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A NOTE ON SURFACES WITH PENCILS OF NON-HYPERELLIPTIC CURVES OF GENUS 3.

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Introduction. Let $f: S \rightarrow B$ be a surjective holomorphic map between a nonsingular projective surface S and a nonsingular projective curve B of genus b. We always assume that it is relatively minimal, that is, there are no (-1)-curves in fibers of f. If a general fiber of f is a non-hyperelliptic curve of genus g, we call it a *non-hyperelliptic fibration of genus g*. The purpose of this paper is to state some results on surfaces with non-hyperelliptic fibration of genus 3.

In § 1, we shall give the lower bound on K^2 of such surfaces. More precisely, we have $K^2 \geq 3\chi(\mathcal{O}_S)+10(b-1)$. This was first obtained by Horikawa [7] and, later, by Chen [4]. Our proof is different from them and rather simple.

In § 2, we construct surfaces with non-hyperelliptic fibrations of genus 3. Though we restrict ourselves to regular surfaces here, our method can be applied to irregular ones as well (with some more effort). We remark that the other examples of such srufaces can be found in [1].

To explain the background of the construction, let $f: S \rightarrow B$ be a non-hyperelliptic fibration of genus 3. Then, we have a canonical birational map of S into a $\boldsymbol{P}^2$-bundle W on B (see, § 1 below). We let V be its image, and consider the fibration $f': V \rightarrow B$ induced by the projection map of W. The difference of the invariants $(\chi(\mathcal{O}_V)-\chi(\mathcal{O}_S),\ \omega_V^2-\omega_S^2)$ can be considered as the contribution of singular fibers of f'. Though we do not have a complete list of possible singular fibers, we at least can expect that they are quite similar to those in [8]. However, a singular fiber arising from a simple elliptic singularity of type $\tilde{E}_7$, which Ashikaga has constructed in [1], seems to be a "special" one ([8, § 9]). What this means may be guaranteed by the fact that the canonical bundle of S cannot be ample in this case. Therefore, there sholud be a way to construct "general" ones. This is the motivation of the present construction.

In § 3, we shall give examples of Type I degenerations, extending a result in [3] and [5]. We hope that such mild degenerations can be used to attack the Torelli problem via degenerate loci (see, [10] for such an approach).

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of general type. After sumbitting this paper, the author recieved a preprint [9] in which our Theorem 1.2 is shown by essentially the same method.

1. The lower bound

Let $f\colon S\to B$ be a relatively minimal fibration of genus g. Recall that $f_*\omega_{S/B}$ is a locally free sheaf of rank g and degree $\chi-(g-1)(b-1)$, where $\omega_{S/B}=\omega_S\otimes f^*\omega_B^{-1}$ is the relative dualizing sheaf and $\chi=\chi(\mathcal{O}_S)$ is the holomorphic Euler-Poincaré characteristic of S.

Lemma 1.1. *Let $f\colon S\to B$ be a relatively minimal fibration of genus g, and let A be a sufficiently ample divisor on B.*

(1) *Put $\mathcal{V}=f_*\omega_{S/B}\otimes\mathcal{O}(A)$ and let $\pi\colon W=\boldsymbol{P}(\mathcal{V})\to B$ be the associated projective bundle. Then,*

$$h^0(W, \mathcal{O}(2T)) = (g+1)(\chi-(g-1)(b-1)+g \deg A)-\frac{1}{2}g(g+1)(b-1).$$

where T is a tautological divisor on W.

(2) $h^0(S, \mathcal{O}(2\omega_{S/B}+2f^*A)) = K^2+\chi-12(g-1)(b-1)+6(g-1)\deg A$.

Proof. Since A is sufficiently ample, we can assume that $H^1(W, \mathcal{O}(2T))\simeq H^1(B, S^2\mathcal{V})=0$. Therefore, (1) follows from the Riemann-Roch theorem on B. We next show (2). By the Riemann-Roch theorem we have

$$\chi(2\omega_{S/B}+2f^*A) = K^2+\chi-12(g-1)(b-1)+6(g-1)\deg A.$$

By the Serre duality, $H^q(2\omega_{S/B}+2f^*A))^*\simeq H^{2-q}(-\omega_S+2f^*(\omega_B-A))$. Therefore, $H^2(2\omega_{S/B}+2f^*A)=0$ and we only have to show that $H^1(-\omega_S+2f^*(\omega_B-A))$ vanishes. Put $D=\omega_S+2f^*(A-\omega_B)$. Since f is relatively minimal and A is sufficiently ample, we can assume that D is a 1-connected divisor satisfying $D^2>0$. By Ramanujam's vanishing theorem, we have $H^1(-D)=0$. Q.E.D.

Theorem 1.2. *Let $f\colon S\to B$ be a non-hyperelliptic fibration of genus 3. Then $K^2\geq 3\chi+10(b-1)$.*

Proof. Let the notation be as in Lemma 1.1. The sheaf homomorphism $f^*\mathcal{V}\to\mathcal{O}(\omega_{S/B}+f^*A)$ defines a rational map $h\colon S\to W$ over B. We put $V=h(S)$ and $n+1=h^0(\omega_{S/B}+f^*A)$. We remark that the rational map $\Phi\colon S\to\boldsymbol{P}^n$ associated with the linear system $|\omega_{S/B}+f^*A|$ factors through W, and we have $\Phi=\phi\circ h$ if $\phi\colon W\to\boldsymbol{P}^n$ denotes the holomorphic map defined by $|T|$. Since A is sufficiently ample, we can assume that ϕ is a quadrically normal embedding. Then we have

$$\begin{aligned} &h^0(2\omega_{S/B}+2f^*A)\\ (1.1)\qquad &\geq \dim \operatorname{Im}\{H^0(\boldsymbol{P}^n, \mathcal{O}(2))\to H^0(V, \mathcal{O}(2))\}\\ &= \dim \operatorname{Im}\{H^0(W, \mathcal{O}(2T))\to H^0(V, \mathcal{O}(2T))\}.\end{aligned}$$

We denote by $\mathcal{I}_V$ the ideal sheaf of V in W and consider the cohomology long exact sequence for

$$0 \to \mathcal{I}_V(2T) \to \mathcal{O}_W(2T) \to \mathcal{O}_V(2T) \to 0 .$$

Since f is a non-hyperelliptic fibration of genus 3, we have $\mathcal{I}_V = \mathcal{O}_W(-4T + \pi^* A_1)$ for some divisor A_1 on B. From this, we get $H^0(\mathcal{I}_V(2T)) = 0$. Therefore, the assertion follows from (1.1) and Lemma 1.1. Q.E.D.

REMARK 1.3. The inequality in Theorem 1.2 is stated in [7, § 3] without proof. It seems that he obtained it as a consequence of the classification of possible singular fibers of f. On the other hand, Chen [4] obtained it comparing ω_S^2 with ω_V^2. Our proof gives us a hope to get the lower bound on non-hyperelliptic fibrations of higher genus. The difficulty is in giving the upper bound on $h^0(\mathcal{I}_V(2T))$.

We collect below some applications of Theorem 1.2. Recall that we have $K^2 \geq (8/3)(\chi + 4(b-1))$ for hyperelliptic fibration of genus 3 (see, [6, Theorem 2.1]). Therefore, we get the following which is also pointed out in [7, § 3].

Corollary 1.4. *In the range* $3\chi + 10(b-1) > K^2 \geq (8/3)(\chi + 4(b-1))$, *surfaces with a hyperelliptic fibration of genus* 3 *cannot be deformed to those with a non-hyperelliptic fibration.*

We call a minimal surface S a *canonical* surface, if the rational map induced by the canonical system $|K|$ is a birational map of S onto its image. Note that Castelnuovo's argument (see, e.g., [2, § 1]) tells us that $K^2 \geq 3p_g + q - 7$ holds for canonical surfaces. On the other hand, we have Horikawa's inequality $K^2 \geq 3\chi$ for irregular canonical surfaces ([6, Theorem 3.1]) since they cannot admit any hyperelliptic fibrations. All of these in mind, it would be worth stating here the following:

Corollary 1.5. *Let S be an irregular canonical surface, and assume that the image of the Albanese map is a curve. Then* $K^2 \geq 3\chi + 10(q-1)$.

Proof. Let $f: S \to B \subset \mathrm{Alb}(S)$ be the Albanese map. By the assumption, B is a curve of genus $q(S)$ and f is a non-hyperelliptic fibration of some genus $g \geq 3$. Then we get $K^2 \geq 3\chi + 10(q-1)$ by Theorem 1.1 and a result of Xiao [11, Theorem 2]. Q.E.D.

It is quite likely that the inequality in Corollary 1.5 holds without the assumption on the Albanese map: If the image of the Albanese map is a surface, then the cotangent sheaf Ω_S^1 is generically generated by its global sections and, therefore, it seems to satisfy a certain "ampleness" condition which forces K^2 to be big. For example, we have the following:

Lemma 1.6. *Let S be a nonsingular projective surface with nef cotangent sheaf. Then $K^2 \geq 6\chi$.*

Proof. We consider the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(\Omega_S^1)$ on S, and let L denote the tautological divisor. By the assumption, L is nef. Therefore, we have $0 \leq L^3 = c_1^2(S) - c_2(S)$, i.e., $K^2 \geq 6\chi$. Q.E.D.

2. Construction

We let W denote the total space of the $\boldsymbol{P}^2$-bundle

$$\pi\colon \boldsymbol{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \to \boldsymbol{P}^1,$$

where a, b, c are integers satisfying $0 \leq a \leq b \leq c$. We let T and F denote a relatively ample tautological divisor and a fiber of π, respectively. Then the Picard group of W is a free abelian group generated by them. Furthermore, we have $T^3 = (a+b+c)T^2F$ in the Chow ring of W. We put

$$p = a+b+c+3 \tag{2.1}$$

and assume that $p \geq 4$. Take an integer s, and let Q be a general member of $|2T+sF|$. We remark that Q is irreducible and has only rational double points of type A_1 if

$$a+c+s \geq 0, \ 2b+s \geq 0\,. \tag{2.2}$$

Choose general k fibers $F_1, \cdots, F_k$ of π. We assume that Q intersects with F_i transversally, and that $Q_i = Q \cap F_i$ is an irreducible conic (in $F_i \simeq \boldsymbol{P}^2$) for each i.

Let $\nu\colon \hat{W} \to W$ be the blowing-up along $\cup Q_i$, and put $\mathcal{E}_i = \nu^{-1}(Q_i)$. Since the normal sheaf of $Q_i \simeq \boldsymbol{P}^1$ in W is isomorphic to $\mathcal{O} \oplus \mathcal{O}(4)$, each $\mathcal{E}_i$ is isomorphic to Σ_4, the Hirzebruch surface of degree 4. We put

$$L = \nu^*(4T - (p-5-k)F) - 2\sum_{i=1}^{k} \mathcal{E}_i\,. \tag{2.3}$$

If we denote by $\hat{Q}$ and $\hat{F}_i$ the proper transforms of Q and F_i, respectively, then we get

$$\begin{aligned} L &\sim \nu^*(4T - (p-5+k)F) + 2\textstyle\sum_{i=1}^{k} \hat{F}_i \\ &\sim 2\hat{Q} + (k+5-p-2s)\nu^*F, \end{aligned}$$

where the symbol $\sim$ means the linear equivalence of divisors. Since $\hat{Q} \cap \hat{F}_i = \emptyset$, we have

Lemma 2.1. *$Bs|L| = \emptyset$ if the following conditions are satisfied.*
(1) $k \geq 2s+p-5$.

(2) *Q does not meet $Bs|4T-(p-5+k)F|$.*

Lemma 2.2. *The condition* (2) *of* Lemma 2.1 *is satisfied if one of the following conditions is satisfied:*

(1) $k\leq 4a-p+5=3a-b-c+2$.

(2) $s=-2a$, $k\leq 4b-p+5=3b-a-c+2$.

Proof. If (1) holds, then $Bs|4T-(p-5+k)F|=\emptyset$. We assume that (2) holds. We let X_0, X_1 and X_2 be sections on W of $[T-aF]$, $[T-bF]$ and $[T-cF]$, respectively, such that they form a system of homogeneous coordinates on each fiber of $\pi: W\rightarrow \boldsymbol{P}^1$. If $4b\geq p-5+k$, then $|4T-(p-5+k)F|$ has no base point outside the rational curve G defined by $X_1=X_2=0$. On the other hand, the equation of Q can be written as

$$q_{00}X_0^2+q_{10}X_0X_1+q_{01}X_0X_2+q_{20}X_1^2+q_{11}X_1X_2+q_{02}X_2^2=0\,,$$

where q_{ij} is a homogeneous form on $\boldsymbol{P}^1$ of degree $(2-i-j)a+ib+jc+s$. If $s=-2a$, then we can assume that q_{00} is a nonzero constant. Then Q does not meet G. Thus (2) is also sufficient to imply (2) of Lemma 2.1. Q.E.D.

Lemma 2.3. *Suppose that $|L|$ contains an irreducible nonsingular member S. Then is a minimal surface satisfying:*

(1) *The canonical map of S is a birational morphism.*

(2) *$p_g(S)=p$, $q(S)=0$ and $K^2=3p_g-7+k$.*

Proof. By the adjunction formula, we have

$$K\sim(K_{\hat{W}}+S)|_S\sim(\nu^*T+\sum_{i=1}^k \hat{F}_i)|_S\,.$$

Since $\hat{F}_i$, $1\leq i\leq k$, does not meet S, we have $\mathcal{O}(K)=\mathcal{O}_S(\nu^*T)$. We next consider the cohomology exact sequences for

$$0\rightarrow\mathcal{O}(K_{\hat{W}})\rightarrow\mathcal{O}(\nu^*T+\sum_{i=1}^k \hat{F}_i)\rightarrow\mathcal{O}(K)\rightarrow 0\,, \tag{2.4}$$

$$0\rightarrow\mathcal{O}(\nu^*T)\rightarrow\mathcal{O}(\nu^*T+\sum_{i=1}^k \hat{F}_i)\rightarrow\oplus_{i=1}^k\mathcal{O}_{\hat{F}_i}(-1)\rightarrow 0\,. \tag{2.5}$$

Since $\hat{W}$ is rational, we have $H^q(\hat{W},\mathcal{O}(K_{\hat{W}})=0$ for $q<3$. Thus we get $H^q(S,\mathcal{O}(K))\simeq H^q(\hat{W},\mathcal{O}(\nu^*T+\sum\hat{F}_i))$ for $q<2$ from (2.4). Then, from (2.5), we get $H^q(K)\simeq H^q(W,\mathcal{O}(T))$ for $q<2$. This shows the formulae for $p_q(S)$ and $q(S)$. We in particular have shown that $|\nu^*T|$ is restricted to $|K|$ isomorphically. Therefore, $Bs|K|=Bs|\nu^*T|=\emptyset$. This implies that S is minimal. Since $p\geq 4$, we at least have $c>0$ by (2.1). Then the holomorphic map induced by $|T|$ gives a birational map of W onto its image. Therefore, (1) follows from what we have established above. Finally, we calculate K^2 to get:

$$K^2=(\nu^*T)^2L=T^2(4T-(p-5-k)F)=3p-7+k\,.$$

Q.E.D.

Proposition 2.4. *Let x and y be positive integers satisfying*

$$x\geq 4,\ 3x-7\leq y\leq \begin{cases} 4x-8 & \textit{if } x \textit{ is odd,} \\ 4x-10 & \textit{if } x \textit{ is even.} \end{cases}$$

Then there exists a minimal, regular surface of general type S with the following properties:

(1) $p_g(S)=x,\ K^2=y$.

(2) *The canonical linear system of S has neither fixed components nor base points, and the canonical map is a birational morphism onto its image.*

(3) *S has a pencil of non-hyperelliptic curves of genus three.*

(4) *The canonical image of S is contained in a threefold of minimal degree.*

Sketch of Proof. Put $(a, b, c)=(a, a, a), (a, a, a+1), (a, a+1, a+1)$ according to $x\equiv 0, 1, 2$ modulo 3, respectively. Then, by Lemma 2.1 and (1) of Lemma 2.2, we can cover the region

$$3x-7\leq y\leq \frac{10}{3}x-\begin{cases} 6, & \text{if } x\equiv 0, \\ \frac{22}{3}, & \text{if } x\equiv 1, \\ \frac{26}{3}, & \text{if } x\equiv 2, \end{cases} \quad (\text{mod. } 3) \tag{2.6}$$

for a suitably chosen s satisfying (2.2).

We next put $s=-2a$ and consider (2) of Lemma 2.2. Putting $a=0$, $b=c$, we can cover the region $4x-12\leq y\leq 4x-8$ with x odd. Similarly, putting $a=1$, $b=c$, we can cover the region $4x-16\leq y\leq 4x-10$ with x even. In this way, by increasing a and putting $b=c$, we can cover the region outside (2.6) as well. The other statements follow from Lemma 2.3 and the construction.

Q.E.D.

REMARK 2.5. The inverse image of Q_i on S is a hyperellitpic curve of genus 3 (see, [8, § 9]). Furthermore, it can be checked that most surfaces we have constructed have ample canonical bundle.

3. Examples of Type I degenerations

In this section, we construct some degenerating family of surfaces with non-hyperelliptic fibrations of genus 3. The central fiber consists of a surface Σ and some $\boldsymbol{P}^2$, intersecting along a conic. Such degenerations are Type I degenerations in the sense that the local monodromy acts trivially on the second cohomology. For the properties, see [5].

We keep the notation of § 2. Moreover, we assume that the conditions in Lemmas 2.1 and 2.2 are satisfied.

(A) Degenerations of Hyperelliptic Type.

Our first example is essentially the same as one in [3, § 2]. Take two integers α, β satisfying

$$p-5-k=2\alpha-\beta, \beta\geq 0. \tag{3.1}$$

We put $M_\alpha=\nu^*(2T-\alpha F)-\sum_{i=1}^k \mathcal{E}_i$. The following can be shown in the same way as Lemmas 2.1 and 2.2.

Lemma 3.1. *$Bs|M_\alpha|=\emptyset$ if the following conditions are satisfied.*

(1) $s+\alpha\leq 0$.

(2) *One of the following conditions is satisfied:*

(i) $2a\geq\alpha+k$.

(ii) $s=-2a$, $2b\geq\alpha+k$.

We assume that the conditions in Lemma 3.1 are satisfied. We choose $q_\alpha\in H^0(\hat{W}, \mathcal{O}(M_\alpha))$ which defines an irreducible nonsingular divisor Q_α. Let ε be a sufficiently small positive number and put $\Delta_\varepsilon=\{z\in \boldsymbol{C};\ |z|<\varepsilon\}$. We consider a family $\{S_t\}_{t\in\Delta}$, of subvarieties of the $\boldsymbol{P}^1$-bundle $X=\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(M_\alpha))\to\hat{W}$ defined by

$$S_t:\begin{cases} a_0Z_0^2+a_1Z_0Z_1+a_2Z_1^2=0 \\ tZ_0=q_\alpha Z_1\end{cases}, \quad t\in\Delta_\varepsilon, \tag{3.2}$$

where (Z_0, Z_1) is a system of homogeneous coordinates on fibers of $X\to\hat{W}$ and $a_j\in H^0(\hat{W}, \mathcal{O}(jM_\alpha+\beta\nu^*F))$, $0\leq j\leq 2$. We assume that a_j are general. If $t\neq 0$, then S_t is biholomorphically equivalent to a surface in $\hat{W}$ defined by the equation

$$a_0q^2+ta_1q+t^2a_2=0. \tag{3.3}$$

By (3.1), we see that S_t is a surface constructed in § 2.

Lemma 3.2. *For a generic choice of q_α and a_j, $0\leq j\leq 2$, S_0 is a divisor with simple normal crossings consisting of a minimal surface Σ with $K^2=3p_g-7+k-\beta$ and β disjoint copies of $\boldsymbol{P}^2$. Furthermore, a double curve $\Sigma\cap\boldsymbol{P}^2$ is a smooth conic (in $\boldsymbol{P}^2$) and $p_g(\Sigma)=p_g(S_t)$, $t\neq 0$.*

Proof. Putting $t=0$ in the second equation of (3.2), we get $Z_1=0$ or $q_\alpha=0$. If $Z_1=0$, then we get $a_0=0$ from the first equation. Since a_0 can be identified with a homogeneous form of degree β on $\boldsymbol{P}^1$, we may assume that its zeros are mutually distinct and give β fibers $F_1, \cdots, F_\beta$ on W. We may identify the

F_i with its inverse image on $\hat{W}$. Thus $Z_1=a_0=0$ defines a section $\tilde{F}_j$ of the $\boldsymbol{P}^1$- bundle $X|_{F_j}\to F_j$ for each j, $1\leq j\leq\beta$ and therefore $\tilde{F}_j\simeq\boldsymbol{P}^2$. On the other hand, if $q_\alpha=0$, the first equation of (3.2) defines a double covering Σ of Q_α. As in [3, § 2], we can show that Σ is a regular minimal surface with $K^2=3p_g(\Sigma)-7+k-\beta$ whose geometric genus equals that of a general fiber S_t. Q.E.D.

In this way, we get Type I degenerations such that a general fiber is a surface with a non-hyperelliptic fibration of genus 3 and the main component of the central fiber is one with a hyperelliptic fibration of genus 3. Note that, putting $\beta=0$, we get a family of deformations of surfaces such that a general fiber is canonical but the central fiber is non-canonical.

(B) Degenerations of Non-hyperelliptic type.

We consider the subsystem $|L'|+\hat{F}_k$ of $|L|$, where

$$L'=\nu^*(4T-(p-5-(k-1)F)-2\sum_{i=1}^{k-1}\mathcal{E}_i-\mathcal{E}_k\,.$$

Since we have

$$\begin{aligned}L'&\sim\nu^*(4T-(p-5+k)F)+2\textstyle\sum_{i=1}^{k-1}\hat{F}_i+\hat{F}_k\\&\sim 2\hat{Q}+(k+4-p-2s)\nu^*F+\mathcal{E}_k\\&\sim\hat{Q}+\nu^*(2T-(p-5+s)F)+\textstyle\sum_{i=1}^{k-1}\hat{F}_i\,,\end{aligned}$$

we see that $Bs|L'|=\emptyset$ if $k\geq 2s+p-4$ and

$$\hat{Q}\cap Bs|\nu^*(4T-(p-5+k)F)|\cap Bs|\nu^*(2T-(p-5+s)F)|=\emptyset\,.$$

Therefore, as in § 2, we can show the following:

Lemma 3.3. *$Bs|L'|=\emptyset$ provided that the following conditions are satisfied:*
(1) $k\geq 2s+p-4$.
(2) *One of the conditions in* Lemma 2.2 *is satisfied.*

We assume that the conditions in Lemma 3.3 are satisfied. Let Σ be a general member of $|L'|$ which is defined by $x'\in H^0(\hat{W},\mathcal{O}(L'))$. Furthermore, we let $\hat{f}_k$ define $\hat{F}_k$. Take a sufficiently general section $x\in H^0(\hat{W},\mathcal{O}(L))$ and consider a degenerating family $\{S_t\}$, $t\in\Delta_\varepsilon$, defined by $S_t=(tx+x'\cdot\hat{f}_k)$. Note that we have $K_\Sigma\sim\nu^*T|_\Sigma$. Therefore, the main component Σ of the central fiber S_0 is a minimal regular surface with $K^2=3p-8+k$, $p_g(\Sigma)=p$. Furthermore, it intersects with $\hat{F}_k\simeq\boldsymbol{P}^2$ along a conic curve.

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