above for \( (1 - g_i)e \), there exist a central idempotent \( g_2 \) and an integer \( n(2) \) which satisfy the same conditions as above. Since \( n(1) \) is the minimal of \( N \), we have \( n(1) < n(2) \). By induction, we can obtain orthogonal central idempotents \( \{g_i\}^\infty_{i=1} \) and an increasing sequence \( \{n(t)\}_{t=1}^\infty \) of integers, which satisfy the following conditions:

(3) \[ e R \supseteq \bigoplus_{i=1}^m g_i e_{n(i)}^* R, \sum_{i=1}^m g_i e_{n(i)-1} R \supseteq \bigoplus_{i=1}^m g_i e R \Bigg( \sum_{i=1}^m g_i e_{n(i)}^* R \Bigg) \]

(4) \[ \bigwedge_{t=1}^\infty g_t = c \cdot c(e), \quad n(1) < n(2) < \cdots \]

Because it follows from \( (1 - g_1) \cdots (1 - g_r) e R \subseteq (1 - g_t) e_{n(t)}^* R \) that \( g^* R \subseteq e_{n(t)}^* R \) for all
$t=1, 2, \ldots$, where $g'=c\cdot c(e)-\sqrt{g}$. Then we obtain $\xi_0(g'eR)\leq \sum_{t=1}^{\infty} \oplus e_{t(t)}^* R \subset \sum_{t=1}^{\infty} e_t R \subset R$. By [2] Corollary 9.23, we obtain that $g'e=0$, i.e. $c\cdot c(e)=\sqrt{g}$ because $eR \geq gte_{t(t)}^* R$ implies $c\cdot c(e) \geq g_t$ for all $t$.

Put $e_1'=\prod_{t=1}^{\infty} g_t e_{t(t)-1}^* e_2'=\prod_{t=1}^{\infty} g_t e_{t(t)}^*$ in $\prod_{t=1}^{\infty} g_t R=\sqrt{g_t R}$. We see from (3), (4) that $\{g_t\}$, $e_1$, $e_2$ satisfy (i), (ii) and (iii) and (iv).

(c) $\Rightarrow$ (d): Let $e$ be an idempotent satisfying (c). By (iv) $c\cdot c(e) \in \mathfrak{M}$, we have $e \in R\mathfrak{M}$, i.e., $e=0$. By (iii) in (c), we have $e_1 R \supseteq e R$. By (iv) $g_t \in \mathfrak{M}$, we obtain that $(1-\sum_{t=1}^{t} g_t) e_t = e_t$ for all $t=1, 2, \ldots$. By (i) and general comparability on $R$, we obtain the following:

\[(1-\sum_{t=1}^{t} g_t) e_t R \leq e_{t(t)}^* R\]

for all $t$. Then the following hold:

\[\xi_0(e R) \leq \xi_0(e_1 R) \quad \text{(from } e_1 R \leq e_1 R)\]

\[\leq \sum_{t=2}^{\infty} (1-\sum_{t=1}^{t} g_t) e_t R \quad \text{(from } e_1 = (1-\sum_{t=1}^{t} g_t) e_1)\]

\[\leq \sum_{t=2}^{\infty} \oplus e_{t(t)+1}^* R \subset R. \quad \text{(from (5)) .}\]

Thus we have $\xi_0(e R) \leq e R$.

(d) $\Rightarrow$ (a): By [2] Theorem 9.32, $R=R/\mathfrak{m}=R/\mathfrak{m}$ is a directly finite, right self-injective simple regular ring. For an idempotent $e$ satisfying (d), we see from $\mu(e R) \leq R$ that $\mu(e R) \leq R$ for all $n=1, 2, \ldots$. By [2] Corollary 9.23, it follows that $\hat{e}=0$, i.e., $e \in \mathfrak{m}$.

(II) It is clear from (I).

Theorem 9. Let $R$ be a directly finite, right self-injective regular ring and $Q$ the maximal left quotient ring of $R$. Let $\mathfrak{M}$ be a maximal ideal of $B(R)$. Let $\mathfrak{H}$ and $\mathfrak{M}$ be the maximal ideals of $Q$ and $R$ including the ideal $\mathfrak{M}R$, respectively. Then the factor ring $Q/\mathfrak{H}$ is the maximal left quotient ring of $R/\mathfrak{M}$.

Proof. By Theorem 7 and [2] Theorem 10.13, there exists a decomposition $Q=Q_1 \times Q_2$ such that $Q_1$ is type $I_f$ and $Q_2$ is type $II_f$. We denote by $R=R_1 \times R_2$ the decomposition of $R$ as same as $Q$. By [2] Proposition 10.4, we have $R_1 \subset Q_1$. Since $R_1$ is left and right self-injective and $Q_1 \cap R_2 = 0$ ([2] Proposition 10.4), we have $R_1=Q_1$. Then every prime ideal contains $R_1$ or $R_2$. So, if $\mathfrak{m}$ contains $R_2$, the assertion is clear. Since $\mathfrak{M}R$ is prime ideal of $R$, we may assume that $R$ is type $II_f$.

First we prove that $R \cap \mathfrak{M} = \mathfrak{m}$. Suppose that the equality does not hold. By
Corollary 8.23, \( m \) is a unique maximal ideal of \( R \) which contains the minimal prime ideal \( \mathfrak{M}R \). Hence \( m \supseteq R \cap \mathfrak{M} \supseteq R \mathfrak{M} \). There exists an idempotent \( e \) in \( m \setminus \mathfrak{M} \cap R \). From \( e \in \mathfrak{M} \) and Proposition 8 (II), there exists a nonzero central idempotent \( g \) in \( B(Q) \) such that \( g \in \mathfrak{M} \) and

\[
Qe \geq Qe^*_ng
\]

for some integer \( m \). From \( e \in m \) and Proposition 8 (I)-(c), there exist orthogonal central idempotents \( \{g_i\}_{i=1}^{\infty} \) and idempotents \( e_i, e_1 \) and integers \( \{n(i)\}_{i=1}^{\infty} \) satisfying the conditions of Proposition 8 (I) (c). From \( e_i R \leq e R \leq e_1 R \), we obtain the following:

\[
Qe_i^* \leq Qe \leq Qe_1^*.
\]

There exists an integer \( t \) such that \( n(i) > m \) for all \( i \geq t \). From \( g_t \in \mathfrak{M} \), we have \( (c \cdot c(e) - \sum_{i=1}^{t-1} g_i)g \neq 0 \). There exists an integer \( s > t \) satisfying

\[
g_s g \neq 0.
\]

Then the following relations hold:

\[
Qg_s = Qg_s g = Qg_s g (\text{from Prop. 8 (c) (i)})
\]

\[
\geq Qe^*_ng (\text{from (2)})
\]

\[
\geq Qe^*_ng (\text{from (1)})
\]

Thus we obtain

\[
Qe^*_n = Qe^*_ng g.
\]

On the other hand we obtain from Proposition 3 that \( 2^{n+1} - n(e_{n+1})^{-1} R = e_{n+1}^* R \). So we have

\[
2^{n+1} - n(Qe_{n+1}^*) = Qe_{n+1}^*.
\]

Hence, from (3), (4) and (5), nonzero \( Qe_{n+1}^* g g \) is isomorphic to a proper direct summand of itself. This contradicts that \( Q \) is directly finite. So we obtain \( m = \mathfrak{M} \subseteq R \).

We prove that \( \bar{R} = R/m \) is essential in \( \bar{Q} = Q/\mathfrak{M} \) as left \( \bar{R} \)-module. From Proposition 6, for a given element \( q \) in \( Q \), we obtain an essential left ideal \( R(1-h) \oplus \sum_{i=1}^{n} R e_i \) in \( B(R) \) such that \( h \in B(R) \) and \( R e_i = R e_i h \) for all integer \( i \). Now \( B(\bar{R}) = \{ \bar{1}, 0 \} \) implies \( (\bar{1} - h) = \bar{1} \) or \( 0 \). If \( (\bar{1} - h) = \bar{1} \), then \( \bar{R} \) contains \( q \), i.e., \( (\bar{R}, q) = \bar{R} \). If \( (\bar{1} - h) = 0 \), then \( (\bar{R}, q) \) contains \( \sum \bar{R} e_i \). Since \( \bar{R} \) is a simple regular ring with a unique rank function \( N, 2^{n}(\bar{e}_n^* \bar{R}) = \bar{R} \) implies \( N(\bar{e}_n^*) = 1/2^n \). Further \( \bar{e}_n^* \bar{R} = \bar{e}_n^* \bar{R} \) implies \( N(\bar{e}_n^*) = N(\bar{e}_n) \). Since \( \{\bar{e}_i\}_{i=1}^{\infty} \) are pairwise
orthogonal, we obtain
\[ 1 = \sum_{i=1}^{\infty} N(e_i) = \sup \{ N(x) | x \in \sum_{i=1}^{\infty} e_i \mathcal{R}_R \} . \]
Hence, \( \sum_{i=1}^{\infty} e_i \mathcal{R}_R \) is an essential left ideal of \( \mathcal{R}_R \) that is, \( (\mathcal{R}_R, \mathcal{Q}) \) is an essential left ideal of \( \mathcal{R}_R \) for every \( \mathcal{Q} \in \mathcal{Q} \). Thus \( \mathcal{R}_R \) is essential in \( \mathcal{Q} \) as left \( \mathcal{R}_R \)-module.

By [2] Theorem 9.32, \( \mathcal{Q} \) is a left self-injective regular ring. Thus \( \mathcal{Q} \) is the maximal left quotient ring of \( \mathcal{R}_R \) from \( \mathcal{R} \).

3. **Left and right self-injective regular ring**

A ring \( R \) is said to be right (resp. left) \( K \)-injective if every homomorphism from a countably generated right (resp. left) ideal of \( R \) into \( R \) extends to an endomorphism of right (resp. left) \( R \)-module \( R \).

By Proposition 6, we obtain the following theorem.

**Theorem 10.** Let \( R \) be a directly finite, right self-injective regular ring. Then \( R \) is a left self-injective ring if and only if \( R \) is left \( K \)-injective.

Proof. Let \( R \) be left \( K \)-injective and \( \mathcal{Q} \) the maximal left quotient ring of \( R \). For any element \( q \) in \( \mathcal{Q} \), there exist a set \( \{ e_n \}_{n=1}^{\infty} \) of orthogonal idempotents and a central idempotent \( g \in B(R) \) such that \( \sum_{i=1}^{\infty} e_i \mathcal{R}_R \) is essential in \( (R, q) \) and \( Re_n = Re_n g \) for all \( n \). Since the right multiplication by \( q \) is a homomorphism from \( \sum \mathcal{R}_R \) to \( R \), there exists an element \( x \) in \( R \) such that \( (\sum \mathcal{R}_R) (q-x) = 0 \). Since \( \mathcal{Q} \) is a nonsingular left \( R \)-module, we obtain that \( R \) contains \( q = x \), i.e., that \( \mathcal{Q} = R \).

The converse is trivial.

A ring is said to satisfy \( K_r \) (resp. \( K_l \)) if every non-essential left (resp. right) ideal has a non-zero right (resp. left) annihilator ideal. We consider one generalization of Kobayashi's theorem. For the end we use the following Utsumi's theorem:

**Theorem.** Let \( R \) be a regular ring and \( \mathcal{Q}_1 \) (resp. \( \mathcal{Q}_r \)) the maximal left (resp. right) quotient ring of \( R \). Then \( \mathcal{Q}_l = \mathcal{Q}_r \) if and only if \( R \) satisfies \( K_l \) and \( K_r \). ([6] Theorem 3.3)

In the following Lemmas 11, 12 and 13 and 14, we denote by \( R \) a right self-injective regular ring of type \( II \) and by \( \mathcal{Q} \) the maximal left quotient ring of \( R \). We use \( \{ e_i \}_{i=1}^{\infty} \) to denote the orthogonal idempotents of \( R \) given by Proposition 3.

**Lemma 11.** Let \( \{ e_i \}_{i=1}^{\infty}, \{ f_i \}_{i=1}^{\infty} \) be pairwise orthogonal idempotents respectively, which satisfy the following conditions:

1. \( Re_i = Re_i f_i \cdot c(c(e_i)) \) for all \( i \in N \),
2. \( \{ i | e_i \neq 0 \} \) is infinite, and for every nonzero \( g \in B(R), ge_i \neq 0 \) for infinite many \( i \),
where \( \{n(i)\}_{i=1}^{t-1} \) is a strictly increasing sequence.

(b). There exists an integer \( t \) such that \( R_{t(i)} = R_{t+1} \) for all \( i \in N \).

(c). \( \sum_{i=1}^{s} \oplus R_{i(i)} \cap \sum_{i=1}^{s} \oplus R_{fi} = 0 \) and \( \sum_{i=1}^{s} \oplus R_{i} \oplus \sum_{i=1}^{s} \oplus R_{fi} \) is essential in \( R \).

Then \( R_{i(i)} = R_{t} \) for all \( i=1, 2, \ldots, t-1 \) and \( R_{t(i)} = R_{t+1} \) for all \( i \geq t \).

Proof. If \( n(i) \neq t \) for all \( i \in N \), then the following relations hold:

\[
\sum_{i=1}^{s} \oplus R_{i(i)} = \sum_{i=1}^{s} \oplus R_{t} \quad \text{(from (a))}
\]

\[
\subseteq \sum_{i=1, t \neq t}^{s} \oplus R_{t} \quad \text{(from \( n(i) \neq t \) for all \( i \))}
\]

\[
\sum_{i=1}^{s} \oplus R_{fi} = \sum_{i=1}^{s} \oplus R_{t} \quad \text{(from (b))}
\]

\[
\subseteq R(1 - \sum_{i=1}^{t} e_{t}^{*}) \simeq R_{t} \quad \text{(from Proposition 3 and its Corollary)}
\]

By Theorem 7, \( R \) satisfy (2) of Proposition 2 for left ideals of \( R \). Since \( \sum_{i=1}^{s} \oplus R_{i(i)} \oplus \sum_{i=1}^{s} \oplus R_{fi} \) is an essential left ideal, it follows from Proposition 2 that \( R_{i(i)} = R_{t} \) for all \( i=1, 2, \ldots, t-1 \), \( R_{t(i)} = R_{t+1} \) for all \( i \geq t \). So we will show that \( n(i) = t \) for all \( i \in N \).

We begin by showing that \( n(i) = i \) for \( i=1, 2, \ldots, t-1 \). Suppose that there exists an integer \( s \) with \( n(i) \neq s \) for all \( i \). Now the following relations hold:

\[
\sum_{i: n(i) > s}^{s} R_{i(i)} = \sum_{i: n(i) > s}^{s} R_{t(i)} c \cdot c(e_{i}) \quad \text{(from (a))}
\]

\[
\subseteq \sum_{i: n(i) > s}^{s} R_{t} \quad \text{(from (a))}
\]

\[
\subseteq R(1 - \sum_{i=1}^{t} e_{t}^{*}) \simeq R_{t} \quad \text{(from Proposition 3 and its Corollary)}
\]

\[
\sum_{i: n(i) < s}^{s} R_{i(i)} = \sum_{i: n(i) < s}^{s} R_{t} c \cdot c(e_{i}) \quad \text{(from (a))}
\]

\[
\subseteq \sum_{i=1}^{s} R_{t} \quad \text{(from (b))}
\]

Consequently essential left ideal \( \sum_{i=1}^{s} \oplus R_{i(i)} \oplus \sum_{i=1}^{s} \oplus R_{fi} \) is subisomorphic to a proper direct summand \( R(e_{t}^{*} + e_{t+1}^{*} + \ldots + e_{t}^{*} + e_{t+1}^{*}) \) of \( R \). This contradicts that \( R \) satisfy (2) of Proposition 2. Thus we obtain \( n(i) = i \) for \( i=1, 2, \ldots, t-1 \).

Here we show that \( R_{t(i)} = \sum_{i=1}^{s} \oplus R_{fi} \) for all \( i < t \). Suppose that there exists an nonzero idempotent \( g \in B(R) \) which \( e_{s} g = 0 \) for some \( s < t \). Using the simillar
argument as above for $e_i g$ and $e^*_ig$, essential left ideal $\bigoplus_{i=1}^\infty \oplus R_{e_i} g \oplus \bigoplus_{i=1}^\infty \oplus R_{f_i} g$ of $Rg$ is subisomorphic to a proper direct summand $R(e^*_1 + \cdots + e^*_t + e^*_S)g$ of $Rg$. This is a contradiction as above. So we obtain that for every $s < t$, $e_sg = 0$ for every nonzero $g \in B(R)$. So we have

$$Re^*_i = Re_i$$

for all $i < t$.

Suppose that $n(t) = t$. Put $h = c \cdot c(e_t) \neq 0$. Now $\bigoplus_{i=1}^t \oplus R_{e_i} h \oplus \bigoplus_{i=1}^\infty \oplus R_{f_i} h = \bigoplus_{i=1}^t \oplus R_{e_i} h \oplus \bigoplus_{i=1}^\infty \oplus R_{f_i} h$ from (1) and (b). By (a), $\bigoplus_{i=1}^t \oplus R_{e_i} h$ is a proper direct summand of $\bigoplus_{i=1}^\infty \oplus R_{e_i} h$, that is, $\bigoplus_{i=1}^t \oplus R_{e_i} h \oplus \bigoplus_{i=1}^\infty \oplus R_{f_i} h$ is not essential in $Rh$. This is a contradiction. Hence we obtain $n(i) \neq t$ for all $i$.

**Lemma 12.** Let $e \in Q$ be an idempotent with $eQ = e^*_Q$ for some integer $n$. There exist orthogonal idempotents $\{f_i\}_{i=1}^\infty$ in $l_\infty(e)$ such that $\bigoplus_{i=1}^\infty \oplus R_{f_i}$ is essential in $l_\infty(e)$, $R_{f_i} = Re^*_i$ for all $i(\neq n) \in N$.

Proof. We see that $l_\infty(e)$ is a centrally closed left ideal of $R$. By Corollary of Proposition 5, there exists a set $\{f_i\}_{i=1}^\infty$ of orthogonal idempotents in $l_\infty(e)$ which satisfy the following conditions:

1. $f_i R = c \cdot c(f_i) e_{n(i)}^* R$ for all $i$ where $\{n(i)\}_{i=1}^\infty$ is a strictly increasing sequence of integers.

2. $\bigoplus_{i=1}^\infty \oplus R_{f_i}$ is essential in $l_\infty(e)$.

Then there exists an essentially closed left ideal $K$ of $R$ such that $\bigoplus_{i=1}^\infty \oplus R_{f_i} \oplus K$ is essential in $K R$. Let $a$ be an element in $eQe^*_n$ such that the right multiplication by $a$ induces a given isomorphism $Qe \cong Qe^*_n$. By Corollary of Proposition 5, there exists a set $\{f_i\}_{i=1}^\infty$ of orthogonal idempotents in $K \cap (R,^*a)$ which satisfies the following conditions:

3. $f_i R = c \cdot c(f_i) e_{n(i)}^* R$ for all $i$ where $\{n(i)\}_{i=1}^\infty$ is a strictly increasing sequence of integers.

4. $\bigoplus_{i=1}^\infty \oplus R_{f_i}$ is essential in $K \cap (R,^*a)$.

Here we claim that $R_{f'_i} = Re^*_{i+1}$ for all $i$. From $l_\infty(a) = Q(1 - e) \supset l_\infty(e) \supset \bigoplus_{i=1}^\infty \oplus R_{f_i}$ and $\bigoplus_{i=1}^\infty \oplus R_{f_i} \cap K = 0$, the right multiplication by $a$ is a monomorphism from $\bigoplus_{i=1}^\infty \oplus R_{f'_i}$ to $Re^*$. Since $\bigoplus_{i=1}^\infty \oplus R_{f_i} \cap (R,^*a)$ is essential left ideal, we obtain

5. $\bigoplus_{i=1}^\infty \oplus R_{f'_i} \subseteq Re^*.$
If \( n'(i) < n \) for some \( i \), then \( n'(i) < n \) implies \( \text{Re}^*_R \supseteq \text{Re}^*_R \) which contradicts that \( R \) is directly finite. So we have \( n'(i) \leq n \) for all \( i \). Suppose that \( n(1) = n \). Put \( g = c \cdot c(f_i) \in B(R) \). If \( f_i \cdot g \neq 0 \) for some \( j \), then we obtain \( \text{Re}^*_R \supseteq \text{Re}^*_R \) from (5) and \( \text{Re}^*_R \supseteq \text{Re}^*_R \) from \( n(1) = n \), and they imply that \( \text{Re}^*_R \supseteq \text{Re}^*_R \) is isomorphic to a proper direct summand of itself. This is a contradiction. So we have \( f_i \cdot g = 0 \) for all \( i \geq 2 \). By Corollary of Proposition 3, the following holds:

(6) \[ \text{Re}^*_R = R(1 - \sum_{i=1}^n e^*_i) \supseteq \sum_{i=n+1}^\infty \oplus \text{Re}^*_R \]

Then \( \text{Re}^*_R \) contains a left ideal which is isomorphic to \( \sum_{i=n+1}^\infty \oplus \text{Re}^*_R \) and is essential in \( \text{Re}^*_R \). Changing suitable \( \{f_i\}_{i=1}^m \) from Lemma D, we may assume that \( n'(i) > n \) for all \( i \). Then we obtain the following relation:

(7) \[ \sum_{i=1}^\infty \oplus \text{Re}^*_R = \sum_{i=n+1}^\infty \oplus \text{Re}^*_R \quad (\text{from } n'(i) > n) \]

Using Proposition 2 and Theorem 7 for two left ideal \( \sum_{i=n+1}^\infty \oplus \text{Re}^*_R \), \( \sum_{i=1}^\infty \oplus \text{Re}^*_R \) from (5), (7)), the homomorphism (7) \( \sum_{i=1}^\infty \oplus \text{Re}^*_R \supseteq \sum_{i=n+1}^\infty \oplus \text{Re}^*_R \) implies that \( \text{Re}^*_R \supseteq \text{Re}^*_R \) for all \( i \).

Suppose that there exists a nonzero central idempotent \( g \in B(R) \) satisfying \( g f_i = 0 \) for all but finite many \( i \). For the sake of simplicity, put \( \{g_i | g_i \neq 0 \} = \{1, 2, \ldots, m\} \). So we have \( \text{Re}^*_R \supseteq \sum_{i=1}^\infty \oplus \text{Re}^*_R \oplus \sum_{i=1}^\infty \oplus \text{Re}^*_R \).

Here we claim that (I): \( \{n(i) | i = 1, 2, \ldots, m\} \supseteq \{1, 2, \ldots, n\} \), (II): \( \text{Re}^*_R = \text{Re}^*_R \) for all \( i \leq n \), (III): \( \{n(i) | 1 \leq i \leq m\} = \{1, 2, \ldots, n\} \).

(I) Suppose that there exists an integer \( s \leq n \) with \( s \neq n(i) \) for all \( 1 \leq i \leq m \). Now we obtain the following from Proposition 3:

(8) \[ \sum_{i=1}^\infty \oplus \text{Re}^*_R \supseteq \text{Re}^*_R \supseteq R(1 - \sum_{i=1}^n e^*_i) g \]

(9) \[ \sum_{i: n(i) < s} \oplus \text{Re}^*_R = \sum_{i: n(i) < s} \oplus \text{Re}^*_R c(g f_i) e^*_i \quad (\text{from } 1) \]

\[ \sum_{i=1}^{n(m)} \oplus \text{Re}^*_R e^*_i Q \]

(10) \[ \sum_{i: n(i) > s} \oplus \text{Re}^*_R = \sum_{i: n(i) > s} \oplus \text{Re}^*_R c(g f_i) \]

\[ \sum_{i=n+1}^\infty \oplus \text{Re}^*_R \supseteq R g \supseteq \sum_{i=1}^\infty \oplus \text{Re}^*_R Q \]
On the other hand, from Proposition 3, \( \bigoplus_{i=1}^{n} \oplus ge_i Q \) is isomorphic to a proper direct summand of \( Re_i g \). Thus we obtain from (8), (9), (10) that \( \bigoplus_{i=1}^{n} \oplus Rf_i g \oplus \bigoplus_{i=1}^{n} \oplus Rf' _i g \) is subisomorphic to a proper direct summand of \( Rg \). This contradict that \( R \) is directly finite. So we have (I).

(II). Suppose that for a given nonzero idempotent \( h=gh \in B(R), hgf_s=0 \) for some \( s \leq n \). Using \( \{f'_i gh, gfh_i, ghe_i\} \) for \( \{f_i, g, f'_i, g, e_i^*\} \), the same argument as above implies that an essential left ideal of \( Rgh \) is subisomorphic to a proper direct summand of \( Rg \). This is a contradiction. So we obtain that \( Rf_i g \simeq Re_i^* g \) for all \( 1 \leq i \leq n \).

(III). Suppose that \( n(m) > n \). Then \( \bigoplus_{i=1}^{n} \oplus Rf_i g \oplus \bigoplus_{i=1}^{n} \oplus Rf'_i g \) is a proper direct summand of \( \bigoplus_{i=1}^{n} \oplus Rf_i g \oplus \bigoplus_{i=1}^{n} \oplus Rf'_i g \). On the other hand we obtain from (I), (II) and Proposition 3 that \( \bigoplus_{i=1}^{n} \oplus Rf_i g \oplus \bigoplus_{i=1}^{n} \oplus Rf'_i g \) is essential in \( Rg \). This is a contradiction. Thus we have (III).

Put \( J=\{g \in B(R) \mid gf_i=0 \text{ for all but finite many } i\} \). Put \( h=\bigvee g \in B(R) \). Then \( Rf_i h \simeq Re_i^* h \) for all \( i=1, 2, \ldots, n \). By Proposition 3, it follows that \( Rhf_i \) contains a left ideal which is isomorphic to \( \bigoplus_{i=n+1}^{n} Re_i^* h \). Changing suitable pairwise orthogonal idempotents \( \{f_i\}_{n+1}^{n} \) from Lemma D, we may assume that for every nonzero central idempotent \( g \in B(R), gfi \neq 0 \) hold for infinite many \( i \in N \).

By Lemma 11, we obtain that \( Rf_i \simeq Re_i^* (i) \) for all \( i \neq n \) and \( n(i)=i \) for all \( i \leq n-1 \) and \( n(i)=i+1 \) for all \( i \geq n \).

Lemma 13. Let \( I \) be an essentially closed right ideal of \( Q \) such that \( I \oplus eQ \) is essential in \( Q, eQ \simeq e_t^* Q \) for some integer \( t \). There exist pairwise orthogonal idempotents \( \{e_n\}_{n=1}^{t} \) of \( I \) such that \( \bigoplus_{n=1}^{t} e_n Q \) is essential in \( I, e_n Q \simeq e_t^* Q \) for all \( n(\pm t) \in N \).

Proof. Since \( I \) is essentially closed, \( I \) is centrally closed in \( Q \). From Corollary of Proposition 5, there exist pairwise orthogonal idempotents \( \{e_i \in I\}_{i=1}^{t} \) which satisfy the following conditions:

1. \( e_i Q \simeq c \cdot c(e_i) e_t^* (i) Q \) for all \( i \) where \( \{n(i)\}_{i=1}^{t} \) is a strictly increasing sequence.

2. \( \bigoplus_{i=1}^{t} e_i Q \) is essential in \( I \).

Suppose that there exists a nonzero central idempotent \( g \in B(R) \) satisfying \( ge_i = 0 \) for all but finite many \( i \). By the similar argument in (I), (II), (III) of Proof of Lemma 12, we obtain that \( ge_i Q \simeq ge_t^* Q \) for all \( i \leq t \) and \( ge_i = 0 \) for all
\[ i > t. \]

Put \( J = \{ g \in B(R) \mid ge_i = 0 \text{ for all but finite many } i \} \). Put \( h = \bigvee g \) in \( B(Q) \). Then \( he_i Q \cong he_i^* Q \) for all \( i = 1, 2, \ldots, t \). By Proposition 3 and its Corollary, it follows that \( he_i Q \) contains a right ideal which is isomorphic to \( \sum_{i=t+1}^{\infty} \oplus he_i^* Q \) and essential in \( he_i Q \). Changing suitable pairwise orthogonal idempotents \( \{ e_i \}_{i=1}^{t} \) from Lemma D, we may assume that for every nonzero central idempotent \( g \in B(R), ge_i \neq 0 \) for infinite many \( i \in \mathbb{N} \). Since \( eQ = e_i^* Q = (1 - \sum_{i=1}^{t} e_i^*) Q \supset \sum_{i=t+1}^{\infty} \oplus e_i^* Q \), it follows that there exist pairwise orthogonal idempotents \( \{ f_i \}_{i=1}^{t} \) satisfying \( eQ \supset \sum_{i=1}^{\infty} \oplus f_i Q \) and \( f_i Q = e_i^* f_i Q \) for all \( i \). Applying Lemma 11 to the present argument, we complete the proof.

**Lemma 14.** Let \( I \) be an essentially closed right ideal of \( Q \) such that \( I \oplus eQ \) is essential in \( Q \), \( eQ \cong e_i^* Q \) for some integer \( t \). Then \( l_Q(I) \) is nonzero.

Proof. By Lemma 13, there exist pairwise orthogonal idempotents \( \{ e_n \}_{n=1, \pm t} \) in \( I \) which satisfy the following for every \( n \neq t \):

1. \( e_n Q \cong e_i^* Q \).
2. \( \sum_{n=1}^{\infty} \oplus e_n Q \subseteq I \).

By Lemma 12, for every \( e_n \), there exist pairwise orthogonal idempotents \( \{ f_{ni} \}_{i=1, \pm n} \) in \( l_R(e_n) \) which satisfy the following for every \( i \neq n \).

3. \( Rf_{ni} = Re_i^* \).

Put \( f_n = \sum_{i=1, \pm n}^{n+t+2} f_{ni} \) in \( l_R(e_n) \) for all \( n \neq t \). From [2] Theorem 4.14, \( R \) satisfy cancellation property. Since \( R \) has two decompositions \( R = \sum_{i=1, \pm n}^{n+t+2} \oplus Rf_{ni} \oplus R(1-f_n) = \sum_{i=1, \pm n}^{n+t+2} \oplus Re_i^* \oplus Re_n^* \oplus R(1-\sum_{i=1}^{n+t+2} f_{ni}) \), we obtain the following from (3):

\[
R(1-f_n) = Re_n^* \oplus R(1-\sum_{i=1}^{n+t+2} f_{ni}) = Re_n^* \oplus Re_{n+t+2}^* \quad \text{(from Proposition 3)}
\]

So we obtain:

4. \( (1-f_n)R \cong e_n^* R \oplus e_{n+t+2}^* R \).

On the other hand \( l_R(e_n) \supset Rf_n \) implies \( r_R l_R(e_n) \subseteq (1-f_n) R \). So we have

5. \( \sum_{n=1, \pm t}^{\infty} r_R l_R(e_n) \subseteq \sum_{n=1, \pm t}^{\infty} (1-f_n) R \).
By Lemma $D$, there exist pairwise orthogonal idempotents \( \{h_n\}_{n=1}^\infty \) in $R$, which satisfy the following for every $n \neq t$.

\[(6) \quad (1-f_n) R \succeq h_n R.\]

\[(7) \quad \sum_{n=1, \neq t}^\infty (1-f_n) R = \sum_{n=1, \neq t}^\infty h_n R.\]

Consequently we obtain:

\[(8) \quad \sum_{n=1, \neq t}^\infty h_n R \succeq \sum_{n=1, \neq t}^\infty (e_n^* R \oplus e_n^* R_{n+1, t+2}) \quad \text{(from (4), (6))}\]

where we denote by \( \oplus \) outer direct sum. Since $R$ satisfy cancellation property, $R=(1-e_n^*) R \oplus e_n^* R = \sum_{n=1}^\infty e_n^* R \oplus (1-\sum_{n=1}^\infty e_n^* ) R$ and $(1-\sum_{n=1}^\infty e_n^*) R = e_n^* R$ implies $(1-e_n^*) R = \sum_{n=1}^\infty e_n^* R$. So we obtain from (8) that $\sum_{n=1, \neq t}^\infty h_n R \succeq \sum_{n=1}^\infty e_n^* R \oplus e_{t+2} R$. Since $R = \sum_{n=1}^\infty e_n^* R \oplus e_{t+2} R \oplus e_{t+1} R \oplus (1-\sum_{n=1}^{t+2} e_n^*) R$, i.e., $\sum_{n=1}^\infty e_n^* R \oplus e_{t+2} R$ is a proper direct summand of $R$, it follows from (7) and Proposition 2 that $\sum_{n=1, \neq t}^\infty (1-f_n) R$ is not essential in $R$.

Let $e'$ be an idempotent in $R$ such that $\sum_{n=1, \neq t}^\infty (1-f_n) R$ is essential in $e'R$. We obtain that $0\neq R(1-e')=l_{e'}(1-f_n) R = R e_n \subseteq l_{e'}(e_n)$ for all $n \neq t$. Thus $\cap_{n=1, \neq t} l_{e_n} R(1-e') = 0$. For any element $q$ in $I$, we obtain from (2) that

\[(q', \sum_{n=1, \neq t}^\infty e_n Q) = 0\]

is an essential right ideal of $Q$. Since $(1-e') q = 0$ and $Q_q$ is nonsingular, we see that $(1-e') q = 0$. Thus $l_q(I) = Q(1-e') = 0$.

**Theorem 15.** Let $R$ be a directly finite, right self-injective regular ring. Then the maximal left quotient ring of $R$ is a left and right self-injective regular ring.

Proof. By [2] Theorem 10.13, $R$ has a decomposition $R=R_1 \times R_2$ such that $R_1$ is type $I_f$ and $R_2$ is type $II_f$. Then $R_1$ is right and left self-injective. So we may assume that $R$ is type $II_f$.

Suppose that $Q$ satisfies $K_r$. Since $Q$ satisfies $K_r$, it follows by Utsumi's Theorem that the maximal right quotient ring of $Q$ is equal to the maximal left quotient ring of $Q$, i.e., $Q$. Thus $Q$ is left and right self-injective. So it is sufficient to show that $Q$ satisfies $K_r$.

Let $I$ be a non-essential right ideal of $Q$ such that $eQ \cap I = 0$ for some non-
zero idempotent $e$ in $Q$. Put $e' = e^2$ in $eQ$ such that $Qe' \subseteq Qe^2$, $g = c \cdot c(e')$ for some integer $t$. We prove that $l_Q(I) \neq 0$. So we may assume that $Qe = Qge^2$, $c \cdot c(e) = g$ and $eQ \oplus I$ is an essential right ideal of $Q$. Since $I \subseteq I'$ implies $l(I') \subseteq l(I)$ and $I' \cap eQ = 0$ where $I'$ is the essential closure of $I$, we may assume that $I$ is essentially closed. From $l_Q(gI) \subseteq l_Q(I)$, we can assume that $c \cdot c(e) = 1$.

By Lemma 14, we have $l_Q(I) \neq 0$, i.e., that $Q$ satisfies $K_r$.

**Corollary.** ([5]) Let $R$ be a regular ring with a rank function $N$. Suppose that $N$ satisfy $1 = \sup \{N(x) | x \in I\}$ for every essential left ideal $I$ of $R$. Then the maximal right quotient ring of the maximal left quotient ring of $R$ is left and right self-injective and is isomorphic to the $N$-completion of $R$ by an extension of natural map $\varphi: R \to \bar{R}$ (See [2]).

Proof. From [2] Theorem 21.17, the maximal left quotient ring $S$ of $R$ is directly finite. By Theorem 15, the maximal right quotient ring $Q$ of $S$ is left and right self-injective. By the hypothesis and [2] Theorem 21.17, we consider that $S$ is a subring of the $N$-completion of $R$ and there exists a rank function $N$ of $S$ as an extension of $N$. For every essential left ideal $I$ of $S$, we have $gI \supseteq I \cap R$ and $R \supseteq I \cap S$. So $1 = \sup \{N(x) | x \in I \cap S\} \leq \sup \{N(x) | x \in I\} \leq 1$, i.e., $\sup \{N(x) | x \in I\} = 1$ for all essential left ideal $I$ of $S$. For a given essential right ideal $J$ of $S$, there exist pairwise orthogonal idempotents $\{g_n\}$ such that $\sum \bigoplus g_n S$. Suppose $\sum \bigoplus g_n S \subseteq S(1 - g)$ Then $ga = \sum g_n a_n \in gS \cap \sum g_n S$ implies $0 = g_n ga = g_n a_n$ for all $n$, so $g = 1$, i.e., $\sum g_n S$ is essential in $S$. Thus $1 = \sum N(g_n) \leq \sup \{N(x) | x \in J\} \leq 1$, i.e., $1 = \sup \{N(x) | x \in J\}$ for all essential right ideal $J$ of $S$. From [2] Theorem 21.17, we consider that $Q$ is a subring of the $N$-completion $\bar{R}$ of $R$ and have a same rank function $N$. In the same way as $S$, we obtain that $1 = \sup \{N(x) | x \in K\}$ for all essential right ideal of $Q$. From [2] Proposition 21.3 and 4, $N$ is countably additive on $Q$. By [2] Theorem 21.7, $Q$ is complete in the $N$-metric. So $R \subseteq Q \subseteq \bar{R}$ implies $Q = \bar{R}$.

We consider again a necessary and sufficient condition for the maximal right quotient ring of a regular ring to be directly finite.

**Theorem 16.** For a regular ring $R$, the following conditions are equivalent.

1) The maximal right quotient ring of $R$ is directly finite.

2) Every right ideal isomorphic to some essential right ideal is essential in $R$.

3) The maximal left quotient ring of the maximal right quotient ring of $R$ is right and left self-injective.

4) There exists a left and right self-injective regular ring $S$ such that $R$ is a subring of $S$ and $S$ is a non-singular right $R$-module.

Proof. 1) $\Rightarrow$ 2): Proposition 2.

1) $\Rightarrow$ 3): Theorem 15 and Proposition 2.
3) \(\Rightarrow\) 4): Let \(Q\) be the maximal right quotient ring of \(R\) and \(S\) the maximal left quotient ring of \(Q\). Suppose that \(x\) is a singular element in \(S\) such that \(r(x)\) is an essential right ideal of \(R\). Since \((Q, x) = \{q \in Q | qx \in Q\}\) is an essential left ideal of \(Q\), \((Q, x) \ r_R(x) = 0\) implies \(0 = Z(Q_R) \supseteq (Q, x) x\). Since \(S\) is a non-singular left \(Q\)-module, it follows that \(x = 0\), i.e., \(S\) is a non-singular right \(R\)-module.

4) \(\Rightarrow\) 1): Let \(S_R\) be non-singular. Since \(R\) is regular, \(S\) is a flat left \(R\)-module. So \(S\) is non-singular injective right \(R\)-module ([2] Lemma 6.17). Put \(Q = \{s \in S | (s, R)\}\text{ is an essential right ideal of } R\}. For every \(0 \neq q \in Q\), we have \(q R \cap R \supseteq q(q, R) \neq 0\) from nonsingularity of \(S_R\). So \(Q\) is essential hull of \(R\) in injective module \(S_R\), i.e., \(Q_R\) is injective.

We will show that \((t, (s, R))\) is an essential right ideal for every \(s, t \in Q\). Suppose that \((t, (s, R)) \cap xR = 0\) for some nonzero \(t, s \in Q\) and \(x \in R\). Then \(0 = txR \cap (s, R)\) and \((s, R) \subseteq R\) implies \(txR = 0\), i.e., \(xR \subseteq (t, (s, R))\), which is a contradiction. So \((t, (s, R))\) is essential right ideal for all \(t, s \in Q\). Thus we obtain that \(Q\) is a subring of \(S\). So \(Q\) is the maximal right quotient ring of \(R\).

Since \(S\) is directly finite from Utsumi [7] (see [2] Theorem 9.29), \(Q\) is directly finite.

Let \(R\) be a regular ring and \(Q\) be the maximal right quotient ring of the maximal left quotient ring of \(R\). Here we consider necessary and sufficient conditions for \(Q\) to be complete in the \(N\)-metric for some rank function \(N\) of \(Q\) (Corollary 1). And, in Corollary 2, we consider a case that \(R\) is a prime regular ring.

**Corollary 1.** Let \(R\) be a regular ring and \(Q\) be the maximal right quotient ring of the maximal left quotient ring of \(R\). Then the following conditions are equivalent.

1). There exists a rank function on \(R\) such that \(1 = \sup \{N(x) | x \in I\}\) for all essential left ideal \(I\) of \(R\).

2). There exists a rank function \(N\) of \(R\) such that the \(N\)-completion of \(R\) is the maximal right quotient ring of the maximal left quotient ring of \(R\).

3). There exists a rank function \(\bar{N}\) on \(Q\) such that \(Q\) is complete in the \(\bar{N}\)-metric.

4). There exists a rank function \(N\) on \(R\) such that the \(N\)-completion \(\bar{R}\) of \(R\) is a nonsingular \(R\)-module.

Proof. Let \(S\) be the maximal left quotient ring of \(R\).

1) \(\Rightarrow\) 2): Corollary of Theorem 15.


3) \(\Rightarrow\) 4): The restriction of \(\bar{N}\) to \(R\) is a rank function on \(R\). From 3) and \(Q \supseteq R\), the \(\bar{N}\)-completion \(\bar{Q} = Q\) of \(Q\) contains the \(N\)-completion \(\bar{R}\) of \(R\) as a sub-
ring. Suppose that $0 \neq q \in Z(RQ)$ is a singular element of $Q$. Then there exists $b \in (q^*, S) = \{ x \in S \mid qx \in S \}$ such that $qb \neq 0$. Then $l_R(q) qb = 0$, which contradicts that $R$ is nonsingular.

4) $\Rightarrow$ 1): For a given essential left ideal $I$ of $R$, there exist pairwise orthogonal idempotents $\{ e_i \}_{i=1}^\infty$ such that $I \supset \sum_{i=1}^\infty Re_i$. Set $f = \lim \sum_{i=1}^\infty e_i$ in $\bar{R}$. Then $\sum_{i=1}^\infty Re_i (1-f) = 0$ implies $1-f = 0$. So, from [2] Theorem 19.6, $1 = N(f) = \lim \sum_{i=1}^\infty N(e_i) = \sum_{i=1}^\infty N(e_i) \leq \sup \{ N(x) \mid x \in I \}$, i.e., $\sup \{ N(x) \mid x \in I \} = 1$.

**Corollary 2.** Let $R$ be a prime regular ring. Then the following conditions are equivalent.

1. There exists a rank function $N$ on $R$ such that $1 = \sup \{ N(x) \mid x \in I \}$ for all essential left ideal $I$ of $R$.
2. The maximal left quotient ring of $R$ is directly finite.
3. The maximal right quotient ring of the maximal left quotient ring of $R$ is right and left self-injective.
4. There exists a rank function $N$ on $R$ such that the $N$-completion of $R$ is a nonsingular left $R$-module.

2) $\Rightarrow$ 3): Theorem 15.
4) $\Rightarrow$ 1): See 4) $\Rightarrow$ 1) in Proof of Corollary 1 of Theorem 16.

A regular ring $R$ is said to satisfy $K^*_f$ if $r_R(I) \oplus r_R(J)$ is an essential right ideal of $R$ for every essential left ideal $I \oplus J$. Note that essentiality of $I \oplus J$ implies $r_R(I) \cap r_R(J) = 0$.

Let $R$ be a subring of a ring $S$ such that $R$ is nonsingular. Then $S$ is said to be a left quotient ring of $R$ if $R$ is essential in $S$.

In the following theorem, we consider necessary and sufficient conditions that the maximal right quotient ring of a regular ring is left and right self-injective.

**Theorem 17.** For a regular ring $R$, the following conditions are equivalent.
1) The maximal right quotient ring of $R$ is right and left self-injective.
2) The maximal left quotient ring of $R$ is directly finite and a right quotient ring of $R$.
3) i) Every left ideal isomorphic to some essential left ideal is an essential left ideal of $R$.
   ii) $R$ satisfies $K^*_f$.

Proof. Let $Q$ be the maximal right quotient ring of $R$.
1) $\Rightarrow$ 2): Let $Q$ be a right and left self-injective ring. By [2] Lemma 6.17,
Q is injective left R-module. Suppose that \( x \) is a singular element of left R-module Q. Since \( R \) is a non-singular left R-module, it follows that \( \text{im}(x) = 0 \) implies \( x(x,R) = 0 \). Since Q is a non-singular right R-module, it follows that \( x = 0 \), i.e., Q is non-singular left R-module.

Put \( S = \{ x \in Q \mid (R,x) \subseteq R \} \). By similar symmetric argument as 3) \( \Rightarrow \) 1) in Proof of Theorem 16, we obtain that \( S \) is the maximal left quotient ring of \( R \) and \( S \) is a right quotient ring of \( R \).

2) \( \Rightarrow \) 1): Let \( S \) be the maximal left quotient ring of \( R \) such that \( S \) is a directly finite, right quotient ring of \( R \), i.e., \( R \subseteq S \). So we can consider that \( S \) is a submodule of \( Q \). For any element \( s, t \) of \( S \), we denote by \( sot \) the multiplication of \( s \) and \( t \) in \( Q \). Put \( I = (t',(t,R)) \) and \( J = (t',R) \). Then \( t(I \cap J) \subseteq (t',R) \) and \( (sot) (I \cap J) = s(t(I \cap J)) = st(I \cap J) \). Since Q is a non-singular right R-module, we have \( sot = st \). Thus \( S \) is a subring of \( Q \). Since Q is right self-injective and a flat left \( R \)-module, it follows from [2] Lemma 6.17 that \( Q \) is right \( S \)-injective module. While \( Q \subseteq S \) implies \( Q \subseteq S \). Thus \( Q \) is the maximal right quotient ring of \( S \). By Theorem 15, it follows that \( Q \) is left and right self-injective.

3) \( \Leftrightarrow \) 2): By Proposition 2, 3)-i) is equivalent that the maximal left quotient ring of \( R \) is directly finite. Let \( S \) be the maximal left quotient ring of \( R \).

Suppose that \( R \) satisfies \( K^r \). By [2] Theorem 13.14, \( S \) has a decomposition \( S = S_1 \times S_2 \) such that \( S_1 \) is strongly regular ring and \( S_2 \) has no non-zero central abelian idempotent. Since \( S_1 \cap R \subseteq S_1 \) as right \( R \)-module, it is sufficient to show that \( S_2 \cap R \subseteq S_2 \) as right \( R \)-modules. By [2] Theorem 13.16, \( S_2 \) is generated as a ring by its idempotents. For a given idempotent \( e \) in \( S_2 \), put \( I = Se \cap R \) and \( J = S(1-e) \cap R \). Then \( I \oplus J \) is an essential left ideal of \( R \). Since \( r(I) \oplus r(J) = (1-e) S \cap R \oplus eS \cap R \) is essential in \( R \), it follows that \( (e',R) \oplus r(I) \oplus r(J) \) are essential right ideals of \( R \). Therefore \( S \) is an essential extention of \( R \).

Conversely, suppose that \( S \) is a right quotient ring of \( R \), i.e., \( S \) is a subring of \( Q \). Let \( I \oplus J \) be an essential left ideal of \( R \). There exists an idempotent \( f \) in \( S \) such that \( I \subseteq Sf \) and \( J \subseteq S(1-f) \). Then \( fQ \cap R \oplus (1-f) Q \cap R \) is essential in \( R \). Now \( fQ \cap R \oplus fS \) is essential in \( R \). While we have \( fQ \cap R \subseteq fR \cap R \) from \( f(Q \cap R) = fQ \cap R \). So \( r(J) = fS \cap R = fQ \cap R \). Similarly \( r(I) = (1-f) Q \cap R \). Therefore \( r(I) \oplus r(J) \) is essential in \( R \). Thus \( R \) satisfies \( K^r \).

Remark. For a regular ring, the condition \( K^r \) implies the condition \( K \). For, let \( R \) be a regular ring satisfying \( K^r \). Suppose that \( I \) is a non essential left ideal of \( R \) with \( r(I) = 0 \). Let \( J \) be a nonzero left ideal of \( R \) such that \( I \oplus J \) is essential in \( R \). Then \( r(I) \oplus r(J) = r(J) \) is an essential right ideal. Since \( R \) is nonsingular, it follows that \( J = 0 \). This is a contradiction.

We don’t know whether the converse hold or not.
Here we consider the same problem as Theorem 17 for a regular ring with a rank function (Corollary 1) and for a prime regular ring (Corollary 2). The equivalence 2) \(\Leftrightarrow\) 4) in Corollary 1 was proved by A. Vogel [8].

**Corollary 1.** For a regular ring \( R \) with a rank function \( N \), the following conditions are equivalent.

1). The maximal left quotient ring \( S \) of \( R \) is a right quotient ring of \( R \) and 
\[
\sup \{N(x) | x \in I\} = 1 \text{ for all essential left ideal } I \text{ of } R.
\]

2). The \( N \)-completion \( \tilde{R} \) of \( R \) is the maximal right quotient ring of \( R \).

3). The maximal right quotient ring \( Q \) of \( R \) is right and left self-injective and there exists a rank function \( \tilde{N} \) on \( Q \) such that \( \tilde{N} \) is an extension of \( N \) and \( Q \) is complete in the \( \tilde{N} \)-metric.

4). For every left ideal \( I \) of \( R \),
\[
\sup \{N(x) | x \in I\} + \sup \{N(x) | x \in r(I)\} = 1.
\]

Proof. 1) \(\Rightarrow\) 2): Since \( S \) is right quotient ring, we have \( Q \supseteq s \supseteq R \supseteq R \), where \( Q \) is the maximal right quotient ring of \( R \). Let \( E \) be an injective hull of \( S \). For every \( a \in E \), \( (a \cdot S) = \{x \in S | ax \in S\} \) is an essential right ideal of \( S \), so \( (a \cdot S) \cap R \) is an essential right ideal of \( R \). Then we have \( E_s \supseteq S \), so we consider \( Q \supseteq E \). For every \( q \in Q \), we have \( (q \cdot S) \supseteq (q \cdot R) \) is essential right ideal of \( R \), so \( E \supseteq Q \). Then \( Q \) is maximal right quotient ring of \( S \). From Corollary 1 of Theorem 16, \( \tilde{R} \) satisfies 2).


3) \(\Rightarrow\) 4): Let \( \{e_i\}_{i=1}^{\infty} \) be pairwise orthogonal idempotents with \( \sum_{i=1}^{\infty} Re_i \subseteq R \).

Set \( f = \lim_{i \to \infty} \sum_{i=1}^{\infty} e_i \) in \( Q \). Then \( 0 = r(Q \cup f) \supseteq (1-f) Q \cap R \) implies \( 1-f=0 \). From [2] Theorem 21.7, \( \tilde{N} \) is countably additive on \( Q \). Thus we obtain \( 1 = \sum \tilde{N}(e_i) = \sum N(e_i) \), i.e., \( 1 = \sup \{N(x) | x \in I\} \) for all essential left ideals \( I \) of \( R \). While, from [2] Theorem 21.7, we have \( 1 = \sup \{N(x) | x \in I'\} \) for all essential right ideals \( I' \) of \( R \).

For a given left ideal \( I \) of \( R \), set \( I \cup J \subseteq R \) and \( J \subseteq R \).

Then \( 1 = \sup \{N(x) | x \in I\} + \sup \{N(x) | x \in J\} \). From (1-f) \( R \supseteq r(I) \) for every \( f = e_i \), we have \( 1 = N(f) \supseteq \sup \{N(x) | x \in r(I)\} \) for every \( f \in I \), i.e., \( 1 \sup \{N(x) | x \in r(I)\} \). Similarly, \( 1 \sup \{N(x) | x \in J\} \).

Thus \( R \) satisfies 4).

4) \(\Rightarrow\) 1): By 4), \( \sup \{N(x) | x \in I\} = 1 \) for every essential left ideal \( I \) of \( R \).

So we have \( \sup \{N(x) | x \in J\} + \sup \{N(x) | x \in J'\} = 1 \) for every essential left ideal \( J \cup J' \) of \( R \). From 4), \( \sup \{N(x) | x \in J\} + \sup \{N(x) | x \in r(J)\} = 1 \) and \( \sup \{N(x) | x \in J'\} + \sup \{N(x) | x \in r(J')\} = 1 \). Hence we have \( \sup \{N(x) | x \in
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\(r(J) + \sup \{N(x) | x \in r(J')\} = 1\), so \(r(J) \oplus r(J')\) is essential in \(R_e\), i.e. \(R\) satisfies 3) of Theorem 17. Thus the maximal left quotient ring of \(R\) is right quotient ring of \(R\) and \(1 = \sup \{N(x) | x \in I\}\) for all essential left ideal \(I\) of \(R\).

**Ccollorary 2.** For a prime regular ring \(R\), following conditions are equivalent.

1. The maximal left quotient ring of \(R\) is a right quotient ring of \(R\) and directly finite.
2. There exists a rank function \(N\) on \(R\) such that the \(N\)-completion of \(R\) is the maximal right quotient ring of \(R\).
3. The maximal right quotient ring of \(R\) is left and right self-injective.
4. There exists a rank function \(N\) on \(R\) such that \(1 = \sup \{N(x) | x \in I\} + \sup \{N(x) | x \in r(I)\}\) for every left ideal \(I\) of \(R\).

**Proof.** 1) \(\Rightarrow\) 2): From [2] Corollary 21.19, there exists a rank function \(N\) on \(R\) such that \(\sup \{N(x) | x \in I\} = 1\) for every essential left ideal \(I\) of \(R\). By Corollary 1 of Theorem 17, \(R\) satisfies 2) by the rank function \(N\).


3) \(\Rightarrow\) 4): By Corollary 21.14, there exists a rank function \(\overline{N}\) of the maximal right quotient ring \(Q\) of \(R\) such that \(Q\) is complete in the \(\overline{N}\)-metric. From Corollary 1 of Theorem 17, \(R\) satisfies 4).

4) \(\Rightarrow\) 1): It is clear from Corollary 1 of Theorem 17 and [2] Corollary 21.19.

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