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THE MODULO 2 HOMOLOGY GROUPS OF THE SPACE OF RATIONAL FUNCTIONS

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1. Introduction and statement of results

We shall denote by $F_k^*(S^2, \boldsymbol{C}P^m)$ the space of based holomorphic maps of degree k from S^2 to $\boldsymbol{C}P^m$. Any element of $F_k^*(S^2, \boldsymbol{C}P^m)$ is clearly an element of $\Omega_k^2 \boldsymbol{C}P^m$, the space of all based continuous maps from S^2 to $\boldsymbol{C}P^m$ of degree k. Let

$$(1.1) \qquad i: F_k^*(S^2, \boldsymbol{C}P^m) \to \Omega_k^2 \boldsymbol{C}P^m$$

be the inclusion. Segal [5] showed that i is a homotopy equivalence up to dimension $k(2m-1)$.

Recently Boyer and Mann [2] introduced a loop sum and a C_2 structure in $\coprod_k F_k^*(S^2, \boldsymbol{C}P^m)$ which are compatible with i. (It is well known [3] that $\Omega^2\boldsymbol{C}P^m$ has a natural loop sum and a C_2 structure). Hence we can naturally define the loop sum $*$ and the Araki-Kudo operation Q_1 [1] in $\bigoplus_k H_*(F_k^*(S^2, \boldsymbol{C}P^m); \boldsymbol{Z}_2)$. By using this method, Boyer and Mann constructed certain elements in $H_*(F_k^*(S^2, \boldsymbol{C}P^m); \boldsymbol{Z}_2)$. Then the following question arises naturally.

QUESTION. Do the elements constructed by loop sums and iterated operations on ι_{2m-1}(ι_{2m-1} will be defined later) form a basis of $H_*(F_k^*(S^2, \boldsymbol{C}P^m); \boldsymbol{Z}_2)$?

We shall study this question. The results are as follows.

Theorem A. *The elements constructed by loop sums and iterated operations on ι_1 form a basis of $H_*(F_2^*(S^2, \boldsymbol{C}P^1); \boldsymbol{Z}_2)$.*

Theorem B. *For $m \geq 2$, the elements constructed by loop sums and iterated operations on ι_{2m-1} form a basis of $H_*(F_2^*(S^2, \boldsymbol{C}P^m); \boldsymbol{Z}_2)$.*

Theorem C. *For $m \geq 2$, the elements constructed by loop sums and iterated operations on ι_{2m-1} form a basis of $H_*(F_3^*(S^2, \boldsymbol{C}P^m); \boldsymbol{Z}_2)$.*

Theorem D. *For $m \geq k+1$, the elements constructed by loop sums and iterated operations on ι_{2m-1} form a basis of $H_*(F_k^*(S^2, \boldsymbol{C}P^m); \boldsymbol{Z}_2)$.*

This paper is organized as follows. In §2 we shall review some results of [2], [3] and [5]. In §3 we shall give a strategy oi proving Theorems B, C and D. In §4 we shall prove Theorems A and B. In §5 we shall prove Theorem D. In §6 we shall prove Theorem C.

The results of this paper were announced in [4]. The author is grateful to Professor A. Hattori for many useful comments.

2. Known results

First we state the Segal's result precisely.

Theorem 2.1 ([5]). *The inclusion*

$$i\colon F_k^*(S^2, CP^m) \to \Omega_k^2 CP^m$$

is a homotopy equivalence up to dimension $k(2m-1)$, *i.e. the induced homomorphism* $i_*\colon \pi_q(F_k^*(S^2, CP^m)) \to \pi_q(\Omega_k^2 CP^m)$ *is bijective for* $q<k(2m-1)$ *and surjective for* $q=k(2m-1)$.

Next we describe the Pontryagin ring structure of $H_*(\Omega^2 CP^m; Z_2)$. Let $\tilde{\iota}_{2m-1}$ be the generator of $H_{2m-1}(\Omega_1^2 CP^m; Z_2)=Z_2$ and let [1] be the generator of $H_0(\Omega_1^2 CP^m; Z_2)$. Then, according to [3], we can state

Theorem 2.2. $H_*(\Omega^2 CP^m; Z_2)=Z_2[[1], \tilde{\iota}_{2m-1}, Q_{I_l}(\tilde{\iota}_{2m-1})]$, *the polynomial algebra over* Z_2, *under loop sum Pontryagin product, on generators* [1], $\tilde{\iota}_{2m-1}$ *and* $Q_{I_l}(\tilde{\iota}_{2m-1})=Q_1Q_1\cdots Q_1(\tilde{\iota}_{2m-1})$, *where* I_l *has length* l *and* l *is an any natural number.*

Finally we review some results of Boyer and Mann. If we regard a function belonging to $F_k^*(S^2, CP^1)$ as a holomorphic function $f\colon S^2\to S^2$ of degree k such that $f(\infty)=1$, then $F_k^*(S^2, CP^1)$ can be described in the following form.

(2.3) $$F_k^*(S^2, CP^1) = \{p(z)/q(z) = (z^k+a_1 z^{k-1}+\cdots+a_k)/(z^k+b_1 z^{k-1} +\cdots+b_k);\ p(z) \text{ and } q(z) \text{ have no common root.}\}$$

Similarly we shall regard $F_k^*(S^2, CP^m)$ as follows.

(2.4) $$F_k^*(S^2, CP^m) = \{[p_0(z), \cdots, P_m(z)];\ p_i(z) \text{ are monic polynomials of degree } k \text{ such that there exists no } \alpha\in C \text{ which satisfies } p_0(\alpha)=0, \cdots, p_m(\alpha)=0.\}$$

Note that $F_1^*(S^2, CP^m)$ is homotopically equivalent to S^{2m-1} by (2.4). Let ι_{2m-1} be the generator of $H_{2m-1}(F_1^*(S^2, CP^m); Z_2)=Z_2$. If we start with ι_{2m-1} and compute iterated operations on ι_{2m-1} and loop sums of such elements, we may contruct many non-zero homology classes in $H_*(F_k^*(S^2, CP^m); Z_2)$. Then by combining Theorems 2.1 and 2.2, the following theorem is known.

Theorem 2.5 ([2]). *Any element* ξ *of* $H_*(F_k^*(S^2, CP^m); Z_2)$ *with* $\deg \xi<k$

$(2m-1)$ *can be constructed by loop sums and iterated operations on* ι_{2m-1}.

3. Strategy of proof

We shall give the strategy of proving Theorems B, C and D. The strategy of proving Theorem A is slightly different. So it will be postponed to §4. In the following, all homology groups, cohomology groups and compact support cohomology groups have coefficients $\boldsymbol{Z}_2$.

In order to prove Theorems B, C and D, it will be enough to compute $H_q(F_k^*(S^2, \boldsymbol{C}P^m))$ for $q \geq k(2m-1)$ by virtue of Theorem 2.5. Let us filter $F_k^*(S^2, \boldsymbol{C}P^m)$ by the closed subspaces

$$(3.1) \qquad F_k^*(S^2, \boldsymbol{C}P^m) = X_k \supset X_{k-1} \supset \cdots \supset X_1$$

where

$$(3.2) \quad X_n = \{[p_0(z), \cdots, p_m(z)] \in F_k^*(S^2, \boldsymbol{C}P^m); p_0(z) \text{ has at most } n \text{ distinict zeros.}\}$$

Let H_c^* be the compact support cohomology. Assume that we have some informations about $H_c^*(X_{n-1})$ and $H_c^*(X_n - X_{n-1})$. Then we obtain new informations about $H_c^*(X_n)$ by using the following compact support cohomology exact sequence of the pair (X_n, X_{n-1}).

$$(3.3) \quad \cdots \to H_c^q(X_n - X_{n-1}) \to H_c^q(X_n) \to H_c^q(X_{n-1}) \to H_c^{q+1}(X_n - X_{n-1}) \to \cdots$$

Moreover assume that we have some informations about $H_c^*(X_{n+1} - X_n)$. Then we obtain new informations about $H_c^*(X_{n+1})$ by using the compact support cohomology exact sequence of the pair (X_{n+1}, X_n).

We repeat this process. Then finally we obtain new informations about $H_c^*(F_k^*(S^2, \boldsymbol{C}P^m))$ which can be converted to those of $H_*(F_k^*(S^2, \boldsymbol{C}P^m))$ by the Poincaré duality. In particular if k and m are taken to be in Theorems B, C and D, then we can determine $H_q(F_k^*(S^2, \boldsymbol{C}P^m))$ for $q \geq k(2m-1)$.

4. Proofs of Theorems A and B

First we prove Theorem B by using the strategy given in §3. Note that in degrees greater than or equal to $4m-2$, the elements constructed by loop sums and iterated operations are given by ι_{2m-1}^2 and $Q_1(\iota_{2m-1})$ (which are non-trivial by Theorem 2.2). Hence it will be enough to show the following proposition in order to prove Theorem B.

Proposition 4.1. $H_q(F_2^*(S^2, \boldsymbol{C}P^m)) = \begin{cases} \boldsymbol{Z}_2 & q = 4m-2, 4m-1 \\ 0 & q \geq 4m . \end{cases}$

We filter $F_2^*(S^2, \boldsymbol{C}P^m)$ as given in §3.

Lemma 4.2. X_1 *is homeomorphic to* $\boldsymbol{C} \times \boldsymbol{C}^m \times (\boldsymbol{C}^m)^*$.

In fact if $[p_0(z), \cdots, p_m(z)]$ belongs to X_1 and $p_0(z)$ has a multiple root α, then $p_i(z)$ $(1\leq i\leq m)$ are completely determined by giving $p_i(\alpha), p_i'(\alpha)$ which are arbitrary except for the constraint $(p_1(\alpha), \cdots, p_m(\alpha))\neq(0, \cdots, 0)$. ■

Let $\tilde{C}_n$ be the space of ordered distinct n-tuples in $\boldsymbol{C}$.

Lemma 4.3. *X_2-X_1 is the quotient of $\{(\boldsymbol{C}^m)^*\times(\boldsymbol{C}^m)^*\}\times\tilde{C}_2$ by a free action of the symmetric group Σ_2.*

In fact if $[p_0(z), \cdots, p_m(z)]$ belongs to X_2-X_1 and $p_0(z)$ has roots α_1, α_2, then $p_i(z)$ $(1\leq i\leq m)$ are completely determined by giving $p_i(\alpha_1), p_i(\alpha_2)$ which are arbitrary except for the constraint $(p_1(\alpha_1), \cdots, p_m(\alpha_1))\neq(0, \cdots, 0)$ and $(p_1(\alpha_2), \cdots, p_m(\alpha_2))\neq(0, \cdots, 0)$. ■

Note that X_1 is homotopically equivalent to S^{2m-1} by Lemma 4.2. Hence we see $H^q(X_1)=0$ for $q\geq 2m$. Note also that $\dim_{\boldsymbol{R}} X_1=4m+2$. Hence by the Poincaré duality, we see

$$H_c^q(X_1) = 0 \qquad \text{for} \quad q\leq 2m+2\,. \tag{4.4}$$

Note also that X_2-X_1 is homotopically equivalent to $(S^{2m-1})^2\underset{\Sigma_2}{\times} S^1$ by Lemma 4.3. We consider the Serre spectral sequence of the fiber bundle

$$(S^{2m-1})^2 \to (S^{2m-1})^2\underset{\Sigma_2}{\times} S^1 \to S^1\,. \tag{4.5}$$

As $H^{2(2m-1)}((S^{2m-1})^2)=\boldsymbol{Z}_2$, the action of $\pi_1(S^1)$ on $H^{2(2m-1)}((S^{2m-1})^2)$ is trivial. By using this fact, spectral sequence argument shows

$$H^q(X_2-X_1) = \begin{cases} \boldsymbol{Z}_2 & q = 4m-2, 4m-1 \\ 0 & q\geq 4m\,. \end{cases} \tag{4.6}$$

As $\dim_{\boldsymbol{R}} X_2=4m+4$, we see the following fact by (4.6) and the Poincaré duality.

$$H_c^q(X_2-X_1) = \begin{cases} \boldsymbol{Z}_2 & q = 5, 6 \\ 0 & q\leq 4\,. \end{cases} \tag{4.7}$$

By using (4.4) and (4.7), the compact support cohomology exact sequence of the pair (X_2, X_1) shows

$$H_c^q(X_2) = \begin{cases} \boldsymbol{Z}_2 & q = 5, 6 \\ 0 & q\leq 4\,. \end{cases} \tag{4.8}$$

Proposition 4.1 follows easily from (4.8) by the Poincaré duality. ■

Next we shall prove Theorem A. We write F_k^* for $F_k^*(S^2, \boldsymbol{C}P^1)$. Let [1] be the generator of $H_0(F_1^*)$. Then the elements constructed by loop sums and iterated operations are given by $\iota_1*[1]$, ι_1^2 and $Q_1(\iota_1)$ (which are non-trivial by Theorem 2.2). Hence it will be enough to show the following proposition in order to prove Theorem A.

Proposition 4.9. $H_q(F_2^*) = \begin{cases} \boldsymbol{Z}_2 & q=0,1,2,3 \\ 0 & q\geq 4 . \end{cases}$

Note that $\pi_1(F_2^*)=\boldsymbol{Z}$ by Theorem 2.1. Hence if we follow the proof of Theorem B in order to prove Theorem A, we will encounter some difficulties. So we first consider the universal covering of F_2^*. We define

$$R\colon F_2^* \to \boldsymbol{C}^* \tag{4.10}$$

as follows. Let $p(z)/q(z)$ be an element of F_2^* and let α_1, α_2 be the roots of $p(z)$, β_1, β_2 be the roots of $q(z)$. Then $R(p(z)/q(z))$ is defined by $\prod_{i,j}(\alpha_i-\beta_j)$. Let Y_2 be $R^{-1}(1)$. Then it is known in [5] that (4.10) is a fiber bundle with simply connected fiber Y_2.

First we shall compute $H^*(Y_2)$. We define the closed subspace Y_1 of Y_2 by

$$Y_1 = \{p(z)/q(z)\in Y_2;\ q(z) \text{ has a multiple root.}\}$$

Lemma 4.11. *Y_1 is homeomorphic to $\boldsymbol{C}^2 \amalg \boldsymbol{C}^2$.*

In fact if $p(z)/q(z)$ belongs to Y_1 and $q(z)$ has a multiple root β, then $p(z)$ is completely determined by giving $p(\beta)$, $p'(\beta)$ which are arbitrary except for the constraint $R(p(z)/q(z))=p(\beta)^2=1$. ■

We think of $\boldsymbol{C}^*$ as $\{(\xi_1, \xi_2)\in(\boldsymbol{C}^*)^2;\ \xi_1\xi_2=1\}$.

Lemma 4.12. *Y_2-Y_1 is the quotient of $\boldsymbol{C}^*\times\tilde{C}_2$ by a free action of the symmetric group Σ_2.*

In fact if $p(z)/q(z)$ belongs to Y_2-Y_1 and $q(z)$ has roots β_1, β_2, then $p(z)$ is completely determined by giving $p(\beta_1)$, $p(\beta_2)$ which are arbitrary except for the constraint $R(p(z)/q(z))=p(\beta_1)p(\beta_2)=1$. ■

We define the involution τ on $S^1\times S^1$ by

$$(z, w)\,\tau = (1/z, -w) \quad (z, w)\in S^1\times S^1 .$$

Then by Lemma 4.12, we see that Y_2-Y_1 is homotopically equivalent to $S^1\times S^1/\tau$. Note that $S^1\times S^1/\tau$ is Klein's bottle. Now by Lemma 4.11 and the Poincaré duality, we see

$$H_c^q(Y_1) = \begin{cases} \boldsymbol{Z}_2\oplus\boldsymbol{Z}_2 & q=4 \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

By Lemma 4.12 and the Poincaré duality, we see

(4.14) $$H_c^q(Y_2 - Y_1) = \begin{cases} \mathbf{Z}_2 & q = 4, 6 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & q = 5 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $H^1(Y_2)=0$. (In fact Y_2 is simply connected). Hence by the Poincaré duality, we see

(4.15) $$H_c^5(Y_2) = 0 .$$

Now by using the compact support cohomology exact sequence of the pair (Y_2, Y_1), we see by (4.13)-(4.15) that

(4.16) $$H_c^q(Y_2) = \begin{cases} \mathbf{Z}_2 & q = 4, 6 \\ 0 & \text{otherwise.} \end{cases}$$

By the Poincaré duality, we see

(4.17) $$H^q(Y_2) = \begin{cases} \mathbf{Z}_2 & q = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

We consider the Serre spectral sequence of (4.10). As $H^2(Y_2)=\mathbf{Z}_2$, the action of $\pi_1(\mathbf{C}^*)$ on $H^2(Y_2)$ is trivial. By using this fact, spectral sequence argument shows Proposition 4.9. ■

As a corollary of Theorem A, we shall determine the $\mathcal{A}(2)$-module structure of $H^*(F_2^*)$. Note that $\{[2], \iota_1 * [1], \iota_1^2, Q_1(\iota_1)\}$ form the basis of $H_*(F_2^*)$ by Theorem A. Let $u \in H^1(F_2^*)$ be the dual of $\iota_1 * [1]$ and $v \in H^2(F_2^*)$ be the dual of ι_1^2. Then we have the following

Corollary 4.18. *$H^*(F_2^*)=\wedge(u, v)$, the exterior algebra over $\mathbf{Z}_2$ on generators u and v. $Sq^1 v = uv$.*

Proof. Note that the following relation holds in $H_1(F_2^*)$ by Theorem 2.1.

(4.19) $$Q_1[1] = \iota_1 * [1] .$$

Let $\Delta: F_k^* \to F_k^* \times F_k^*$ be the diagonal. Then the following relations are well known [3].

(4.20) $$\Delta_* Q_1(a) = \sum_s \{Q_1(a_s') \otimes (a_s'')^2 + (a_s')^2 \otimes Q_1(a_s'')\}$$

where $\Delta_* a = \sum_s a_s' \otimes a_k''$.

(4.21) (Nishida relation) $\beta Q^j(a) = (j-1)\, Q^{j-1}(a)$

where β is the Bockstein operation.

Then the ring structure is proved by observing the following Kronecker products.

$$\langle u^2, \iota_1^2\rangle = 0\,, \quad \langle uv, Q_1(\iota_1)\rangle = 1\,.$$

The fact $Sq^1\, v = uv$ is proved by observing the following Kronecker product.

$$\langle Sq^1\, v, Q_1(\iota_1)\rangle = 1\,. \ \blacksquare$$

5. Proof of Theorem D

We prove Theorem D by using the strategy given in §3. We filter $F_k^*(S^2, CP^m)$ as given in §3. In general $X_n - X_{n-1}$ has one component for each partition of k into n pieces. Let $k = \nu_1 + \cdots + \nu_n$ be one of such partitions. We shall study the component which corresponds to this partition. Let $\mu_1, \cdots, \mu_s$ be the numbers distinct to each other which appear among the ν_i. We can assume μ_1 appears with multiplicity i_1, μ_2 appears with multiplicity i_2, $\cdots$, μ_s appears with multiplicity i_s so that $i_1 + \cdots + i_s = n$. We define the subgroup G of Σ_n by $G = \Sigma_{i_1} \times \Sigma_{i_2} \times \cdots \times \Sigma_{i_s}$. Then by the same argument as the proof of Lemma 4.3, we see the following

Lemma 5.1. *The component which corresponds to the partition $k = \nu_1 + \cdots + \nu_n$ as above is homotopically equivalent to $(S^{2m-1})^n \times_G \tilde{C}_n$.*

By using Lemma 5.1, we shall show the following

Proposition 5.2. $H_c^q(X_{k-1}) = 0$ *for* $q \leq 2m + k - 2$.

Proof. We shall admit the following lemma for a moment.

Lemma 5.3. $H_c^q(X_n - X_{n-1}) = 0$ *for* $q \leq n + 2m(k-n) - 1$.

Then we see by Lemma 5.3

$$H_c^q(X_1) = 0 \qquad \text{for} \quad q \leq 2m(k-1)$$

and

$$H_c^q(X_2 - X_1) = 0 \qquad \text{for} \quad q \leq 2m(k-2) + 1\,.$$

Hence by using the compact support cohomology exact sequence of the pair (X_2, X_1), we see

$$H_c^q(X_2) = 0 \qquad \text{for} \quad q \leq 2m(k-2) + 1\,.$$

If we repeat this process, we can inductively prove the following fact.

$$H_c^q(X_n) = 0 \qquad \text{for} \quad q \leq n + 2m(k-n) - 1\,.$$

In particular we see

$$H_c^q(X_{k-1}) = 0 \qquad \text{for} \quad q \leq 2\mathrm{m}+k-2 . \blacksquare$$

Proof of Lemma 5.3. By Lemma 5.1, each component of $X_n - X_{n-1}$ is homotopically equivalent to $(S^{2m-1})^n \times_G \tilde{C}_n$ where G is a subgroup of Σ_n. Note that $\dim_{\mathbf{R}}((S^{2m-1})^n \times_G \tilde{C}_n) = 2mn+n$. Hence we see

$$(5.4) \qquad H^q(X_n - X_{n-1}) = 0 \qquad \text{for} \quad q \geq 2mn+n+1 .$$

Note that $\dim_{\mathbf{R}} X_n = 2km+2n$. Hence by the Poincaré duality, we see

$$H_c^q(X_n - X_{n-1}) = 0 \qquad \text{for} \quad q \leq (2km+2n)-(2mn+n+1) = n+2m(k-n)-1 . \blacksquare$$

Next by using Proposition 5.2, we shall show the following

Proposition 5.5. *$H^q(X_k) \simeq H^q(X_k - X_{k-1})$ for $q \geq k(2m-1)$.*

Proof. By Proposition 5.2, we know

$$H_c^q(X_{k-1}) = 0 \qquad \text{for} \quad q \leq 2m+k-2 .$$

Hence by the compact support cohomology exact sequence of the pair (X_k, X_{k-1}), we see

$$(5.6) \qquad H_c^q(X_k) \simeq H_c^q(X_k - X_{k-1}) \qquad \text{for} \quad q \leq 2m+k-2 .$$

Note that $\dim_{\mathbf{R}} X_k = 2k(m+1)$. Hence by the Poincaré duality, we see

$$(5.7) \qquad H^q(X_k) \simeq H^q(X_k - X_{k-1}) \qquad \text{for} \quad q \geq 2k(m+1)-(2m+k-2) = 2m(k-1)+k+2 .$$

Note that we assumed $m \geq k+1$. Hence by (5.7), we see

$$H^q(X_k) \simeq H^q(X_k - X_{k-1}) \qquad \text{for} \quad q \geq 2mk+k+2-2(k+1) = k(2m-1) . \blacksquare$$

Proposition 5.8. *We have the following isomorphism as graded $\mathbf{Z}_2$ vector spaces.*

$$\bigoplus_{q \geq k(2m-1)} H^q(X_k - X_{k-1}) \simeq H^*(\tilde{C}_k/\Sigma_k) \otimes H^{k(2m-1)}((S^{2m-1})^k) .$$

Proof. First note that $X_k - X_{k-1}$ is homotopically equivalent to $(S^{2m-1})^k \times_{\Sigma_k} \tilde{C}_k$ by Lemma 5.1. We consider the Serre spectral sequence of the fiber bundle

$$(5.9) \qquad (S^{2m-1})^k \to (S^{2m-1})^k \times_{\Sigma_k} \tilde{C}_k \to \tilde{C}_k/\Sigma_k .$$

As $H^{k(2m-1)}((S^{2m-1})^k) = \mathbf{Z}_2$, the action of $\pi_1(\tilde{C}_k/\Sigma_k)$ on $H^{k(2m-1)}((S^{2m-1})^k)$ is trivial.

Note that $\dim_{R}\tilde{C}_k/\Sigma_k=2k$. Then we see the following facts.

(5.10) $$\bigoplus_{p\leq 2k} E_2^{p,k(2m-1)}\simeq H^*(\tilde{C}_k/\Sigma_k)\otimes H^{k(2m-1)}((S^{2m-1})^k).$$

(5.11) $$E_2^{p,q}=0 \qquad \text{for} \quad p>2k.$$

(5.12) $$E_2^{p,q}=0 \qquad \text{for} \quad (k-1)(2m-1)<q<k(2m-1) \quad \text{or} \quad k(2m-1)<q.$$

Note that we assumed $m\geq k+1$. Then by (5.10)-(5.12), we see

(5.13) $$E_2^{p,k(2m-1)}\simeq E_\infty^{p,k(2m-1)} \qquad \text{for all} \quad p.$$

If we use the consition $m\geq k+1$ once more, we can easily prove Proposition 5.8. ■

Now by Propositions 5.5 and 5.8, we see

$$\bigoplus_{q\geq k(2m-1)} H^q(X_k)\simeq H^*(\tilde{C}_k/\Sigma_k)\otimes H^{k(2m-1)}((S^{2m-1})^k).$$

Equivalently

(5.14) $$\bigoplus_{q\geq k(2m-1)} H_q(X_k)\simeq H_*(\tilde{C}_k/\Sigma_k)\otimes H_{k(2m-1)}((S^{2m-1})^k).$$

Hence it will be enough to show the following proposition in order to prove Theorem D.

Proposition 5.15. *The elements of* $\bigoplus_{q\geq k(2m-1)} H_q(X_k)$ *constructed by loop sums and iterated operations correspond bijectively to the elements of* $H_*(\tilde{C}_k/\Sigma_k)\otimes H_{k(2m-1)}$ $((S^{2m-1})^k)$.

We shall prove Proposition 5.15. First we shall study the elements constructed by loop sums and iterated operations. We define $l\in N$ to be $2^{l+1}>k\geq 2^l$. Let $[s]$ be the generator of $H_0(F_s^*(S^2, CP^m))$. Then the elements constructed by loop sums and iterated operations are given by the following two types.

(5.16) $$\iota_{2m-1}^{\alpha_0}*Q_1(\iota_{2m-1})^{\alpha_1}*\cdots*Q_{I_l}(\iota_{2m-1})^{\alpha_l}$$

(5.17) $$\iota_{2m-1}^{\alpha_0}*Q_1(\iota_{2m-1})^{\alpha_1}*\cdots*Q_{I_l}(\iota_{2m-1})^{\alpha_l}*[s]$$

for some $s\in N$.

Lemma 5.18. *The degree of an element of type* (5.17) *is less than* $k(2m-1)$. *While the degree of an element of type* (5.16) *is greater than or equal to* $k(2m-1)$.

Proof. We prove the first half. The second half can be proved similarly. We assume that an element

$$x=\iota_{2m-1}^{\alpha_0}*Q_1(\iota_{2m-1})^{\alpha_1}*\cdots*Q_{I_l}(\iota_{2m-1})^{\alpha_l}*[s]$$

of type (5.17) has degree greater than or equal to $k(2m-1)$. As x is an element of $H_*(F_k^*(S^2, CP^m))$, we have the following fact.

$$s+\alpha_0+2\alpha_1+\cdots+2^l\alpha_l=k. \tag{5.19}$$

As deg $x\geq k(2m-1)$, we have the following fact. We write M for $2m-1$.

$$\alpha_0M+\alpha_1(2M+1)+\alpha_2(4M+3)+\cdots+\alpha_l(2^lM+2^l-1)\geq kM. \tag{5.20}$$

Combining (5.19) and (5.20), we see

$$\begin{aligned}\alpha_0M+\alpha_1(2M+1)+\alpha_2(4M+3)+\cdots+\alpha_l(2^lM+2^l-1)\\ \geq sM+\alpha_0M+2\alpha_1M+\cdots+2^l\alpha_lM.\end{aligned} \tag{5.21}$$

(5.21) is equivalent to

$$\alpha_1+3\alpha_2+\cdots+(2^l-1)\,\alpha_l\geq sM. \tag{5.22}$$

By (5.19), we have the following inequality.

$$\alpha_1+3\alpha_2+\cdots+(2^l-1)\,\alpha_l\leq k-s. \tag{5.23}$$

Combining (5.22) and (5.23), we see $k-s\geq sM$. Hence

$$k\geq s(M+1)=2ms. \tag{5.24}$$

Note that we assumed $m\geq k+1$. Hence we see $s=0$ by (5.24). This is a contradiction. This completes the proof of the first half of Lemma 5.18. ■

We write ζ_i for $Q_{I_i}(l_{2m-1})$. Then by Lemma 5.18, the elements of $\bigoplus_{q\geq k(2m-1)} H_q(X_k)$ constructed by loop sums and iterated operations correspond to

$$\{\zeta_0^{\alpha_0}\zeta_1^{\alpha_1}\cdots\zeta_l^{\alpha_l};\ \alpha_i\geq 0,\ \alpha_0+2\alpha_1+\cdots+2^l\,\alpha_l=k\}. \tag{5.25}$$

(Note that the elements of (5.25) are linearly independent by Theorem 2.2).

Next we shall study the elements of $H_*(\tilde{C}_k/\Sigma_k)\otimes H_{k(2m-1)}((S^{2m-1})^k)$. $H_*(\tilde{C}_k/\Sigma_k)$ is described in [3]. We follow the notation of [3].

Proposition 5.26. $H_*(\tilde{C}_k/\Sigma_k)=Z_2[\xi_j]/I$.
Where deg $\xi_j=2^j-1$ *and I is the two sided ideal generated by* $(\xi_{j_1})^{k_1}\cdots(\xi_{j_t})^{k_t}$, *here* $\sum_{i=1}^t k_i\,2^{j_i}>k$.

By Proposition 5.26, the basis of $H_*(\tilde{C}_k/\Sigma_k)$ is given as follows.

$$\{\xi_1^{k_1}\xi_2^{k_2}\cdots\xi_l^{k_l};\ k_i\geq 0,\ 2k_1+4k_2+\cdots+2^l\,k_l\leq k\}. \tag{5.27}$$

Let $[(S^{2m-1})^k]$ be the fundamental class of $(S^{2m-1})^k$. Then by (5.27), the elements of $H_*(\tilde{C}_k/\Sigma_k)\otimes H_{k(2m-1)}((S^{2m-1})^k)$ correspond to

(5.28) $\{\xi_1^{k_1}\xi_2^{k_2}\cdots\xi_l^{k_l}\otimes[(S^{2m-1})^k];\ k_i\geq 0,\ 2k_1+4k_2+\cdots+2^l k_l\leq k\}$.

We see that (5.25) and (5.28) correspond to each other. This completes the proof of Proposition 5.15 and, consequently, of Theorem D. ■

6. Proof of Theorem C

In order to prove Theorem C, the case we need to consider is $F_3^*(S^2, \boldsymbol{C}P^2)$ and $F_3^*(S^2, \boldsymbol{C}P^3)$ by virtue of Theorem D. We shall prove the former. The latter can be proved similarly. Note that in degrees greater than or equal to 9, the elements constructed by loop sums and iterated operations in $H_*(F_3^*(S^2, \boldsymbol{C}P^2))$ are given by ι_3^3 and $\iota_3 * Q_1(\iota_3)$ (which are non-trivial by Theorem 2.2). Hence it will be enough to show the following proposition in order to prove Theorem C in the case $F_3^*(S^2, \boldsymbol{C}P^2)$.

Proposition 6.1. $H_q(F_3^*(S^2, \boldsymbol{C}P^2))=\begin{cases}\boldsymbol{Z}_2 & q=9, 10\\ 0 & q\geq 11\,.\end{cases}$

We filter $F_3^*(S^2, \boldsymbol{C}P^2)$ as given in §3. Then by the same argument as the proof of Lemmas 4.2 and 4.3, we see the following lemmas.

Lemma 6.2. *X_1 is homotopically equivalent to S^3.*

Lemma 6.3. *X_2-X_1 is homotopically equivalent to $(S^3)^2\times S^1$.*

Lemma 6.4. *X_3-X_2 is homotopically equivalent to $(S^3)^3\underset{\Sigma_3}{\times}\tilde{C}_3$.*

Note that $\dim_{\boldsymbol{R}} X_3=18$, $\dim_{\boldsymbol{R}} X_2=16$ and $\dim_{\boldsymbol{R}} X_1=14$. First we compute $H_c^*(X_2)$.

Lemma 6.5. $H_c^q(X_2)=\begin{cases}\boldsymbol{Z}_2 & q=9\\ 0 & q\leq 8\,.\end{cases}$

Proof. By Lemma 6.2 and the Poincaré duality, we see

(6.6) $$H_c^q(X_1)=0 \qquad \text{for}\quad q\leq 10\,.$$

By Lemma 6.3 and the Poincaré duality, we see

(6.7) $$H_c^q(X_2-X_1)=\begin{cases}\boldsymbol{Z}_2 & q=9\\ 0 & q\leq 8\,.\end{cases}$$

Hence Lemma 6.5 follows from the compact support cohomology exact sequence of the pair (X_2, X_1). ■

Next we compute $H^*(X_3-X_2)$. Note that X_3-X_2 is homotopically equivalent to $(S^3)^3\underset{\Sigma_3}{\times}\tilde{C}_3$ by Lemma 6.4. In order to compute $H^*((S^3)^3\underset{\Sigma_3}{\times}\tilde{C}_3)$, we decompose the covering space

(6.8) $$\Sigma_3 \to (S^3)^3 \times \tilde{C}_3 \to (S^3)^3 \underset{\Sigma_3}{\times} \tilde{C}_3$$

into the following two covering spaces. We embed $\boldsymbol{Z}_3$ in Σ_3 as the alternating group. Note that the following extension holds.

(6.9) $$1 \to \boldsymbol{Z}_3 \to \Sigma_3 \to \boldsymbol{Z}_2 \to 1 .$$

Then (6.8) is decomposed as follows.

(6.10) $$\boldsymbol{Z}_3 \to (S^3)^3 \times \tilde{C}_3 \to (S^3)^3 \underset{\boldsymbol{Z}_3}{\times} \tilde{C}_3 .$$

(6.11) $$\boldsymbol{Z}_2 \to (S^3)^3 \underset{\boldsymbol{Z}_3}{\times} \tilde{C}_3 \to (S^3)^3 \underset{\Sigma_3}{\times} \tilde{C}_3 .$$

As for (6.10), we see

(6.12) $$H^*((S^3)^3 \underset{\boldsymbol{Z}_3}{\times} \tilde{C}_3) \simeq H^*((S^3)^3 \times \tilde{C}_3)^{\boldsymbol{Z}_3} .$$

In order to compute (6.12), we need to know $H^*(\tilde{C}_3)$. $H^*(\tilde{C}_3)$ is described in [3]. We follow the notation of [3].

Proposition 6.13.

(1) $H^1(\tilde{C}_3) = \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ *and a basis is* $\{\alpha_{11}^*, \alpha_{21}^*, \alpha_{22}^*\}$.
(2) $H^2(\tilde{C}_3) = \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ *and a basis is* $\{\alpha_{11}^* \alpha_{21}^*, \alpha_{11}^* \alpha_{22}^*\}$.
(3) $\alpha_{21}^* \alpha_{22}^* = \alpha_{11}^*(\alpha_{21}^* + \alpha_{22}^*)$.
(4) *Let* $\sigma = (2\ 3)(1\ 2)$ *be the generator of* $\boldsymbol{Z}_3$. *Then* $\sigma^*\alpha_{11}^* = \alpha_{22}^*$, $\sigma^*\alpha_{21}^* = \alpha_{11}^*$ *and* $\sigma^*\alpha_{22}^* = \alpha_{21}^*$.
(5) $H^q(\tilde{C}_3) = 0$ *for* $q \geq 3$.

Now by using (6.12) and Proposition 6.13, we have the following

Lemma 6.14. $$H^q((S^3)^3 \underset{\boldsymbol{Z}_3}{\times} \tilde{C}_3) = \begin{cases} \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 & q = 8 \\ \boldsymbol{Z}_2 & q = 9, 10 \\ 0 & q \geq 11 . \end{cases}$$

Let $(\mathcal{G}_1)$ be the Gysin exact sequence of (6.11) and let $(\mathcal{G}_2)$ be the compact support cohomology exact sequence of the pair (X_3, X_2). By inspecting $(\mathcal{G}_1)$ and $(\mathcal{G}_2)$, we shall prove Proposition 6.1. We write X for $X_3 - X_2$.

Step 1. $H^q(X) = 0$ for $q \geq 11$.

In fact by the fact $H^q((S^3)^3 \underset{\boldsymbol{Z}_3}{\times} \tilde{C}_3) = 0$ for $q \geq 11$ (Lemma 6.14), we see $H^q(X) \simeq H^{11}(X)$ for $q \geq 11$ by $(\mathcal{G}_1)$. As X is a finite dimensional manifold, Step 1 holds.

Step 2. $H^q(X_3) = 0$ for $q \geq 11$.

In fact we see $H_c^q(X) = 0$ for $q \leq 7$ by Step 1 and the Poincaré duality. Note that $H_c^q(X_2) = 0$ for $q \leq 8$ (Lemma 6.5). Hence we see $H_c^q(X_3) = 0$ for $q \leq 7$ by $(\mathcal{G}_2)$. By the Poincaré duality, we see $H^q(X_3) = 0$ for $q \geq 11$.

In order to complete the proof of Proposition 6.1, it will be enough to determine $H^9(X_3)$ and $H^{10}(X_3)$ by virtue of Step 2.

Step 3. $H^{10}(X) = \boldsymbol{Z}_2$ and $H^9(X) \to H^{10}(X)$ is surjective in $(\mathcal{G}_1)$.

In fact by the fact $H^{11}(X)=0$ (Step 1) and $H^{10}((S^3)^3 \times_{\boldsymbol{Z}_3} \tilde{C}_3)=\boldsymbol{Z}_2$ (Lemma 6.14), we can write $(\mathcal{G}_1)$ in the following form.

$$\to H^9(X) \to H^{10}(X) \to \boldsymbol{Z}_2 \to H^{10}(X) \to 0$$

By the exactness, Step 3 follows.

Before we proceed to Step 4, we shall state a fact about $H^8(X_3)$.

$$H^8(X_3) = 0\,. \tag{6.15}$$

((6.15) is easily proved by using Theorems 2.1 and 2.2.)

Step 4. $H_c^{10}(X) = \boldsymbol{Z}_2$. Hence $H^8(X) = \boldsymbol{Z}_2$.

In fact by the fact $H^8((S^3)^3 \times_{\boldsymbol{Z}_3} \tilde{C}_3)=\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ (Lemma 6.14), we see $H^8(X) \neq 0$ by $(\mathcal{G}_1)$. Hence $H_c^{10}(X) \neq 0$ by the Poincaré duality. Note that $H_c^9(X_2)=\boldsymbol{Z}_2$ (Lemma 6.5). Note also that $H_c^{10}(X_3)=0$ ((6.15) and the Poincaré duality). Hence we see $H_c^{10}(X)=\boldsymbol{Z}_2$ by $(\mathcal{G}_2)$. By the Poincaré duality, $H^8(X)=\boldsymbol{Z}_2$.

Step 5. $H_c^8(X) \simeq H_c^8(X_3)$, $H_c^9(X) \simeq H_c^9(X_3)$.

In fact as $H_c^7(X_2) \simeq H_c^8(X_2)=0$ (Lemma 6.5), we see $H_c^8(X) \simeq H_c^8(X_3)$ by $(\mathcal{G}_2)$. In $(\mathcal{G}_2)$, we see $H_c^9(X_2) \to H_c^{10}(X)$ is an isomorphism by Step 4. Hence we see $H_c^9(X) \simeq H_c^9(X_3)$ by $(\mathcal{G}_2)$.

Step 6. $H^{10}(X_3) = \boldsymbol{Z}_2$.

In fact by the fact $H^{10}(X)=\boldsymbol{Z}_2$ (Step 3), we see $H_c^8(X)=\boldsymbol{Z}_2$ by the Poincaré duality. Hence we see $H_c^8(X_3)=\boldsymbol{Z}_2$ by Step 5. Then we see $H^{10}(X_3)=\boldsymbol{Z}_2$ by the Poincaré duality.

Step 7. $H^9(X_3) = \boldsymbol{Z}_2$.

In fact by the fact $H^8(X)=\boldsymbol{Z}_2$ (Step 4), $H^9((S^3)^3 \times_{\boldsymbol{Z}_3} \tilde{C}_3)=\boldsymbol{Z}_2$ (Lemma 6.14), $H^{10}(X)=\boldsymbol{Z}_2$ and $H^9(X) \to H^{10}(X)$ is surjective in $(\mathcal{G}_1)$ (Step 3), we can write $(\mathcal{G}_1)$ in the following form.

$$\to \boldsymbol{Z}_2 \to H^9(X) \to \boldsymbol{Z}_2 \to H^9(X) \to \boldsymbol{Z}_2 \to 0$$

By the exactness, we see $H^9(X)=\boldsymbol{Z}_2$. Hence $H_c^9(X)=\boldsymbol{Z}_2$ by the Poincaré duality. Then $H_c^9(X_3)=\boldsymbol{Z}_2$ by Step 5 so $H^9(X_3)=\boldsymbol{Z}_2$ by the Poincaré duality. This completes the proof of Proposition 6.1 and, consequently, of Theorem C in the case $F_3^*(S^2, CP^2)$. ■

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