Murata, T.
Osaka J. Math.
28 (1991), 623–637

HOLOMORPHIC MAPPINGS FROM THE UNIT DISK TO ALGEBRAIC VARIETIES

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(Reseived Febraury 16, 1990)

Introduction

In 1925, R. Nevalinnna established the second main theorem for meromorphis functions on the complex plane $\boldsymbol{C}$ and developed the value distribution thoery. His theory was extended by many authors. In particular, H. Cartan proved the second main theorem for holomorphic mpas from $\boldsymbol{C}$ to the complex projective space $P^n(\boldsymbol{C})$ (cf., e.g., S. Lang [7]). And J. Noguchi [9] studied holomorphic maps from $\boldsymbol{C}$ to algebraic verieties and showed a version of second main theorem for these maps. Nevanlinna's lemma on the logarithmic derivative plays a crucial role in these theory.

On the other hand R. Nevanlinna also gave the second main theorem on a disk of finite radius (cf., e.g., W. Hayman [5]).

In this paper we shall study holomorphic maps from a disk of finite radius into an algebraic variety and derive a version of the second main theorem.

Let V be a nonsingular projective algebraic variety and Σ an effective divisor of simply normal crossing. Let Ω be a Kahler form on V and $\Delta(R)$ the disk of $\boldsymbol{C}$ around the origin with radius R. In this paper, we assume that R is greater than 1 for technical reasons. Let us denote by $T_f(r)$ and $\bar{N}_f(r, \Sigma)$ the charactearistic function of f relative to Ω and the counting function for Σ without multiplicities (see §1) respectively. Suppose that V-Σ satisfy condition (A) in §1; namely, there exists a system of logarithmic 1-forms $\{\omega_i\}_{i=1}^{n+1}$ along Σ such that $\omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1}$ are linearly independent over $\boldsymbol{C}$, where n is the dimension of V. A holomorphic map f: $\Delta(R) \rightarrow V$ is by definition *degenerate with respect to* $\{\omega_i\}_{i=1}^{n+1}$ if the image $f(\Delta(R)-f^{-1}(\Sigma))$ is contained in a subvariety

$$\{x \in V-\Sigma: \sum_{i=1}^{n+1} a_i(\omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1})_x = 0\}$$

with $(a_1, \cdots, a_{n+1}) \neq (0, \cdots, 0)$ $(a_i \in \boldsymbol{C})$. Then the main theorem of this paper is stated as follows.

Theorem A. *Let* V, Σ *and* $\{\omega_i\}_{i=1}^{n+1}$ *be as above. Let* f: $\Delta(R) \rightarrow V$ *be a* *holomorphic map which is nondegenerate with respect to* $\{\omega_i\}_{i=1}^{n+1}$. *Then*

$$\kappa T_f(r) \leq \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \| , \qquad \text{(I)}$$

where κ *is a constant independent of* r *and* f. *Furthermore if* f *is of finite order, then*

$$\kappa T_f(r) \leq \bar{N}_f(r, \Sigma) + O(\log \frac{1}{R-r}) + O(1) , \qquad \text{(II)}$$

Throughout this paper we shall write, $\varphi(r) \leq \phi(r) \|$ when $\varphi(r) \leq \phi(r)$ except on an open set E with $\int_E (R-r)^{-1} dr < \infty$. As applications of Theorem A, we shall prove the following two results.

Theorem B. *Under the same assumptions as in Theorem* A, *if*

$$\int_I^R \bar{N}_f(t, \Sigma) (R-t)^{\mu-1} dt < \infty$$

for a positive number μ, *then the holomorphic map* f *is of finite order and*

$$\int_I^P T_f(t) (R-t)^{\mu-1} dt < \infty$$

holds.

Corollary C. *Let* V, Σ, $\{\omega_i\}_{i=1}^{n+1}$ *and* f *be as in Theorem* A, *and let* supp $f^*\Sigma = \{a_1, a_2, \cdots\}$. *Suppose that*

$$\sum_i (R - |a_i|)^{\lambda+1} < \infty \quad (\lambda > 0) .$$

Then for any effective divisor D *such that* $f(\Delta(R)) \not\subset$ supp D, *we have*

$$\sum_i (R - |b_i|)^{\lambda+1} < \infty ,$$

where $f^*D = b_1 + b_2 + \cdots$.

We remark that when $V = P^1(\boldsymbol{C})$, Theorem B and Corollary C are well known (cf., e.g., [17] P. 140, [13] P. 104).

In §1 we recall some definitions and known results. §2 is devoted to the extension of Nevanlinna's lemma on the logarithmic derivative. Theorem A (resp. Theorem B and Corollary C) will be proved in §3 (resp. §4). In §5, we shall discuss holomorphic maps from $\Delta(R)$ into an algebraic variety with bounded charactaristic functions.

I would like to thank Professor S. Murakami for his encouragement, Professor J. Noguchi for his guidance in preparing this paper and Professor A. Kasue for his reading the manuscript of this paper.

1. Preliminaries

(1) We call a meromorphic 1-form ω a logarithmic 1-form along Σ if any point $a \in V$ we can take a holomorphic coordinate system U, $(x_1, \cdots, x_n)$ around a such that $\{x_1 \cdots x_k = 0\} = \Sigma \cap U$ $(k \leq n)$ and

$$\omega = a_1(x) \frac{dx_1}{x_1} + \cdots + a_k(x) \frac{dx_k}{x_k} + \eta \text{ on } U,$$

where $a_1(x), \cdots, a_k(x)$ are holomorphic functions on U and η is a holomorphic 1-form on U. Let $H^0(V, \Omega_V^1(\log \Sigma))$ be the vector space of logarithmic 1-forms along Σ on V. An element of $H^0(V, \Omega_V^1(\log \Sigma))$ is d-closed on V-Σ and

$$\dim H^0(V, \Omega_V^1) + \dim H^0(V, \Omega_V^1(\log \Sigma)) = \dim H_1(V - \Sigma, \boldsymbol{C})$$

where Ω_V^1 denotes the sheaf of germs of holomorphic 1-forms (see Delgine [2]). We assume the following condition (A):

(A) "Three exists a system $\{\omega_i\}_{i=1}^{n+1}$ of $n+1$ logarithmic 1-forms ω_i in $H^0(V, \Omega_V^1(\log \Sigma))$ such that the n-forms

$$\omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1} \quad (i = 1, \cdots, n+1)$$

are linearly independent over $\boldsymbol{C}$, where n is the dimension of V."

A holomorphic map f: $\Delta(R) \rightarrow V$ is by definition *degenerate* with respect to $\{\omega_i\}_{i=1}^{n+1}$ if the image $f(\Delta(R) - f^{-1}(\Sigma))$ is contained in a subvariety

$$\{x \in V - \Sigma : \sum_{i=1}^{n+1} a_i(\omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1})_x = 0\},$$

where $(a_1, \cdots, a_{n+1}) \neq (0, \cdots, 0)$ $(a_i \in \boldsymbol{C})$. If f is degenerate with respect to $\{\omega_i\}_{i=1}^{n+1}$, then $f(\Delta(R))$ is cotained in the support of an element of the complete linear system $|K_V + \Sigma|$.

(2) We denote by supp D the support of a divisor D on $\Delta(R)$ or V. For an effect,ve divisor D on V such that $f(\Delta(R)) \not\subset \operatorname{supp} D$ we denote by $n_f(t, D)$ the sum of orders of the divisor $f^*D \cap \Delta(t)$. We define $\bar{n}_f(t, D)$ the number of points of supp f^*D in $\Delta(t)$. We denote $n_f(0, D)$ the order of f^*D at 0 and $\bar{n}_f(0, D)$ the order of supp f^*D at 0. Set

$$N_f(r, D) = \int_0^r \{n_f(t, D) - n_f(0, D)\} \frac{dt}{t} + n_f(0, D) \log r,$$

$$\bar{N}_f(r, D) = \int_0^r \{\bar{n}_f(t, D) - \bar{n}_f(0, D)\} \frac{dt}{t} + \bar{n}_f(0, D) \log r.$$

(3) For a meromorphic function α in $\Delta(R)$ we write

$$m(r, \alpha) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\alpha(re^{i\theta})| d\theta ,$$

$$N(r, \alpha) = \int_0^r \{n(r, \alpha) - n(0, \alpha)\} \frac{dt}{t} + n(0, \alpha) \log r ,$$

where $\log^+|\alpha| = \max\{\log|\alpha|, 0\}$, and $n(t, \alpha)$ denotes the number of poles of α in $\Delta(t)$ with counting multiplicities and $n(0, \alpha)$ the order of α at 0. We also set

$$T(r, \alpha) = m(r, \alpha) + N(r, \alpha) .$$

(4) Let f be a holomorphic map from $\Delta(R)$ to V. The characteristic function $T_f(r)$ of f relative to Ω is defined by

$$T_f(r) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^*\Omega .$$

We say that f is of finite order $\lambda \in [0, \infty)$ if

$$\limsup_{r \to R} \frac{\log T_f(r)}{\log \dfrac{1}{R-r}} = \lambda$$

and of infinite order if

$$\limsup_{r \to R} \frac{\log T_f(r)}{\log \dfrac{1}{R-r}} = \infty .$$

We note that f is of finite order λ if and only if

$$\int_1^R T_f(t)(R-t)^{\mu-1} dt = \begin{cases} \infty & (\text{for } \mu < \lambda) \\ \text{finite} & (\text{for } \mu > \lambda) . \end{cases}$$

(5) Let $[D] \to V$ be the line bundle over V defined by a divisor D on V. Let $\Psi \in c_1([D])$ be the first Chern form defined by a fiber metric $\|\cdot\|$ in $[D]$ and take a section $\sigma \in \Gamma(V, [D])$ such that σ defines the divisor D and $\|\sigma\| \leq 1$, then the first main theorem says that

$$T_f(r, c_1([D])) = N_f(r, D) + m_f(r, D) + O(1) ,$$

where $T_f(r, c_1([D])) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^*\Psi$

and $$m_f(r, D) = \frac{1}{2\pi}\int_0^{2\pi} \log(\|(\sigma \circ f)(re^{i\theta})\|^{-1}) d\theta$$

(cf., e.g., Shabat [16], p.61). Since Ω is positive definite and V is compact, there exists a constant $K>0$ such that

$$(1.1) \qquad T_f(r, c_1([D])) \leq K\, T_f(r)\,.$$

Let $\mathfrak{R}(V)$ be the field of rational functions over V and $\{\phi_1, \cdots, \phi_l\}$ the generators of $\mathfrak{R}(V)$ such that each $f^*\phi_j$ is defined. Put

$$\tilde{T}_f(r) = \max_{1\leq j\leq l} \{T(r, f^*\phi_j)\}\ .$$

Then we have

$$(1.2) \qquad B'\, T_f(r) + O(1) \leq \tilde{T}_f(r) \leq B\, T_f(r) + O(1)\,,$$

where B and B' are positive constants (cf., [12]).

(6) Let V and Σ be as in Theorem A. We denote by $\mathfrak{M}_V^*(\Sigma)$ the sheaf of germs of non-zero meromorphic functions whose zeros and poles are contained in Σ, and $H^0(V, \mathfrak{A}_V(\log \Sigma))$ the Z module of meromorphic closed 1-forms whose germs concide with $d \log \zeta$ where $\zeta \in \mathfrak{M}_V^*(\Sigma)$. Let $\Sigma_i (i=1, 2, \cdots)$ be the irreducible components of Σ and $\mathring{\Sigma}$ the seto of regular points of Σ. For each point of $\mathring{\Sigma} \cap \Sigma_i$ we can take a neighborhood U and a holomrorphic coordinate system $(x_1, \cdots, x_n)$ such that $\{x_1=0\} = \Sigma_i \cap U$. Then every seation ω in $H^0(V, \mathfrak{A}_V(\log \Sigma))$ is written in U as

$$\omega = \nu_i \frac{dx_1}{x_1} + \eta\,,$$

where ν_i is an integer and η is a holomorphic 1-form. The integer ν_i is independent of the choice of a local coordinate system $(x_1, \cdots, x_n)$. Since $\Sigma_i \cap \mathring{\Sigma}$ is connected ν_i is constant on $\Sigma_i \cap \mathring{\Sigma}$. We define the residue of ω on $\Sigma_i \cap \mathring{\Sigma}$ by

$$\operatorname{res}(\omega, \Sigma_i) = \nu_i\,.$$

Thus we get a divisor $D = \Sigma \operatorname{res}(\omega, \Sigma_i)\, \Sigma_i$.

(7) **Proposition.** *There exists a basis* $\{\omega_i\}$ *of the vector space* $H^0(V, \Omega_V^1$ $(\log \Sigma))$ *over* $\boldsymbol{C}$ *such that every* ω_i *is an element of* $H^0(V, \mathfrak{A}_V(\log \Sigma))$

Proof. See Iitaka [6] Sections 2∼4.

(8) **Ochai's theorem** (cf., [14], [9].). *Suppose that there exists a system* $\{\omega_i\}_{i=1}^{n+1}$ *of logarithmic* 1*-forms on* V *satisfying* (A). *Let* f *be a holomorphic map from* $\Delta(R)$ *to* V *which is non-degenerate with respect to* $\{\omega_i\}_{i=1}^{n+1}$. *Then for every rational function* $\phi \in \mathfrak{R}(V)$ *such that* $f^*\phi$ *is defined, the meromorphic function* $f^*\phi$ *is algebraic over the field generated by* $\{\zeta_i^{(k)}: 0 \leq k \leq n-1, 1 \leq i \leq n+1\}$, *where* ζ_i *is defined by* $f^*\omega_i = \zeta_i\, dz$ *and* $\zeta_i^{(k)}$ *denotes tne k-the derivative of* ζ_i.

(9) **Proposition.** *Let F be a meromorphic function on $\Delta(R)$ and A_i $(i=0, \cdots, l)$ holomorphic functions on $\Delta(R)$ such that $A_0 \not\equiv 0$ and*

$$A_0 F^l + A_1 F^{l-1} + \cdots + A_{l-1} F + A_l = 0 .$$

Then

$$T(r, F) \leq \sum_{j=0}^{l} T(r, A_i) + O(1) .$$

Proof. See Noguchi-Ochiai [12] Lemma (6.1.5).

2. A generalization of Nevanlinna's lemma on logarithmic derivative

In this section we give a generalization of Nevanlinna's lemma on logarithmic derivative.

Lemma 2.1. *Let $\varphi(r)$ be a positive valued C^1 function with non-negative derivatives on $[0, R)$. Then*

$$\varphi^{(1)}(r) \leq \{\varphi(r)\}^2 \frac{1}{R-r} \, || \, .$$

Proof. Suppose that $\varphi^{(1)}(r)/(\varphi(r))^2 > (R-r)^{-1}$ for a subset E of $[0, R)$. Then

$$\int_{E \cap [1,R)} \frac{dr}{R-r} \leq \int_{E \cap [1,R)} \frac{\varphi^{(1)}(r)}{(\varphi(r))^2} dr$$
$$\leq [-(\varphi(r)^{-1})]_1^R \leq \varphi(1)^{-1} . \qquad \text{Q.E.D.}$$

Let V, Σ, and f be as in Theorem A, and let Σ_i denote the irreducible components of Σ. For an element ω of $H^0(V, \mathfrak{A}_V(\log \Sigma))$ we set $f^*\omega = \zeta(z)\, dz$. Then $\xi(z)$ is a meromorphic function with poles of order one and their residues are integers. Set

$$G(z) = \int_0^n f^*\omega \quad (\text{mod. } 2\pi i) , \quad g(z) = \exp(G(z)) .$$

By the same arguments as in [9: lemma 2.2] we get the following

Lemma 2.2. *There exists constants $K(>0)$, A and B such that*

$$T(r, g) \leq K \left\{\left(\frac{1}{2\pi}\right)^{1/2} (r \frac{d}{dr} T_f(r) + A)^{1/2} + T_f(r)\right\} + B .$$

Next we shall prove

Main lemma 2.3. *Let f, ω and ζ be as above. Then*

$$m(r, \zeta) \leq O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \; || \, .$$

Furthermore, if f is a map of finite order, then ζ is of finite order and

$$m(r, \zeta) \leq O(\log \frac{1}{R-r}) + O(1) \, .$$

Proof. Applying Lemma 2.2 and the classical Nevanlinna's lemma on the logarithmic derivative to $\zeta(z)$, we have

$$\begin{aligned} m(r, \zeta) &= m(r, g^{(1)}/g) \\ &\leq O(\log^+ T(r, g)) + O(\log \frac{1}{R-r}) + O(1) \; || \\ &\leq O(\log^+ (K \{(2\pi)^{-1/2} (r \frac{d}{dr} T_f(r) + A)^{1/2} + T_f(r)\} + B)) \\ &\quad + O(\log \frac{1}{R-r}) + O(1) \; || \, . \end{aligned} \tag{2.3.1}$$

It follows from Lemma 2.1 and (2.3.1) that

$$m(r, \zeta) \leq O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \; || \, .$$

This is the first part of the lemma.

By Lemma 2.1 the inequality

$$\frac{d}{dr} T_f(r) \leq (T_f(r))^2 (R-r)^{-1}$$

holds except for a disjoint union $E = \cup_j I_j$ of intervals $I_j = (a_j, b_j)$ such that $\int_E (R-r)^{-1} \, dr = M < \infty$. We define r' by

$$\begin{aligned} r' &= r \quad \text{if} \quad r \notin E \, , \\ r' &= b_j \quad \text{if} \quad r \in I_j = (a_j, b_j) \, . \end{aligned}$$

Then

$$\begin{aligned} r \left(\frac{d}{dr} T_f\right)(r) &\leq r' \left(\frac{d}{dr} T_f\right)(r') \leq R (T_f(r'))^2 (R-r')^{-1} \\ &\leq R (R-r')^{-2\mu-2} \leq R \left(\frac{R-r}{R-r'}\right)^{2\mu+2} \left(\frac{1}{R-r}\right)^{2\mu+2} , \end{aligned}$$

where μ is the order of f. On the other hand,

$$\int_r^{r'} \frac{dt}{R-t} \leq \int_E \frac{dt}{R-t} = M < \infty \, ,$$

and hence

$$\frac{R-r}{R-r'} \leq e^M .$$

Therefore we obtain

$$r\left(\frac{d}{dr} T_f\right)(r) \leq R\, e^{(2\mu+2)M} \left(\frac{1}{R-r}\right)^{2\mu+2} .$$

Since f is of finite order, it follows from Lemma 2.2 that g is also of finite order. Hence by the classical Nevanlinna's lemma on logarithmic derivative, we get

$$m(r, \zeta) = m(r, g^{(1)}/g) \leq O(\log \frac{1}{R-r}) + O(1) .$$

Moreover

$$N(r, \zeta) \leq \bar{N}_f(r, \Sigma) \leq N_f(r, \Sigma) = O(T_f(r)) .$$

It follows that $N(r, \zeta)$ is of finite order. Therefore ζ is of finite order. Q.E.D.

For the sake of convenience we use the following notation:

$$S_f(r) = O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \ ||$$

if f is of infinite order, and

$$S_f(r) = O(\log \frac{1}{R-r}) + O(1)$$

if f is of finite order.

Corollary 2.4. *Let ζ and ζ bs as above. Then*

$$T(r, \zeta) \leq \bar{N}_f(r, D) + S_f(r) ,$$

where $D = \Sigma\ |\mathrm{res}(\omega, \Sigma_i)|\, \Sigma_i$.

Proof. Sinse ζ has a pole at z only if $f(z)$ belongs to supp D and every pole of ζ has order one, we have

$$N(r, \zeta) \leq \bar{N}_f(r, D) .$$

Hence our assertion follows from Main lemma 2.3. Q.E.D.

Corollary 2.5. *Let ω be an element of $H^0(V, \Omega_V^1(\log \Sigma))$ and put $f^*\omega = \zeta(z)\, dz$. Then*

$$T(r, \zeta) \leq \bar{N}_f(r, \Sigma) + S_f(r) .$$

Proof. By the proposition of Section 1 (7) there exist

$$\omega_j \in H^0(V, \mathfrak{A}_V(\log \Sigma))$$

and $c_j \in \boldsymbol{C}$ $(j=1, \cdots, q)$ such that

$$\omega = c_1 \omega_1 + c_2 \omega_2 + \cdots + c_q \omega_q .$$

We set $f^*\omega_j = \zeta_j(z)\, dz$. Then

$$\zeta = c_1 \zeta_1 + c_2 \zeta_2 + \cdots + c_q \zeta_q .$$

By Main lemma 2.3 we obtain

$$m(r, \zeta) \leq \sum_{i=1}^{q} m(r, \zeta_j) + O(1) = S_f(r) .$$

Since any pole of ζ is of order one and ζ has a pole at z only if $f(z)$ belongs to supp Σ, We have

$$N(r, \zeta) \leq \bar{N}_f(r, \Sigma) .$$

Hence we ha have

$$T(r, \zeta) \leq \bar{M}_f(r, \Sigma) + S_f(r) . \qquad \text{Q.E.D.}$$

REMARK. Using the same idea as in Noguchi [11: p. 224, Remark (1)], we obtain from Corollary 2.5 the classical Nevanlinna's second main theorem for meromorphic functions on $\Delta(R)$.

3. Proof of Thorem A.

We keep the same notation as in Theorem A. Let ζ_i be the meromorphic function on $\Delta(R)$ defined by $f^*\omega_i = \zeta_i\, dz$. Applying the classical Nevanlinna's lemma on logarithmic derivative to k-th derivative of ζ_i we have

$$\begin{aligned} T(r, \zeta_i^{(k)}) \leq (k+1)\, T(r, \zeta_i) + O(\log^+ T(r, \zeta_i)) \\ + O(\log \frac{1}{R-r}) + O(1) \,\| . \end{aligned} \tag{3.1}$$

Moreover, combining Corollary 2.5 with the first main theorem and (1.1) we have

$$\begin{aligned} \log^+ T(r, \zeta_i) &\leq \log^+ \bar{N}_f(r, \Sigma) + \log^+ S_f(r) + O(1) \\ &\leq \log^+ T_f(r, [\Sigma]) + S_f(r) + O(1) \\ &\leq O(\log^+ T_f(r)) + S_f(r) + O(1) \\ &= O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) . \end{aligned}$$

This inequality and (3.1) imply

$$T(r, \zeta_i^{(k)}) \leq (k+1)\, T(r, \zeta_i) + O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \,||\,.$$

Hence by Coroallary 2.5

$$T(r, \zeta_i^{(k)}) \leq (k+1)\, \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \,||\,. \tag{3.2}$$

Let $\{\phi_j\}_{j=1,\cdots,l}$ be a system of generators of the rational function field $\mathfrak{R}(V)$ over $\boldsymbol{C}$ such that $f^*\phi_j$ are defined. Then by the Ochiai's theorrm (§1 (8)) there are algebraic relations

$$(f^*\phi_j)^{m_j} + R_{j1}(\zeta_i^{(k)})\,(f^*\phi_j)^{m_j-1} + \cdots + R_{jm_j}(\zeta_i^{(k)}) = 0 \tag{3.3}$$

$j=1, \cdots, l$ where $R_{j\nu}(\zeta_i^{(p)})$ are rational functions of $\zeta_i^{(k)}$ $k=0, \cdots, n-1, i=1, \cdots, n+1$. By making use of [10] we see that there is a positive constant K independent of r and f such that

$$T(r, f^*\phi_j) \leq K\, \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \,||$$

for all j. Thus we have

$$\begin{aligned} \tilde{T}_f(r) &= \max_j T(r, f^*\phi_j) \\ &\leq K\, \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) \\ &\quad + O(\log \frac{1}{R-r}) + O(1) \,||\,. \end{aligned} \tag{3.4}$$

Inequalities (1.2) and (3.4) yield the inequality (I). This completes the first part of Theorem A. By the assumption that f is of finite order, and by Lemma 2.3 we see that ζ_i is of finite order. Then by the classical Nevanlinna's lemma on logarithmic derivative, we have

$$T(r, \zeta_i^{(k)}) \leq (k+1)\, T(r, \zeta_i) + O(\log \frac{1}{R-r}) + O(1)\,.$$

And by the same arguments as in the first part of Theorem A we obtain

$$\tilde{T}_f(r) \leq K\, \bar{N}_f(r, \Sigma) + O(\log \frac{1}{R-r}) + O(1)\,. \tag{3.5}$$

Thus inequalities (1.2) and (3.5) yield the inequality (II). Q.E.D.

4. Proof of Theorem B and Corollary C

Proof of Theorem B. Without loss of generality we may assume $T_f(r) \to \infty$

as $r \to R$. We shall prove that f is of finite order. By the assumption, we have

$$N_f(r, \Sigma) = O\left(\left(\frac{1}{R-r}\right)^{\mu+1}\right),$$

and hence by Theorem A we get

$$\kappa T_f(r) \leq L_1 \left(\frac{1}{R-r}\right)^{\mu+1} + M_1 \log^+ T_f(r)$$
$$+ M_2 \log \frac{1}{R-r} + M_3 \,||\,,$$

where L_1, M_1, M_2 and M_3 are constants. On the other hand, there exists a constant δ (>0) such that

$$\kappa x - M_1 \log^+ x > \delta x$$

for sufficiently large x. Therefore

$$\delta\, T_f(r) \leq L_1 \left(\frac{1}{R-r}\right)^{\mu+1} + M_2 \log \frac{1}{R-r} + M_3$$

except the countable disjoint union of open intervals $E = \bigcup_j I_j$ such that

$$\int_E \frac{dr}{R-r} = M_4 < \infty\,.$$

We define r' as follow

$$r' = r \quad \text{if} \quad r \notin E$$
$$r' = b_j \quad \text{if} \quad r \in I_j = (a_j, b_j)\,.$$

Then we see that

$$\delta\, T_f(r) \leq \delta\, T_f(r') \leq L_1 \left(\frac{1}{R-r'}\right)^{\mu+1} + M_1 \log \frac{1}{R-r'} + M_3\,.$$

Moreover, since

$$M_4 \geq \int_r^{r'} \frac{dt}{R-t} = \log \frac{1}{R-r'} - \log \frac{1}{R-r}\,,$$

we obtain

$$\frac{R-r}{R-r'} \leq \exp(M_4)\,.$$

Therefore we get

$$\delta\, T_f(r) \leq L_1 \exp((\mu+1) M_4) \left(\frac{1}{R-r}\right)^{\mu+1} + M_2 \log \frac{1}{R-r} + M_6\,,$$

where M_5 is a constant. Hence f is a map of finite order. Therefore it follows from Theorem A that

$$\kappa\, T_f(r)\leq \bar{N}_f(r, \Sigma)+\bar{M}_1 \log\frac{1}{R-r}+\bar{M}_2\,,$$

where $\bar{M}_1$, $\bar{M}_2$ are constants. Therefore

$$\kappa\int_1^R T_f(t)(R-t)^{\mu-1}\,dt\leq\int_1^R \bar{N}_f(t, \Sigma)\,(R-t)^{\mu-1}\,dt$$
$$+\bar{M}_1\int_1^R \log\frac{1}{R-t}\,(R-t)^{\mu-1}\,dt+\bar{M}_2\int_1^R (R-t)^{\mu-1}\,dt\,.$$

On the other, hand by the assumption

$$\int_1^R \bar{N}_f(t, \Sigma)\,(R-t)^{\mu-1}\,dt<\infty\,,$$

we obtain

$$\int_1^R T_f(t)\,(R-t)^{\mu-1}\,dt<\infty\,. \qquad \text{Q.E.D.}$$

Proof of Corollary C. This is an immediate consequence of Thoerem B and the following

Lemma 4.1. *Let μ be a positive number and D an effective divisor on V which satisfies $f(\Delta(R))\not\subset \operatorname{supp} D$, Then for* $\operatorname{supp} f^*D=\{a_1, a_2, a_3, \cdots\}$,

$$\int_1^r \bar{N}_f(t, D)\,(R-t)^{\mu-1}\,dt$$
$$\int_1^r \bar{n}_f(t, D)\,(R-t)^{\mu}\,dt\,, \quad \text{and}$$
$$\sum_{1\leq|a_i|\leq r}(R-|a_i|)^{\mu+1}$$

are convergent or divergent at the same time as $r\to R$.

Proof. The same argument as in Shimizu [17] (p.107) can be applied for this case.

REMARK If $g(z)=\exp\dfrac{1}{1-z}$ then

$$\varlimsup_{r\to1}\log T(r, g)/\log\frac{1}{1-r}=0\,,$$

and

$$\varlimsup_{r\to1}\log\log M(r, g)/\log\frac{1}{1-r}=1\,,$$

where $M(r, g)=\underset{|z|=r}{\text{Max}}\ |g(z)|$.

5. Holomorphic maps of bounded characteristic functions

In this section we study holomorphic maps from the unit disk to an algebraic variety with $T_f(r)=O(1)$. Here we shall prove some analogous result to the classical Fatou's and Blaschke's theorems concerning bounded holomorphic functions on the unit disk.

Let f be a holomorphic Map from $\Delta(R)$ to a nonsinglar algebraic variety V and Ω a Kähler form on V.

Proposition 5.1. *Suppose that* $r\left(\frac{d}{dr}T_f\right)(r)=O(1)$ *as* $r\rightarrow R$. *Then the length of a curve of the image of* $\{te^{i\theta};\ 0<t<R\}$ *by* f *with respect to the Hermitian metric* h *determined by* Ω *is finite for almost all* θ, *i.e.*

$$\lim_{r\to R}\int_0^r (s(te^{i\theta}))^{1/2}\,dt \quad (s(z)\,dzd\bar{z}=f^*h)$$

is finite for almost all θ.

Proof. By Schwarz's inequality

$$\int_0^{2\pi}\Big(\int_0^r (s(te^{i\theta}))^{1/2}\,dt\Big)^2 d\theta\le\int_0^{2\pi}\Big(r\int_0^r s(te^{i\theta})\,dt\Big)\,d\theta\le R\int_0^{2\pi}\int_0^r s(te^{i\theta})\,dtd\theta$$
$$\le R\int_0^{2\pi}\int_0^r s(te^{i\theta})\,tdtd\theta+R\int_0^{2\pi}\int_0^1 s(te^{i\theta})\,dtd\theta$$
$$=r\Big(\frac{d}{dr}T_f\Big)(r)\cdot R+B\cdot R$$
$$\Big(B=\int_1^{2\pi}\int_0^1 s(te^{i\theta})\,dtd\theta\Big).$$

Hence we get using the assumption

$$\lim_{r\to R}\int_0^{2\pi}\Big(\int_1^r (s(te^{i\theta}))^{1/2}\,dt\Big)^2 d\theta<\infty\,.$$

Therefore we see

$$\int_0^{2\pi}\Big(\lim_{r\to R}\int_0^r (s(te^{i\theta}))^{1/2}\,dt\Big)^2 d\theta<\infty\,.$$

Thus

$$\lim_{r\to R}\int_0^r (s(te^{i\theta}))^{1/2}\,dt$$

are finite for almost all θ. Q.E.D.

Before showing a Blaschke-type theorem we need a lemma (cf.,e.g., Shimizu

[17] p.107).

Lemma 5.2. *Let D betan effective divisor on V with* $f(\Delta(R))\not\subset \operatorname{supp} D$. *Then for* $f^*D=a_1+a_2+\cdots$

$$N_f(r, D), \int_1^r n_f(t, D)\,dt\,, \quad \text{and} \quad \sum_{1\leq |a_i| \leq r}(R-|a_i|)$$

are convergent or divergent at the same time as $r\to R$.

Theorem 5.3. *Suppose that* $T_f(r)=O(1)$ *as* $r\to R$ *Then for any effective divisor D on V which satisfies* $f(\Delta(R))\not\subset \operatorname{supp} D$ *we have*

(5.3.1) $$N_f(r, D) = O(1) \quad (r\to R)$$

(5.3.2) $$n_f(r, D) = o\left(\frac{1}{R-r}\right)(r\to R)$$

(5.3.3) $$\sum_{|a_i|<R}(R-|a_i|)<\infty\,,$$

where $f^*D=a_1+a_2+\cdots$.

Proof. By the first main theorem and (1.1) we obtain (5.3.1). and from Lemma 5.2, we have

$$\int_1^R n_f(t, D)\,dt<\infty\,.$$

Since $n_f(t, D)$ is non-decreasing (5.3.2) holds. Moreover by Lemma 5.2 we obtain (5.3.3). Q.E.D.

Finally we shall given an example related to the above results. W. Rudin [15] gave an example of a holomorphic map from $\Delta(1)$ to $\Delta(1)$ with

$$\int_0^1 |f^{(1)}(re^{i\theta})|\,dr = \infty \qquad \text{for almost all } \theta\,.$$

this map is an example such that

$$\int_1^1 \frac{|f^{(1)}(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\,dr = \infty$$

for almost all θ. This map has the property $r\left(\frac{d}{dr}T_f\right)(r)\to\infty$ as $r\to 1$. But clearly $T_f(r)=O(1)$ as $r\to 1$.

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