Nagamachi, S. and Nishimura, T.
Osaka J. Math.
28 (1991), 579-607

NONSTANDARD ANALYSIS OF LINEAR CANONICAL TRANSFORMATIONS ON A FERMION FOCK SPACE WITH AN INDEFINITE METRIC

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(Received June 10, 1989)

0. Introduction

It is well known that an indefinite metric Hilbert space is necessary in order to describe the quantum electromagnetic field (See Strocchi, F. and A.S. Wightman [11]). Ito, K.R. [5] investigated two dimensional quantum electrodynamics in the indefinite metric formulation, where the theory of linear canonical transformation on Boson Fock space with an indefinite metric was used which was developed in Ito, K.R. [4]. On the other hand, indefinite metric Hilbert space are not necessary for Dirac field in usual formulation. But in the Euclidean formulation sometimes appears an indefinite metric Hilbert space. Nagamachi, S. and N. Mugibayashi [7] studied the Euclidean formulation of Dirac field and its Euclidean covariance. There appeared a Fermion Fock space with an indefinite metric and canonical transformations which represent Euclidean transformations of field operators. Fortunately, since these canonical transformations do not mix creation and annihilation operators and moreover the operators Φ, Ψ which determine the canonical transformation commute with the operator η giving the indefinite metric in the form $[x, y]=(x, \eta y)$, they are implementable by bounded operators which are isometric with respect to the indefinite metric, which we call Λ-unitary operators (Remark 7.10). In generalizing the theory of Clifford group of Sato, M., T. Miwa and M. Jimbo [17] to an infinite dimensional case, Palmer, J. [8] found the condition under which an automorphism of Clifford algebra is implementable by some operator in the Fock space. Similar results were obtained by Araki, H. [1]. Their results have an intimate connection with ours but do not concern the implementability by an isometry operator with respect to the indefinite metric inner product which we call a Λ-unitary operator.

In this paper, we extensively use nonstandard analysis and Berezin calculus to investigate the linear canonical transformations in an infinite dimansional Fermion Fock space with an indefinite metric, especially their implementability by a Λ-unitary operator. In the same time we want to show how the Berezin calculus on a finite dimensional superspace can be applied to analysis on an infinite dimensional Fermion Fock space by using nonstandard analysis. In the indefinite metric case, even if the standard part of a nonstandard Λ-unitary operator exists, it is not necessarily a bounded operator (Example 7.11), and so we introduce the notion of weakly Λ-unitary operators (Definition 1.1). Then we give a sufficient condition for the linear canonical transformation to be weakly Λ-unitarily implementable.

This paper is organized as follows. In §1, we define the Fermion Fock space $\mathcal{F}(\mathcal{H})$ over a Hilbert space $\mathcal{H}$ and introduce an indefinite metric on $\mathcal{H}$ and notion of linear canonical transformations with respect to this metric. In §2, we summarize the differential and integral calculus of functions with finite number of Grassmann variables which was developed in Berezin, F.A. [2], Rogers, A. [9] and Kobayashi, Y. and S. Nagamachi [6], and called the Berezin calculus. In §3, we explain some important notions of nonstandard analysis and investigate the relation between $*$-finite dimensional Grassmann algebra $\mathcal{Q}(F)$ and the Fock space $\mathcal{F}(\mathcal{H})$ over $\mathcal{H}$ (Theorem 3.6) by introducing a new concept 'almost standard', where F is a $*$-finite dimensional subspace of the nonstandard extension $^*\mathcal{H}$ of $\mathcal{H}$ containing $\mathcal{H}$ ($\mathcal{H}\subset F\subset {}^*\mathcal{H}$). In §4, we use nonstandard analysis and the Berezin calculus to define for a canonical transformation an operator U on $\mathcal{Q}(F)$ which is Λ-isometric on a certain subspace of $\mathcal{Q}(F)$ (Proposition 4.2). In §5, we prove that, under some condition on the canonical transformation, the operator U has the standard part (Proposition 5.6), where the concept 'almost standard' introduced in §3 plays a crucial role. In §6, we prove that the operator U implements the canonical transformation (Proposition 6.2). The arguments in §§4–6 depend on an orthonormal basis $\{e_i\}$, since we must fix an orthonormal basis to apply Berezin calculus. As a result, the operator U depends on $\{e_i\}$. In §7, we define a weakly Λ-unitary operator U_1 on the Fock space $\mathcal{F}(\mathcal{H})$ which does not depend on $\{e_i\}$ using the operator U on $\mathcal{Q}(F)$. Thus we obtain the main theorem (Theorem 7.7), which states that under certain conditions the linear canonical transformation is weakly Λ-unitarily implementable.

1. Linear Canonical Transformations

In this section we introduce the notion of linear canonical transformations of annihilation and creation operators on a Fermion Fock space with an indefinite metric, and give a definition of its Λ-unitary implementability, the investigation of which is a main thema of this paper.

Let $\mathcal{H}$ be a Hilbert space over the complex number field $\boldsymbol{C}$ with an inner product (x, y) which is linear in x and conjugate linear in y. We assume that $\mathcal{H}$ has an involution $*$, i.e., a mapping $*: \mathcal{H}\to\mathcal{H}$ satisfying $x^{**}=x$, $(x+y)^*=x^*+y^*$, $(\alpha x)^*=\bar{\alpha}x^*(x, y\in\mathcal{H}, \alpha\in\boldsymbol{C})$, and that the involution satisfies

$$(x^*, y^*) = \overline{(x, y)} . \tag{1.1}$$

A Hilbert space which has an involution satisfying (1.1) is called a *Hilbert space with an involution.* We introduce a symmetric bilinear form $\langle x, y\rangle$ by

$$\langle x, y\rangle = (x, y^*) .$$

For a bounded operator A on $\mathcal{H}$ we denote its adjoint operator by $A^\dagger$ and define its *complex conjugate operator* $\bar{A}$ and *transposed operator* A' by

$$\bar{A}x = (Ax^*)^*$$
$$A'x = \overline{A^\dagger}x .$$

Then we have

$$\overline{\langle Ax, y\rangle} = \langle \bar{A}x^*, y^*\rangle$$
$$\langle Ax, y\rangle = \langle x, A'y\rangle$$

Let $\mathcal{G}(\mathcal{H})=\oplus_{m=0}^{\infty} \mathcal{G}(\mathcal{H})_m$ be the Grassmann algebra over $\mathcal{H}$, where $\mathcal{G}(\mathcal{H})_0 = \boldsymbol{C}$ and the element of $\mathcal{F}(\mathcal{H})_m$ is a finite linear combination of $g_1\wedge\cdots\wedge g_m$ with $g_i\in\mathcal{H}$. We introduce an inner product on $\mathcal{G}(\mathcal{H})$ y by setting

$$(g, h) = \begin{cases} \det[(g_i, h_j)] & \text{for } g = g_1\wedge\cdots\wedge g_m, h = h_1\wedge\cdots\wedge h_m\in\mathcal{G}(\mathcal{H})_m \\ g\bar{h} & \text{for } g, h\in\mathcal{G}(\mathcal{H})_0 \\ 0 & \text{for } g\in\mathcal{G}(\mathcal{H})_m, h\in\mathcal{G}(\mathcal{H})_n \text{ with } m\neq n \end{cases} \tag{1.2}$$

The completion of $\mathcal{G}(\mathcal{H})$ with respect to this inner product is called the *Fermion Fock space* over $\mathcal{H}$ and is denoted by $\mathcal{F}(\mathcal{H})$.

We can extend the involution * on $\mathcal{H}$ so that it becomes a continuous involution on $\mathcal{F}(\mathcal{H})$ satisfying

$$(g_1\wedge\cdots\wedge g_n)^* = g_n^*\wedge\cdots\wedge g_1^* .$$

By this involution thus extended (which we also denote by the same symbol *), $\mathcal{F}(\mathcal{H})$ becomes a Hilbert space with an involution, and as in the case of $\mathcal{H}$ every bounded linear operator on $\mathcal{F}(\mathcal{H})$ has its complex conjugate operator and transposed operator.

We introduce an *indefinite inner product* on $\mathcal{H}$. Let η be a real, Hermitian and unitary operator on $\mathcal{H}$ i.e., η satisfies

$$\bar{\eta} = \eta^\dagger = \eta^{-1} = \eta .$$

We define an indefinite inner product $[x, y]$ on $\mathcal{H}$ by

$$[x, y] = (\eta x, y)$$

for $x, y\in\mathcal{H}$. The operator η also naturally induces a real, hermitian and unitary operator Λ on $\mathcal{F}(\mathcal{H})$ defined by

$$\Lambda g = (\eta g_1)_\wedge \cdots {}_\wedge(\eta g_n), \quad \Lambda g_0 = g_0 \tag{1.3}$$

for $g=g_{1\wedge}\cdots{}_\wedge g_n$ and $g_0 \in C = \mathcal{G}(\mathcal{H})_0$. To be exact, by (1.3) and linearity, Λ is defined as an isometric operator on $\mathcal{G}(\mathcal{H})$, and Λ can be extended continuously to all the elements of $\mathcal{F}(\mathcal{H})$. Λ is also real, Hermitian and unitary, i.e.,

$$\bar{\Lambda} = \Lambda^\dagger = \Lambda^{-1} = \Lambda .$$

Now we define an indefinite inner product $[\cdot, \cdot]$ on the Fermion Fock space $\mathcal{F}(\mathcal{H})$ by

$$[g, h] = (\Lambda g, h)$$

for $g, h \in \mathcal{F}(\mathcal{H})$.

The creation operator $a^\dagger(f)$ and the annihilation operator $a(f)$ for $f \in \mathcal{H}$ are defined first on $\mathcal{G}(\mathcal{H})$ by

$$a^\dagger(f)\, g = f_\wedge g_{1\wedge} \cdots {}_\wedge g_n$$
$$a(f)\, g = \sum_{j=1}^{n} (-1)^{j-1} \langle f, g_j \rangle g_{1\wedge} \cdots {}_\wedge \hat{g}_{j\wedge} \cdots {}_\wedge g_n$$

for $g=g_{1\wedge}\cdots{}_\wedge g_n$ $(g_i \in \mathcal{H})$, where the circumflex "$\wedge$" means that the symbol beneath it is to be omitted. Then, as these operators are bounded (see Berezin [2], p. 13), they are extended to the whole space $\mathcal{F}(\mathcal{H})$ and satisfy the relation

$$(a(f)\, g, h) = (g, a^\dagger(f^*)\, h)$$

for $g, h \in \mathcal{F}(\mathcal{H})$. We define the operator $a^{(\Lambda)}(f)$ by

$$a^{(\Lambda)}(f) = \Lambda a^\dagger(f)\, \Lambda \quad (= a^\dagger(\eta f))$$

for $f \in \mathcal{H}$. Then we have

$$[a(f)\, g, h] = [g, a^{(\Lambda)}(f^*)\, h]$$

for $g, h \in \mathcal{F}(\mathcal{H})$. In other words, $a^{(\Lambda)}(f^*)$ is adjoint to $a(f)$ with respect to the inner product $[\cdot, \cdot]$.

From the definitions of $a(\cdot)$ and $a^\dagger(\cdot)$ follow the canonical anti-commutation relations of these operators:

$$\begin{aligned} \{a(f), a^\dagger(g)\} &= \langle f, g \rangle \\ \{a(f), a(g)\} &= \{a^\dagger(f), a^\dagger(g)\} = 0 \end{aligned} \tag{1.4}$$

for $f, g \in \mathcal{H}$, where we used the notation $\{A, B\} \equiv AB + BA$. For $a(\cdot)$ and $a^{(\Lambda)}(\cdot)$, the corresponding anti-commutation relations are

$$\begin{aligned}\{a(f), a^{(\Lambda)}(g)\} &= \langle f, \eta g\rangle \\ \{a(f), a(g)\} &= \{a^{(\Lambda)}(f), a^{(\Lambda)}(g)\} = 0\end{aligned} \tag{1.5}$$

for $f, g \in \mathcal{H}$. These follow immediately from (1.4).

Let Φ and Ψ be bounded operators on $\mathcal{H}$. Using Φ and Ψ, we transform $a(\cdot)$ and $a^{(\Lambda)}(\cdot)$ into another pair $b(\cdot)$ and $b^{(\Lambda)}(\cdot)$ by

$$\begin{aligned}b(f) &= a(\Phi' f) + a^{(\Lambda)}(\Psi' f) \\ b^{(\Lambda)}(f) &= a(\Psi^\dagger f) + a^{(\Lambda)}(\Phi^\dagger f)\end{aligned} \tag{1.6}$$

for $f \in \mathcal{H}$. The transformation (1.6) is called a *linear canonical transformation* if it satisfies 1) the relations:

$$\begin{aligned}\{b(f), b^{(\Lambda)}(g)\} &= \langle f, \eta g\rangle \\ \{b(f), b(g)\} &= \{b^{(\Lambda)}(f), b^{(\Lambda)}(g)\} = 0\end{aligned} \tag{1.7}$$

for $f, g \in \mathcal{H}$, and 2) the invertibility condition: By other operators Φ_1 and Ψ_1, the pair $b(\cdot)$, $b^{(\Lambda)}(\cdot)$ is transformed into the original pair $a(\cdot)$, $a^{(\Lambda)}(\cdot)$ through the formula (1.6). It is easy to see from (1.5) that (1.7) is equivalent to the relation

$$\begin{aligned}\Phi\eta\Phi^\dagger + \Psi\eta\Psi^\dagger &= \eta \\ \Phi\eta\Psi' + \Psi\eta\Phi' &= 0\,.\end{aligned} \tag{1.8}$$

In terms of the matrices

$$\mathcal{A} = \begin{bmatrix}\Phi & \Psi \\ \bar{\Psi} & \bar{\Phi}\end{bmatrix}, \quad \mathcal{A}' = \begin{bmatrix}\Phi' & \Psi^\dagger \\ \Psi' & \Phi^\dagger\end{bmatrix}$$

the invertibility condition is equivalent to the invertibility of $\mathcal{A}$, and (1.8) is written as follows:

$$\mathcal{A}E\mathcal{A}' = E\,, \tag{1.9}$$

where

$$E = \begin{bmatrix}0 & \eta \\ \eta & 0\end{bmatrix}.$$

So, the formula (1.6) is a linear canonical transformation if and only if $\mathcal{A}$ has an inverse and (1.9) holds. From now on we assume that (1.6) is an arbitrary but fixed linear canonical transformation and discuss its properties.

Since the matrix $\mathcal{A}$ satisfies (1.9) we have

$$\mathcal{A}E\mathcal{A}'\,E = 1 \text{ (identity)}\,. \tag{1.10}$$

Then, as $\mathcal{A}$ has an inverse (1.10) implies $\mathcal{A}'E\mathcal{A} = E$, from which we have the relations:

$$\begin{aligned} \Phi^\dagger\eta\Phi+\Psi'\eta\overline{\Psi} &= \eta \\ \Phi^\dagger\eta\Psi+\Psi'\eta\overline{\Phi} &= 0\,. \end{aligned} \tag{1.11}$$

We introduce the following notions.

DEFINITION 1.1. A linear transformation U from $\mathcal{G}(\mathcal{H})$ to $\mathcal{F}(\mathcal{H})$ is called a weakly Λ-unitary operator if $[Ug, Uh]=[g, h]$ for any $g, h\in\mathcal{G}(\mathcal{H})$. If, in addition, U is a bounded operator, we call it a Λ-unitary operator.

DEFINITION 1.2. The canonical transformation (1.6) is said to be (resp. weakly) Λ-unitarily implementable if there exists a (resp. weakly) Λ-unitary operator U such that

$$\begin{aligned} b(f)\,Uh &= Ua(f)\,h \\ b^{(\Lambda)}(f)\,Uh &= Ua^{(\Lambda)}(f)\,h \end{aligned}$$

for $f\in\mathcal{H}$, $h\in\mathcal{G}(\mathcal{H})$.

2. Calculus on functions of Grassmann variables

In this section we summarize the differential and integral calculus of functions with finite number of Grassmann variables which is developed by Berezin, F.A. [2] and Rogers, A. [9] (see also Kobayashi & Nagamachi [6] for complex superspaces). Let B_L be the Grassmann algebra over the complex number field $\boldsymbol{C}$ with sufficiently large number L of generators. Then B_L is the direct sum of the even part $B_{L,0}$ and the odd part $B_{L,1}$. Assume that B_L has an involution $^-$ satisfying $\bar{a}\in B_{L,\alpha}$ for any $a\in B_{L,\alpha}(\alpha=0, 1)$. Let $X_n=(B_{L,1})^n$ be an n-dimensional complex odd superspace. The involution $^-$ is extended to the superspace X_n by $\bar{z}=(\bar{z}_1, \cdots, \bar{z}_n)$ for $z=(z_1, \cdots, z_n)\in X_n$. Let $H^\infty(X_n)$ be the set of smooth functions f of $z\in X_n$ having the form

$$f(z, \bar{z}) = \sum_{s,t=0}^{n} \sum f_{i_1,\cdots,i_s\,;\,j_1,\cdots,j_t}\, \bar{z}_{i_1}\cdots\bar{z}_{i_s}\, z_{j_1}\cdots z_{j_t}$$

with $f_{i_1,\cdots,i_s\,;\,j_1,\cdots,j_t}\in\boldsymbol{C}$. For $f\in H^\infty(X_n)$ above the involution $^-$ is defined by

$$\overline{f(z, \bar{z})} = \sum_{s,t=0}^{n} \sum \overline{f_{i_1,\cdots,i_s\,;\,j_1,\cdots,j_t}}\, \bar{z}_{j_t}\cdots\bar{z}_{j_1}\, z_{i_s}\cdots z_{j_1}\,,$$

where $\overline{f_{i_1\cdots,i_s\,;\,j_1,\cdots,j_t}}$ is the complex conjugate of $f_{i_1,\cdots,i_s\,;\,j_1,\cdots,j_t}$.

In order to define integrals we introduce the symbols dz_i, $d\bar{z}_i$ which anti-commute with each other and anti-commute with variables z_i, $\bar{z}_i$. We define integrals by

$$\begin{aligned} \int z_i\, dz_i &= \int \bar{z}_i\, d\bar{z}_i = 1 \\ \int dz_i &= \int d\bar{z}_i = 0 \end{aligned}$$

(see Berezin [2] p. 59). Then the following formula is well known:

$$\int \exp\{-\frac{1}{2}(z,\bar{z})\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}z\\ \bar{z}\end{bmatrix}+(w,\bar{w})\begin{bmatrix}z\\ \bar{z}\end{bmatrix}\}\prod_{j=1}^{n} dz_i\, d\bar{z}_i$$
$$(2.1)\qquad = (\det\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix})^{1/2}\exp\{-\frac{1}{2}(w,\bar{w})\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}^{-1}\begin{bmatrix}w\\ \bar{w}\end{bmatrix}\}$$

for $w\in X_n$, where A_{ij} $i,j=1,2$ are $n\times n$ matrices satisfying

$$A_{ij} = -A'_{ji}\ i,j=1,2$$

and A' denotes the transposed matrix of A.

Since expressions like those in the exponent of (2.1) often appear in this paper, we explain their meanings here. For example, the exponent in the l.h.s. of (2.1) should be understood as

$$-\frac{1}{2}(zA_{11}z+zA_{12}\bar{z}+\bar{z}A_{21}z+\bar{z}A_{22}\bar{z}+wz+\bar{w}\bar{z}).$$

Here, wz stands for $\sum_{i=1}^{n} w_i z_i$, and for an $n\times n$ matrix $A_{11}=(a_{ij})$, we used the notation

$$zA_{11}\bar{z} = \sum_{i,j=1}^{n} z_i a_{ij}\bar{z}_j .$$

If we consider z and $\bar{z}$ as column vectors or $n\times 1$ matrices defined by

$$z = \begin{bmatrix}z_1\\ \vdots\\ z_n\end{bmatrix},\quad \bar{z} = \begin{bmatrix}\bar{z}_1\\ \vdots\\ \bar{z}_n\end{bmatrix}$$

then $zA_{11}\bar{z}$ should be written, using the transposed matrix z^T of z, as $z^T A_{11}\bar{z}$. Also zw should be $z^T w$. But for simplicity we employ the above notation like $zA_{11}\bar{z}$ or zw following the convention in Berezin's book [2]. Notations like $\bar{z}A_{21}z$, $\cdots$, $\bar{w}\bar{z}$, ... should be understood similarly.

Let $h(\bar{z})=\bar{z}_{i_1}\cdots\bar{z}_{i_k}$, $g(\bar{z})=\bar{z}_{j_1}\cdots\bar{z}_{j_l}$ for $i_1<\cdots<j_k$, $j_1<\cdots<j_l$. Then it follows from the definition of integrals that

$$(2.2)\qquad \int h(\bar{z})\overline{g(\bar{z})}\, e^{-z\bar{z}}\prod_{i=1}^{n} dz_i\, d\bar{z}_i = \delta_{kl}\prod_{s=0}^{k}\delta_{i_s j_s}.$$

Let F be an n-dimensional Hilbert space over $\boldsymbol{C}$ and $\mathcal{G}(F)$ be the Grassmann algebra over F. Let $\{f_1,\cdots,f_n\}$ be an orthonormal basis of F. Then $\mathcal{G}(F)$ is a 2^n-dimensional vector space over $\boldsymbol{C}$ with a basis

$$1,\ f_{j_1}\wedge\cdots\wedge f_{j_k},\quad j_1<j_2<\cdots<j_k,\ k\leq n .$$

We denote by $H_a^\infty(X_n)$ the set of polynomials $g(\bar{z})=\sum_{k=0}^{n}\sum g_{i_1,\dots,i_k}\bar{z}_{i_1}\cdots\bar{z}_{i_k}$ of $\bar{z}$. For each $g(\bar{z})\in H_a^\infty(X_n)$ of the above form we define the element $g\in\mathcal{G}(F)$ by

$$g=\sum_{k=0}^{n}\sum g_{i_1,\dots,i_k}f_{i_1}\wedge\cdots\wedge f_{i_k}\,.$$

Then we have a natural correspondence between $\mathcal{G}(F)$ and the subset $H_a^\infty(X_n)$ of $H^\infty(X_n)$ by the mapping $g(\bar{z})\to g$ and we have for $h(\bar{z})$, $g(\bar{z})\in H_a^\infty(X_n)$

$$\int h(\bar{z})\,\overline{g(\bar{z})}\,e^{-z\bar{z}}\prod dz_i\,d\bar{z}_i=(h,g)\,,$$

since for $h=f_{i_1}\wedge\cdots\wedge f_{i_k}$, $g=f_{j_1}\wedge\cdots\wedge f_{j_l}\in\mathcal{G}(F)$, the inner product (h,g) is equal to the r.h.s. of (2.2) by (1.2).

Moreover, there exists a natural correspondence between operators in $\mathcal{G}(F)$ and operators in $H_a^\infty(X_n)$. To see this, we define the left and right differentiation by

$$\frac{\partial}{\partial x_p}x_{i_1}\cdots x_{i_s}=\sum_{k=1}^{s}(-1)^{k-1}\,\delta_{i_k,p}\,x_{i_1}\cdots\hat{x}_{i_k}\cdots x_{i_s}$$

and

$$x_{i_1}\cdots x_{i_s}\frac{\partial}{\partial x_p}=\sum_{k=1}^{s}(-1)^{s-k}\,\delta_{i_k,p}\,x_{i_1}\cdots\hat{x}_{i_k}\cdots x_{i_s}$$

where each x_i stands for one of Grassmann variables $\bar{z}_j$, z_j, $\bar{w}_j$, w_j in X_n. Then $a_i=a(f_i)$ corresponds to the left differentiation $\partial/\partial\bar{z}_i$ and $a_i^\dagger=a^\dagger(f_i)$ to the multiplication by $\bar{z}_i$ on the left i.e., $(a_i\,h)(\bar{z})=(\partial/\partial\bar{z}_i)\,h(\bar{z})$ and $(a_i^\dagger\,h)(\bar{z})=\bar{z}_i\,h(\bar{z})$ for $h\in\mathcal{G}(F)$.

Let A be an operator in $\mathcal{G}(F)$ which corresponds to an operator in $H_a^\infty(X_n)$ defined by the kernel $A(\bar{z},w)$, i.e.,

$$(Af)(\bar{z})=\int A(\bar{z},w)\,f(\bar{w})\,e^{-w\bar{w}}\prod_{i=1}^{n}dw_i\,d\bar{w}_i\,.$$

for $f\in\mathcal{G}(F)$. In this case we say A corresponds to $A(\bar{z},w)$ and write $A\leftrightarrow A(\bar{z},w)$. Every linear operator A in $\mathcal{G}(F)$ has its kernel $A(\bar{z},w)$.

It is well known that there exists the following correspondence between operators and their kernels:

$$\begin{aligned}
&A\leftrightarrow A(\bar{z},w)\\
&a_j\,A\leftrightarrow\frac{\partial}{\partial\bar{z}_j}A(\bar{z},w)\\
&Aa_j\leftrightarrow -A(\bar{z},w)\,w_j\\
&Aa_j^\dagger\leftrightarrow -A(\bar{z},w)\frac{\partial}{\partial w_j}\\
&a_j^\dagger\,A\leftrightarrow\bar{z}_j\,A(\bar{z},w)\,,
\end{aligned}\tag{2.3}$$

(see Berezin [2]) and from these we have

$$
\begin{aligned}
&A\Lambda \leftrightarrow A(\bar{z}, \eta w) \\
&\Lambda A \leftrightarrow A(\eta\bar{z}, w) \\
(2.4) \qquad &a_j^{(\Lambda)} A \leftrightarrow \sum_k \eta_{jk} \bar{z}_k A(\bar{z}, w) \\
&A a_j^{(\Lambda)} \leftrightarrow -\sum_k A(\bar{z}, w) \eta_{jk} \frac{\partial}{\partial w_k},
\end{aligned}
$$

where $\eta w = [\eta] w$, $\eta\bar{z} = [\eta]\bar{z}$ and $[\eta]$ is the $n \times n$ matrix whose (j, k) entry $\eta_{jk} = \langle f_j, \eta f_k \rangle$.

3. Nonstandard Analysis

One of the main tools in this paper is nonstandard analysis. We use the notations and conventions in the book of Davis [3]. In this section we introduce a $*$-finite dimensional Grassmann algebra $\mathcal{G}(F)$ ($\subset {}^*\mathcal{F}(\mathcal{H})$) and give a condition for an element g of $\mathcal{G}(F)$ to be near standard by using a new concept 'almost standard'.

In the nonstandard universe, there exists a $*$-finite dimensional subspace E of the nonstandard extension ${}^*\mathcal{H}$ of $\mathcal{H}$ satisfying

$$\mathcal{H} \subset E \subset {}^*\mathcal{H}$$

(see Davis [3], p. 150). The nonstandard extension ${}^*\eta$ of η is a mapping from ${}^*\mathcal{H}$ to ${}^*\mathcal{H}$. In such a case it is usual in nonstandard analysis to denote the mapping ${}^*\eta$ simply by η. Similarly the nonstandard extension of the involution $*$ is laso denoted simply by $*$. Such a convention will be used throughout this paper without permission. Let

$$F = E + E^* + \eta(E + E^*).$$

Then F is a $*$-finite dimensional subspace of ${}^*\mathcal{H}$ invariant under η and the involution $*$. Hereafter n denotes the dimension of F. Let e_i, $i \in N$ be a complete real orthonormal system of $\mathcal{H}$. Here by the word *real* we mean

$$(3.1) \qquad e_i^* = e_i.$$

Then e_i is automatically defined for $i \in {}^*\boldsymbol{N}$ in the nonstandard universe and the system $e_i \in {}^*\mathcal{H}$, $i \in {}^*\boldsymbol{N}$ is also a complete real orthonormal system in the nonstandard sense (i.e., *$*$-real orthonormal system*). Since the set $\{i \in {}^*\boldsymbol{N} \mid e_i \in F\}$ is internal by the Internality Theorem (see Davis [3] p. 39) and contains $\boldsymbol{N}$, theie exists an $l \in {}^*\boldsymbol{N} \setminus \boldsymbol{N}$ such that $e_i \in F$ for $i \leq l$. We add an (internal) sequence of $n-l$ vectors $f_{l+1}, \cdots, f_n$ to $e_1, \cdots, e_l$ so that $e_1, \cdots, e_l, f_{l+1}, \cdots, f_n$ is an internal real orthonormal basis of F. Then we have

Theorem 3.1. *Let e_i, $i \in \boldsymbol{N}$ be a real complete orthonormal system of $\mathcal{H}$*

satisfying (3.1), *then there exists an internal real orthonormal basis* $f_i, i=1, \cdots, n$ *of* F *satisfying* (3.1) *such that for some* $l \in {}^*N \backslash N$, $f_i = e_i (i \leq l)$.

Let $\mathcal{G}(F)$ be the Grassmann algebra over F. This is an internal subalgebra of ${}^*\mathcal{G}(\mathcal{H})$ and is a 2^n-dimensional vector space over ${}^*C$ with a basis

$$1, f_{j_1} \wedge \cdots \wedge f_{j_k}, \quad j_1 < j_2 < \cdots < j_k, k \leq n .$$

Note that the relation $\mathcal{G}(\mathcal{H}) \subset \mathcal{G}(F) \subset {}^*\mathcal{F}(\mathcal{H})$ holds. Let $h_i \in \mathcal{H} \subset F (1 \leq i \leq k \leq n)$ and write

$$h_i = \sum_{j=1}^{n} h_{ji} f_j , \quad h_{ji} = \langle f_j, h_i \rangle . \tag{3.2}$$

We may write

$$h_1 \wedge \cdots \wedge h_k = (k!)^{-1/2} \sum_{j_1, \cdots, j_k = 1}^{n} K_{j_1, \cdots, j_k} f_{j_1} \wedge \cdots \wedge f_{j_k} , \tag{3.3}$$

where the coefficients $K_{j_1, \cdots, j_k}$ are given by

$$K_{j_1, \cdots, j_k} = (k!)^{-1/2} \sum_{i_1, \cdots, i_k = 1}^{n} \operatorname{sgn} (i_1, \cdots, i_k) \, h_{j_1 i_1} \cdots h_{j_k i_k}$$

and are anti-symmetric under the permutation of $j_1, j_2, \cdots, j_k$. We have from (1.2)

$$\|h_1 \wedge \cdots \wedge h_k\|^2 = \sum_{j_1, \cdots, j_k = 1}^{n} |K_{j_1, \cdots, j_k}|^2 .$$

Generally, the element g of $\mathcal{G}(F)$ can be written as

$$g = \sum_{k=0}^{n} (k!)^{-1/2} \sum_{i_1, \cdots, j_k = 1}^{n} K^{(k)}_{i_1, \cdots, i_k} f_{i_1} \wedge \cdots \wedge f_{i_k} \tag{3.4}$$

with anti-symmetric $K^{(k)}_{i_1, \cdots, i_k} \in {}^*C$, and

$$\|g\|^2 = \sum_{k=0}^{n} \sum_{i_1, \cdots, i_k = 1}^{n} |K^{(k)}_{i_1, \cdots, i_k}|^2 . \tag{3.5}$$

The following notion *almost standard* is new and useful to give a condition that g is near standard.

DEFINITION 3.2. The element $g \in \mathcal{G}(F)$ is said to be almost standard if there exists a standard sequence $B^{(k)}(x_1, \cdots, x_k)$, $(k=0, 1, \cdots)$ of bounded anti-symmetric k-linear forms on $\mathcal{H}$ such that

$$K^{(k)}_{i_1, \cdots, i_k} = B^{(k)}(f_{i_1}, \cdots, f_{i_k}) \tag{3.6}$$

gives the coefficients in the expression (3.4) of g.

Proposition 3.3. *The elements of* $\mathcal{G}(\mathcal{H})$ *are almost standard.*

Proof. We have only to show that each monomial in $\mathcal{Q}(\mathcal{H})$ is almost standard. For the monomial belonging to $\mathcal{Q}(\mathcal{H})$ of the form (3.3) with (3.2) where each $h_i \in \mathcal{H}$, we define anti-symmetric k-linear form on $\mathcal{H}$ by

$$B^{(k)}(x_1, \cdots, x_k) = (k!)^{-1/2} \sum \operatorname{sgn}(p, q, \cdots, r) \langle x_1, h_p \rangle \langle x_2, h_q \rangle \cdots \langle x_k, h_r \rangle ,$$

where the summation extends over all the permutations $p, q, \cdots, r$ of $1, 2, \cdots, k$. Then $K_{i_1, \cdots, i_k} = B^{(k)}(f_{i_1}, \cdots, f_{i_k})$ holds. Setting $B^{(j)}=0$ for $j \neq k$, we form a sequence $B^{(j)}$, $j=0, 1, \cdots$. Then it is obvious that (3.3) is almost standard.

Proposition 3.4. *Let $g, h \in \mathcal{Q}(F)$ be almost standard. Then $g_\wedge h$ is also almost standard.*

Proof. Let $B^{(k)}(x_1, \cdots, x_k)$, $C^{(l)}(x_1, \cdots, x_l)$ be two sequences of anti-symmetric multilinear form which correspond to g and h respectively. Let $D^{(m)}(x_1, \cdots, x_m)$ be the anti-symmetrization of the multilinear form

$$\sum_{k+l=m} (k!\, l!)^{-1/2} B^{(k)}(x_1, \cdots, x_k)\, C^{(l)}(x_{k+1}, \cdots, x_m) \,.$$

Then $D^{(m)}(x_1, \cdots, x_m)$ is the sequence of anti-symmetric multilinear form corresponding to $g_\wedge h$.

Proposition 3.5. *Let A be a bounded operator. Then*

$$g = \exp\Big(\sum_{i,j=1}^{n} \langle f_i, Af_j \rangle f_{i\wedge} f_i\Big)$$

is almost standard.

Proof. Let $B^{(2k)}(x_1, \cdots, x_{2k})$ be the anti-symmetrization of multilinear form

$$\langle x_1, Ax_2 \rangle \cdots \langle x_{2k-1}, Ax_{2k} \rangle \,.$$

Then the sequence of multilinear form

$$(k!)^{-1}(2k!)^{1/2} B^{(2k)}(x_1, \cdots, x_{2k}) \,,$$

$k=0,1, 2, \cdots$ corresponds to g.

Theorem 3.6. *Let $g \in \mathcal{Q}(F)$ be almost standard. If the norm (3.5) of g is finite then g is a near standard point in ${}^*\mathcal{F}(\mathcal{H})$, that is, there exists an element h of $\mathcal{F}(\mathcal{H})$ with $\|g-h\| \simeq 0$.*

Proof. Write g as in (3.4) and assume that the coefficients are given by (3.6). We form a standard sequence g_m, $m=1, 2, \cdots$ by

$$g_m = \sum_{k=0}^{m} (k!)^{-1/2} \sum_{i_1, \cdots, i_k=1}^{m} B^{(k)}(e_{i_1}, \cdots, e_{i_k})\, e_{i_1\wedge} \cdots {}_\wedge e_{i_k}$$

Let l be the infinitely large number which appeared in Theorem 3.1. Since $f_i=e_i$ for $i\leq l$, we have, for $m\in \boldsymbol{N}$,

$$\|g_m\|^2=\sum_{k=0}^{m}\sum_{i_1,\cdots,i_k=1}^{m}|B^{(k)}(e_{i_1},\cdots,e_{i_k})|^2$$
$$=\sum^{m}\sum^{m}|K_{i_1,\cdots,i_k}|^2\leq\|g\|^2.$$

Since $\|g\|$ is finite by the assumption, it follows from the above inequality that $\|g_m\|$, $m=1,2,\cdots$ is bounded. Combining this with the equality $\|g_{m_1}-g_{m_2}\|^2=\|g_{m_1}\|^2-\|g_{m_2}\|^2(m_1\geq m_2)$ we deduce that $\{g_m\}$ is Cauchy sequence in $\mathcal{G}(\mathcal{H})$. So we put $h=\lim g_m\in\mathcal{F}(\mathcal{H})$. As $\|h-g_l\|\simeq 0$, we have only to show $\|g_l-g\|\simeq 0$. Adding suitable vectors $f_{n+1},\cdots$ to $f_1,\cdots,f_n$ we have a complete orthonormal system $f_1,\cdots,f_n,\cdots$ of ${}^*\mathcal{H}$ in the nonstandard sense. Using the relations $e_i=\sum_{j=1}^{\infty}\alpha_{ij}f_j$ and $\sum_{i=1}^{\infty}\alpha_{ij}\,\alpha_{ik}=\delta_{jk}$, we obtain

$$\sum_{k=0}^{\infty}\sum_{i_1,\cdots,i_k=1}^{\infty}|B^{(k)}(e_{i_1},\cdots,e_{i_k})|^2$$
$$=\sum_{k=0}^{\infty}\sum_{i_1,\cdots,i_k=1}^{\infty}|B^{(k)}(f_{i_1},\cdots,f_{i_k})|^2$$
$$\geq\sum^{n}\sum^{n}|B^{(k)}(f_{i_1},\cdots,f_{i_k})|^2=\|g\|^2\geq\|g_l\|^2,$$

where the infinite sums are, of course, in the nonstandard sense. The first infinite sum above is the limit in the nonstandard sense of the nonstandard extension $\{\|g_m\|^2\}_{m\in{}^*N}$ of $\{\|g_m\|^2\}_{m\in N}$, and it coincides with the usual limit $\lim_{m\to\infty}\|g_m\|^2=\|h\|^2={}^\circ\|g_l\|^2$ where ${}^\circ r$ denotes the standard part of $r\in{}^*R$. Thus we have ${}^\circ\|g_l\|^2\geq\|g\|^2\geq\|g_l\|^2$ and so $\|g_l\|^2\simeq\|g\|^2$ holds, and combining this with $\|g-g_l\|^2=\|g\|^2-\|g_l\|^2$, we conclude that $\|g-g_l\|\simeq 0$.

4. Nonstandard Λ-isometric Operator

The purpose of this and the following sections is to construct an operator which implements the canonical transformation (1.6). In this section especially, we define an operator U in $\mathcal{G}(F)$ and show that its restriction to a standard set $\mathcal{G}\{e_i\}$ is Λ-isometric (Proposition 4.2). We extensively use the nonstandard extension of the Berezin calculus. More precisely, by the Transfer Principle (see Davis [3] p. 28) we apply the Berezin calculus on the finite dimensional superspace which is stated in §2 to (internal) functions on $*$-finite dimensional superspaces.

Here we introduce the notion of *approximating matrices* which will be frequently used in this paper. Let L be an internal linear mapping with its domain containing F and range in ${}^*\mathcal{H}$. We define an $n\times n$ matrix $[L]$ by setting its (i,j) entry $[L]_{ij}=\langle f_i,Lf_j\rangle$ and call this the approximating matrix of L.

For a bounded operator A on $\mathcal{H}$, we denote $[{}^*A]$ by $[A]$, since in our convention ${}^*A$ is denoted by A. From now on P will denote the projection of ${}^*\mathcal{H}$ onto F. For an internal linear mapping, the approximating matrix $[L]$ is the matrix representation of the operator PLP restricted to F with respect to the basis $f_1, \cdots, f_n$ of F.

Now, for the operators Ψ and Φ we define a kernel $U(\bar{z}, w)$ by

$$U(\bar{z}, w) = C\cdot\exp\{-\frac{1}{2}(\bar{z}, w)\begin{bmatrix}[\Phi^{-1}\Psi\eta] & [\Phi^{-1}] \\ -[\Phi'^{-1}] & [\eta\bar{\Psi}\Phi^{-1}]\end{bmatrix}\begin{bmatrix}\bar{z} \\ w\end{bmatrix}\}. \tag{4.1}$$

Here we assume that Φ has a bounded inverse Φ^{-1}. Let U be the operator of $\mathcal{G}(F)$ defined by the integral kernel $U(\bar{z}, w)$, i.e.,

$$(Uf)(\bar{z}) = \int U(\bar{z}, w) f(\bar{w})\, e^{-w\bar{w}} \prod_{i=1}^{n} dw_i\, d\bar{w}_i\,. \tag{4.2}$$

for $f \in \mathcal{G}(F)$.

DEFINITION 4.1. We define $\mathcal{G}\{e_i\}$ to be the standard Grassmann algebra generated by $\{e_i \mid i \in N\}$.

The element of $\mathcal{G}\{e_i\}$ is the linear combination of finite products of e_i's.

Proposition 4.3. *If* $\|\Phi^{-1}\Psi\| < 1$, *then for a suitable* $C \in {}^*\boldsymbol{C}$ *in* (4.1), *we have*

$$(h, g) = (\Lambda U h, U \Lambda g) \tag{4.3}$$

for $h, g \in \mathcal{G}\{e_i\}$.

Proof. It suffices to show that

$$(h, g) = (\Lambda U^\dagger \wedge U h, g)\,, \tag{4.4}$$

for $h = e_{i_1} \wedge \cdots \wedge e_{i_k}$, $g = e_{j_1} \wedge \cdots \wedge e_{j_m}$ with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_m$, $k, m, i_s, j_s \in \boldsymbol{N}$. Note that the kernel corresponding to $U^\dagger$ is $\overline{U(\overline{w}, z)}$, and put $V = \Lambda U^\dagger \Lambda U$. Then using the integral formula (2.1) and the relations:

$$\begin{aligned} &\eta\Phi^{-1}\Psi + \Psi'\Phi'^{-1}\eta = 0 \\ &\eta\bar{\Psi}\Phi^{-1} + \Phi'^{-1}\Psi^\dagger\eta = 0\,, \end{aligned} \tag{4.5}$$

which follow from (1.8) and (1.11), the kernel of V is calculated as follows:

$$\begin{aligned} V(\bar{z}, w) &= \int \overline{U(\bar{u}, \eta\bar{z})}\, U(\eta\bar{u}, w)\, e^{-u\bar{u}} \prod_{i=1}^{n} du_i\, d\bar{u}_i \\ &= |C^2| \int \exp\{-\frac{1}{2}(u, \bar{u})[T]\begin{bmatrix}u \\ \bar{u}\end{bmatrix} + (\phi_1, \phi_2)\begin{bmatrix}u \\ \bar{u}\end{bmatrix}\} \prod du_i\, d\bar{u}_i \end{aligned} \tag{4.6}$$

$$\times \exp\{-\frac{1}{2}(\bar{z}, w)[R]\begin{bmatrix}\bar{z}\\ w\end{bmatrix}\}$$

$$= |C|^2(\det[T])^{1/2}\exp\{-\frac{1}{2}(\phi_1, \phi_2)[T]^{-1}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\}$$

$$\times \exp\{-\frac{1}{2}(\bar{z}, w)[R]\begin{bmatrix}\bar{z}\\ w\end{bmatrix}\},$$

where $\phi_1=-[\Phi^{-1}\eta]\bar{z}$, $\phi_2=[\eta\Phi^{-1}]w$,

$$T=\begin{bmatrix}-\bar{\Phi}^{-1}\bar{\Psi}\eta & 1\\ -1 & \eta\Phi^{-1}\Psi\end{bmatrix}, \quad [T]=\begin{bmatrix}-[\bar{\Phi}^{-1}\bar{\Psi}\eta] & 1\\ -1 & [\eta\Phi^{-1}\Psi]\end{bmatrix} \tag{4.7}$$

(we will see the existence of $[T]^{-1}$ later) and

$$R=\begin{bmatrix}\eta\Phi^{\dagger-1}\Psi' & 0\\ 0 & \Phi'^{-1}\Psi^{\dagger}\eta\end{bmatrix}, \quad [R]=\begin{bmatrix}[\eta\Phi^{\dagger-1}\Psi'] & 0\\ 0 & [\Phi'^{-1}\Psi^{\dagger}\eta]\end{bmatrix}.$$

In general, for an internal linear operator

$$M=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}$$

with domain containing $F\oplus F$ and range in ${}^*\mathcal{H}\oplus{}^*\mathcal{H}$, we define its approximating matrix $[M]$ by

$$[M]=\begin{bmatrix}[M_{11}] & [M_{12}]\\ [M_{21}] & [M_{22}]\end{bmatrix}.$$

We introduce the projection operator $\tilde{P}=\begin{bmatrix}P & 0\\ 0 & P\end{bmatrix}$ of ${}^*\mathcal{H}\oplus{}^*\mathcal{H}$ onto $F\oplus F$, and denote the restriction of $\tilde{P}M\tilde{P}$ to $F\oplus F$ by $\tilde{P}M\tilde{P}_{|F\oplus F}$. Then $[M]$ is the matrix representation of $\tilde{P}M\tilde{P}_{|F\oplus F}$ with respect to the internal real orthonormal basis $f_1\oplus 0, \cdots, f_n\oplus 0, 0\oplus f_1, \cdots, 0\oplus f_n$ of $F\oplus F$.

Let M, N be two operators with domain containing $F\oplus F$ and range in ${}^*\mathcal{H}\oplus{}^*\mathcal{H}$. Then we have the following rules:

1) $[M]=[N]$ if and only if $\tilde{P}M\tilde{P}=\tilde{P}N\tilde{P}$.
2) $[M][N]=[M\tilde{P}N]$.
3) $N(F\oplus F)\subset F\oplus F$ implies $[M][N]=[MN]$.
4) if the domain of M is ${}^*\mathcal{H}\oplus{}^*\mathcal{H}$, then $\tilde{P}M=M\tilde{P}$ implies $[M][N]=[MN]$.

These will be used later.

The existence of $[T]^{-1}$ in (4.6) is equivalent to that of an inverse of the operator $\tilde{P}T\tilde{P}_{|F\oplus F}$, which follows from the following lemma if one takes $A=P\bar{\Phi}^{-1}\bar{\Psi}\eta P_{|F}$ and $B=P\eta\Phi^{-1}\Psi P_{|F}$.

Lemma 4.3. *Let A, B be two operators on F with $||A||<1$, $||B||<1$. Then, the operator* $X=\begin{bmatrix} -A & 1 \\ -1 & B \end{bmatrix}$ *on $F\oplus F$ has an inverse.*

Proof. The existence of $(1-AB)^{-1}$ and $(1-BA)^{-1}$ follows from the assumption. A left and a right inverses of X are

$$\begin{bmatrix} (1-BA)^{-1}B & -(1-AB)^{-1} \\ (1-AB)^{-1} & -(1-BA)^{-1}A \end{bmatrix}$$

and

$$\begin{bmatrix} B(1-AB)^{-1} & -(1-BA)^{-1} \\ (1-AB)^{-1} & -A(1-BA)^{-1} \end{bmatrix}.$$

They coincide and give the inverse of X.

We continue the proof of Proposition 4.2. We have just seen the existence of an inverse of $\tilde{P}T\tilde{P}_{|Y\oplus F}$. So, we set

$$S_1 = (\tilde{P}T\tilde{P}_{|F\oplus F})^{-1}.$$

Note that $[T]^{-1}=[S_1]$. In order to rewrite the r.h.s. of (4.6) we introduce the following operators

$$Q = \begin{bmatrix} \Phi^{-1}\eta & 0 \\ 0 & -\eta\Phi^{-1} \end{bmatrix}$$

and

$$A = Q'\tilde{P}S_1\tilde{P}Q+R.$$

Then by the rule 2), we have $[A]=[Q'][S_1][Q]+[R]=[Q'][T]^{-1}[Q]+[R]$ and hence, from (4.6),

$$V(\bar{z}, w) = |C|^2(\det[T])^{1/2}\exp\{-\frac{1}{2}(\bar{z}, w)[A]\begin{bmatrix} \bar{z} \\ w \end{bmatrix}\}. \tag{4.8}$$

In order to calculate the r.h.s. of (4.4) we define a function $G(\xi, \bar{\xi})$ by

$$G(\xi, \bar{\xi}) = \int V(\bar{z}, w)\, e^{\xi\bar{w}}\, e^{\bar{\xi}z}\, e^{-z\bar{z}}\, e^{-w\bar{w}} \prod dz_i\, d\bar{z}_i\, dw_i\, d\bar{w}_i, \tag{4.9}$$

where we introduced new variables $\xi=(\xi_1, \cdots, \xi_n)\in X_n$. Since

$$h(\bar{z}) = \bar{z}_{i_1}\cdots\bar{z}_{i_k} = \frac{\partial}{\partial\xi_i}\cdots\frac{\partial}{\partial\xi_{i_k}} e^{\xi\bar{z}}{}_{|\xi=0},$$

the r.h.s. of (4.4) is

$$(Vh, g) = \int V(\bar{z}, w)\, h(\bar{w})\, \overline{g(\bar{z})}\, e^{-z\bar{z}}\, e^{-w\bar{w}} \prod dz_i\, d\bar{z}_i\, dw_i\, d\bar{w}_i$$

$$= \frac{\partial}{\partial \xi_{i_1}} \cdots \frac{\partial}{\partial \xi_{i_k}} \frac{\partial}{\partial \bar{\xi}_{j_1}} \cdots \frac{\partial}{\partial \bar{\xi}_{j_l}} G(\xi, \bar{\xi})_{|\xi=\bar{\xi}=0}\,.$$

We calculate the integral (4.9) by using (2.1). Then we obtain

$$G(\xi, \bar{\xi}) = |C|^2(\det[T])^{1/2} \int \exp\{-\frac{1}{2}(\bar{z}, w, z, \bar{w}) \begin{bmatrix} [A] & -\sigma \\ \sigma & 0 \end{bmatrix} \begin{pmatrix} \bar{z} \\ w \\ z \\ \bar{w} \end{pmatrix} + (0, 0, \bar{\xi}, \xi) \begin{pmatrix} \bar{z} \\ w \\ z \\ \bar{w} \end{pmatrix}\} \prod dz_i\, d\bar{z}_i\, dw_i\, d\bar{w}_i$$

$$= |C|^2(\det[T])^{1/2} \exp\{-\frac{1}{2}(\bar{\xi}, \xi)\, \sigma[A]\, \sigma \begin{bmatrix} \bar{\xi} \\ \xi \end{bmatrix}\}\,,$$

where

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, we have

$$(4.10) \qquad (Vh, g) = \frac{\partial}{\partial \xi_{i_1}} \cdots \frac{\partial}{\partial \xi_{i_k}} \frac{\partial}{\partial \bar{\xi}_{j_1}} \cdots \frac{\partial}{\partial \bar{\xi}_{j_l}} G(\xi, \bar{\xi})_{|\xi=\bar{\xi}=0}$$

$$= |C|^2(\det[T])^{1/2} \frac{\partial}{\partial \xi_{i_1}} \cdots \frac{\partial}{\partial \xi_{i_k}} \frac{\partial}{\partial \bar{\xi}_{j_1}} \cdots \frac{\partial}{\partial \bar{\xi}_{j_l}} \exp\{-\frac{1}{2}(\bar{\xi}, \xi)\, \sigma[A]\, \sigma \begin{bmatrix} \bar{\xi} \\ \xi \end{bmatrix}\}_{|\xi=\bar{\xi}=0}\,.$$

Since $i_1, \cdots, i_k, j_1, \cdots, j_k$ are all in $\boldsymbol{N}$, only the terms of $\xi_i\, \xi_j$, $\xi_i\, \bar{\xi}_j$ or $\bar{\xi}_i\, \bar{\xi}_j$ with $i, j \in \boldsymbol{N}$ in the quadratic form in the exponent of the r.h.s. in (4.10) have an effect on the result of calculation. For the calculation of (4.10) we prepare the following lemma.

Lemma 4.4. *Let A, B be operators on F such that for all $f \in \mathcal{H} \subset F$, $Af = Bf$. Then*

$$[A]_{ij} = [B]_{ij} \qquad \text{for} \quad 1 \leq i \leq n,\ j \in \boldsymbol{N}\,.$$

Proof. Let $j \in \boldsymbol{N}$. Then $f_j \in \mathcal{H}$ and hence $Af_j = Bf_j$. Thus for i with $1 \leq i \leq n$, $[A]_{ij} = \langle f_i, Af_j \rangle = [B]_{ij}$. This completes the proof of the lemma.

We can show that for any $f \in \mathcal{H} \oplus \mathcal{H}$,

$$(4.11) \qquad -\sigma A \sigma f = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f$$

holds. To see this we put

$$S=\begin{bmatrix} -\Psi'\eta\overline{\Phi} & -\Phi^{\dagger}\eta\Phi\eta \\ \eta\Phi'\eta\overline{\Phi} & \eta\Psi^{\dagger}\eta\Phi\eta \end{bmatrix}.$$

By direct computation using the relations (1.8) and (1.11) we can see that

$$ST=1 \quad\text{and}\quad TS=1\,. \tag{4.12}$$

Now let $f\in\mathcal{H}\oplus\mathcal{H}$ be arbitrary. Then from (4.12) we see that $\tilde{P}T\tilde{P}Sf=f$ holds and we have

$$S_1 f=(\tilde{P}T\tilde{P}_{|F\oplus F})^{-1}f=Sf\,. \tag{4.13}$$

Replacing f in (4.13) with $\tilde{P}\underset{\sim}{Q}f$ which is also in $\mathcal{H}\oplus\mathcal{H}$ and multiplying $\underset{\sim}{Q}'\tilde{P}$ on the left, we have

$$\underset{\sim}{Q}'\tilde{P}S_1\tilde{P}\underset{\sim}{Q}f=\underset{\sim}{Q}'\tilde{P}S\tilde{P}\underset{\sim}{Q}f=\underset{\sim}{Q}'S\underset{\sim}{Q}f\,.$$

We combine this with the relation $\underset{\sim}{Q}'S\underset{\sim}{Q}=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}-R$ which is obtained by direct computation. Then we have

$$Af=(\underset{\sim}{Q}'\tilde{P}S\tilde{P}\underset{\sim}{Q}+R)f=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}f\,.$$

Replacing f by σf which is also in $\mathcal{H}\oplus\mathcal{H}$ and multiplying $-\sigma$ on the left we obtain (4.11).

Since σ commutes with $\tilde{P}$, by the rule 4) we see that $-[\sigma A\sigma]=-[\sigma]\,[A]\,[\sigma]=-\sigma[A]\,\sigma$ holds. So, by (4.11) and Lemma 4.4, we see that

$$-\frac{1}{2}\,(\xi,\bar{\xi})\,\sigma[A]\,\sigma\begin{bmatrix}\xi \\ \bar{\xi}\end{bmatrix}=\sum_{i=1}^{n}\bar{\xi}_i\,\xi_i+r(\xi,\bar{\xi})$$

where $r(\xi,\bar{\xi})$ is a bilinear form which does not contain terms of $\bar{\xi}_i\,\bar{\xi}_j$, $\bar{\xi}_i\,\xi_j$, $\xi_i\,\xi_j$ with $i,j\in\boldsymbol{N}$. Then we see, by (4.10), that the r.h.s. of (4.4) is

$$|C|^2(\det[T])^{1/2}\,\delta_{k,l}\prod_{s=1}^{k}\delta_{i_s,j_s}\,.$$

Let $h=g=1$ which represents the vacuum. Then we have

$$(\Lambda U1,\,U1)=(\Lambda U1,\,U\Lambda 1)=\,|C|^2(\det[T])^{1/2} \tag{4.14}$$

As the l.h.s. of (4.14) is real, $(\det[T])^{1/2}$ is real and $\det[T]$ must be positive. Therefore we cna choose $C\in{}^*\boldsymbol{C}$ such that

$$|C|^2(\det[T])^{1/2}=1$$

Since $(h,g)=\delta_{k,l}\prod_{s=1}^{k}\delta_{i_s,j_s}$, we have (4.4). This completes the proof of Proposition 4.2.

5. Near Standard Operator

The purpose of this section is to prove Proposition 5.6 which says in effect that when the operator Ψ is Hilbert Schmidt the internal operator U in $\mathcal{G}(F)$ defined in §4 has a 'standard part' as an operator in $\mathcal{G}(\mathcal{H})$ whose domain is $\mathcal{G}\{e_i\}$. This is the most important part for the application of nonstandard analysis and Theorem 3.6 plays a crucial role for that purpose. This standard part gives rise to the operator U_1 in §7 which is weakly Λ-unitary and implements the canonical transformation (1.6). We begin with the following lemmas.

Lemma 5.1. *Let A, B be operators on $\mathcal{H}$ such that*

$$||A||<1,\ ||B||<1$$

and P be a projection of $\mathcal{H}$. Then for $x, y\in\mathcal{H}$,

$$\begin{aligned}&|(x, \log(1-PAPBP)y)|\\ &\leq -\frac{1}{||A||\cdot||B||}\log(1-||A||\,||B||)\,||A^{\dagger}Px||\,||BPy||\end{aligned} \tag{5.1}$$

Proof. $|(x, \log(1-PAPBP)y)| = |(x, \sum_{k=1}^{\infty}(-1/k)(PAPBP)^k y)|$

$$\begin{aligned}&\leq \sum_{k=1}^{\infty}(1/k)\,||A||^{k-1}\,||B||^{k-1}\,||A^{\dagger}Px||\,||BPy||\\ &= (-1/||A||\,||B||)\log(1-||A||\,||B||)\,||A^{\dagger}Px||\,||BPy||\,.\end{aligned}$$

In the above lemma the projection P is a standard operator of $\mathcal{H}$, but for later use of the lemma, here we note that by the transfer principle the lemma is valid for an internal projection P. In the following lemmas the nonstandard extension $^*A$ of the operator A is also denoted by A as usual.

Lemma 5.2. *Let A be a Hilbert Schmidt operator on $\mathcal{H}$ and let $f_1, \cdots, f_n$ be the orthonormal basis of F in Theorem* 3.1. *Then for any positive real number $\varepsilon\in\boldsymbol{R}$, there exists $k\in\boldsymbol{N}$ such that*

$$\sum_{i=k}^{n}||Af_i||^2<\varepsilon\,. \tag{5.2}$$

Proof. Let $\{e_i\}$ be the complete orthonormal system of $\mathcal{H}$ which appeared in Theorem 3.1. Since A is a Hilbert Schmidt operator, there exists $k\in\boldsymbol{N}$ such that

$$\sum_{i=k}^{\infty}||Ae_i||^2<\varepsilon\,.$$

Since the Hilbert Schmidt norm $\sum_{i=1}^{\infty}||Ae_i||^2$ is independent of the choice of the orthonormal basis, we have (5.2).

Lemma 5.3. *Let A, B be Hilbert Schmidt operators on $\mathcal{H}$. Let $f_1, \cdots, f_n$ be the orthonormal basis of F which appeared in Theorem* 3.1. *Then for any positive $\varepsilon \in \boldsymbol{R}$ there exists $k \in \boldsymbol{N}$ such that*

$$\sum_{i=k}^{n} \|A^{\dagger} f_i\| \, \|Bf_i\| < \varepsilon . \tag{5.3}$$

Proof. From the Lemma 5.2, there exists $k \in \boldsymbol{N}$ such that

$$\sum_{i=k}^{n} \|A^{\dagger} f_i\|^2 < \varepsilon \quad \text{and} \quad \sum_{i=k}^{n} \|Bf_i\|^2 < \varepsilon .$$

Therefore we have $\sum_{i=k}^{n} \|A^{\dagger} f_i\| \, \|Bf_i\| < \varepsilon$.

Proposition 5.4. *Let A, B satisfy the conditions of Lemmas* 5.1 *and* 5.3. *Let P be the projection of ${}^*\mathcal{H}$ onto F. Then we have*

$$\operatorname{Tr} \log (1-AB) \simeq \operatorname{Tr} \log (1-PAPBP) \tag{5.4}$$

Proof. From (5.1) and (5.3), for any positive $\varepsilon \in \boldsymbol{R}$ there exists $k \in \boldsymbol{N}$ such that

$$\sum_{i=k}^{\infty} |(e_i, \log (1-AB)\, e_i)| < \varepsilon$$

and

$$\sum_{i=k}^{n} |(f_i, \log (1-PAPBP) f_i)| < \varepsilon .$$

Since

$$\sum_{i=1}^{k-1} (e_i, \log (1-AB)\, e_i) = \sum_{i=1}^{k-1} (f_i, \log (1-PAPB) f_i) ,$$

we have

$$\operatorname{Tr} \log (1-AB) - \operatorname{Tr} \log (1-PAPBP) | < 2\varepsilon .$$

This shows (5.4).

Corollary 5.5. *Let Ψ be a Hilbert Schmidt operator and $\|\Phi^{-1} \Psi\| < 1$, then* $\det(T)$ *for* (T) *of* (4.7) *is finite and its standard part is* $\det(\overline{\Phi}^{-1} \eta \Phi'^{-1} \eta)$ *which does not vanish.*

Proof.

$$\begin{aligned}
\det [T] &= (-1)^n \det \begin{bmatrix} -1 & [\eta \Phi^{-1} \Psi] \\ -[\overline{\Phi}^{-1} \overline{\Psi} \eta] & 1 \end{bmatrix} \\
&= \det (1-[\overline{\Phi}^{-1} \overline{\Psi}]\, [\Phi^{-1} \Psi]) = \exp \operatorname{Tr} \log (1-[\overline{\Phi}^{-1} \overline{\Psi}]\, [\Phi^{-1} \Psi]) \\
&= \exp \operatorname{Tr} \log (1-P\overline{\Phi}^{-1} \overline{\Psi} P \Phi^{-1} \Psi P) .
\end{aligned}$$

By Proposition 5.4, its standard part is

$$\begin{aligned}(5.5)\qquad &\exp \operatorname{Tr} \log (1-\bar{\Phi}^{-1}\bar{\Psi}\Phi^{-1}\Psi) \\ &= \exp \operatorname{Tr} \log (\bar{\Phi}\eta\Phi'^{-1}\eta) = \det (\bar{\Phi}^{-1}\eta\Phi'^{-1}\eta)\,,\end{aligned}$$

where we used the relation (1.8). Since Ψ is a Hilbert Schmidt operator, Tr $\log(1-\bar{\Phi}^{-1}\bar{\Psi}\Phi^{-1}\Psi)$ is finite, so (5.5) does not vanish.

Now, we determine the value of the constant C in (4.1) by

$$C = {}^{\circ}(\det [T])^{-1/4}\,.$$

Then we have the following porposition.

Proposition 5.6. *Let* Ψ *be a Hilbert Schmidt operator and* $\|\Phi^{-1}\Psi\|<1$. *Let* U *be the operator defined by* (4.1) *and* (4.2). *Then for any* $h\in\mathcal{G}\{e_i\}$, Uh *is a near standard point of* ${}^*\mathcal{F}(\mathcal{H})$.

Proof. Let $h(\bar{z})=\bar{z}_{i_1}\cdots\bar{z}_{i_k}$, with $k\in\boldsymbol{N}$, and $i_1,\cdots,i_k\in\boldsymbol{N}$. In view of Theorem 3.6 we have only to show that Uh is almost standard and has a finite norm.

We calculate the following integral:

$$\begin{aligned}(5.6)\qquad &\int U(\bar{z},w)\,e^{\xi\bar{w}}\,e^{-w\bar{w}}\prod dw_i\,d\bar{w}_i \\ &= C\exp\{-(1/2)\,\bar{z}\,[\Phi^{-1}\Psi\eta]\,\bar{z}\} \\ &\quad\times\int \exp\Big\{-\frac{1}{2}(w,\bar{w})\begin{bmatrix}[\eta\bar{\Psi}\Phi^{-1}] & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}w\\ \bar{w}\end{bmatrix} \\ &\qquad\qquad +(-[\Phi'^{-1}]\,\bar{z},\xi)\begin{bmatrix}w\\ \bar{w}\end{bmatrix}\Big\}\prod dw_i\,d\bar{w}_i \\ &= \Psi_0(\bar{z})\,e^{-\phi(\xi,\bar{z})}\,,\end{aligned}$$

where

$$\Psi_0(\bar{z}) = C\exp\{(-1/2)\,\bar{z}\,[\Phi^{-1}\Psi\eta]\,\bar{z}\}$$

and

$$\phi(\xi,\bar{z}) = \xi\,[\Phi'^{-1}]\,\bar{z}+(-1/2)\,\xi\,[\eta\bar{\Psi}\Phi^{-1}]\,\xi\,.$$

Then, since $\dfrac{\partial}{\partial\xi_{i_1}}\cdots\dfrac{\partial}{\partial\xi_{i_k}}\,e^{\xi\bar{w}}{}_{|\xi=0}=\bar{w}_{i_1}\cdots\bar{w}_{i_k}=h(\bar{w})$, we have

$$(5.7)\qquad (Uh)\,(\bar{z}) = \frac{\partial}{\partial\xi_i}\cdots\frac{\partial}{\partial\xi_{i_k}}\,\Psi_0(\bar{z})\,e^{-\phi(\xi,\bar{z})}{}_{|\xi=0}\,.$$

Since we have chosen C to be the standard part of $(\det[T])^{-1/4}$, it follows from the Propositions 3.3, 3.4 and 3.5 that Uh which corresponds to the r.h.s.

of (5.7) is almost standard. In order to estimate the norm of Uh, we calculate the following integral (5.8). Note that, in view of (5.7), $||Uf||^2$ is obtained by differentiating the l.h.s. of (5.8) with respect to $\xi_{i_j}, \bar{\xi}_{i_j} (1 \leq j \leq k)$ and putting $\xi = \bar{\xi} = 0$.

$$\begin{aligned}
&\int \Psi_0(\bar{z})\, e^{-\phi(\xi,\bar{z})} (\overline{\Psi_0(\bar{z})\, e^{-\phi(\xi,\bar{z})}})\, e^{-z\bar{z}} \prod dz_i\, d\bar{z}_i \\
&= |C|^2 \exp\{(-1/2)\,(\xi\,[\eta\bar{\Psi}\Phi^{-1}]\,\xi + \bar{\xi}\,[\Phi^{\dagger -1}\,\Psi'\eta]\,\bar{\xi})\} \\
&\quad \times \int \exp\{-\frac{1}{2}\,(z, \bar{z}) \begin{bmatrix} [\eta\Psi^\dagger\Phi^{\dagger -1}] & 1 \\ -1 & [\Phi^{-1}\,\Psi\eta] \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix} \qquad (5.8) \\
&\qquad + (-[\bar{\Phi}^{-1}]\,\bar{\xi},\, -[\Phi^{-1}]\,\xi) \begin{bmatrix} z \\ \bar{z} \end{bmatrix}\} \prod dz_i\, d\bar{z}_i \\
&= |C|^2 \exp\{(-1/2)\,(\xi\,[\eta\bar{\Psi}\Phi^{-1}]\,\xi + \bar{\xi}\,[\Phi^{\dagger -1}\,\Psi'\,\eta]\,\bar{\xi})\} \\
&\quad [\det(1 + [\Phi^{-1}\,\Psi]\,[\Psi^\dagger\,\Phi^{\dagger -1}])]^{1/2} \\
&\quad \times \exp\{-\frac{1}{2}\,([\bar{\Phi}^{-1}]\,\bar{\xi},\, [\Phi^{-1}]\,\xi) \begin{bmatrix} [\eta\Psi^\dagger\Phi^{\dagger -1}] & 1 \\ -1 & [\Phi^{-1}\,\Psi\eta] \end{bmatrix}^{-1} \begin{bmatrix} [\bar{\Phi}^{-1}]\,\bar{\xi} \\ [\Phi^{-1}]\,\xi \end{bmatrix}\}.
\end{aligned}$$

In (5.8) $\det(1+[\Phi^{-1}\,\Psi]\,[\Psi^\dagger\Phi^{\dagger -1}])$ is finite since Ψ is Hilbert Schmidt and the existence of

$$\begin{bmatrix} [\eta\Psi^\dagger\Phi^{\dagger -1}] & 1 \\ -1 & [\Phi^{-1}\,\Psi\eta] \end{bmatrix}^{-1}$$

is assured by the assumption $||\Phi^{-1}\,\Psi|| < 1$ and Lemma 4.3.

We set

$$\begin{bmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{bmatrix} = \begin{bmatrix} [\Phi^{\dagger -1}\Psi'\eta] & 0 \\ 0 & [\eta\bar{\Psi}\Phi^{-1}] \end{bmatrix} + \begin{bmatrix} [\Phi^{\dagger -1}] & 0 \\ 0 & [\Phi'^{-1}] \end{bmatrix} \begin{bmatrix} [\eta\Psi^\dagger\Phi^{\dagger -1}] & 1 \\ -1 & [\Phi^{-1}\Psi\eta] \end{bmatrix}^{-1} \begin{bmatrix} [\bar{\Phi}^{-1}] & 0 \\ 0 & [\Phi^{-1}] \end{bmatrix}.$$

Then,

$$(5.8) = |C|^2 \{\det(1 + P\Phi^{-1}\,\Psi P \Psi^\dagger \Phi^{\dagger -1} P)\}^{1/2} \exp\{-\frac{1}{2}\,(\bar{\xi}, \xi) \begin{bmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ \xi \end{bmatrix}\}$$

Now, it can be seen that

$$\frac{\partial}{\partial \xi_{i_1}} \cdots \frac{\partial}{\partial \xi_{i_k}} \frac{\partial}{\partial \bar{\xi}_{i_k}} \cdots \frac{\partial}{\partial \bar{\xi}_{i_1}} \exp\{-\frac{1}{2}\,(\bar{\xi}, \xi) \begin{bmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ \xi \end{bmatrix}\}_{|\xi = \bar{\xi} = 0}$$

is a polynomial of $A^{(j)}_{i_r i_s}$, $1 \leq j \leq 4$, $1 \leq r, s \leq k$. So, if these entries of $A^{(j)}$ are standard complex numbers then it is obvious that the norm $||Uh||^2$ is finite and the proof will be completed.

Consider the two operators defined by

$$B=\begin{bmatrix} B^{(1)} & B^{(2)} \\ B^{(3)} & B^{(4)} \end{bmatrix}=\begin{bmatrix} P\Phi^{\dagger-1}\Psi'\eta P & 0 \\ 0 & P\eta\overline{\Psi}\Phi^{-1}P \end{bmatrix}$$
$$+\begin{bmatrix} P\Phi^{\dagger-1}P & 0 \\ 0 & P\Phi'^{-1}P \end{bmatrix}\begin{bmatrix} P\eta\Psi^{\dagger}\Phi^{\dagger-1}P & 1 \\ -1 & P\Phi^{-1}\Psi\eta P \end{bmatrix}^{-1}\begin{bmatrix} P\overline{\Phi}^{-1}P & 0 \\ 0 & P\Phi^{-1}P \end{bmatrix}$$

and

$$C=\begin{bmatrix} C^{(1)} & C^{(2)} \\ C^{(3)} & C^{(4)} \end{bmatrix}=\begin{bmatrix} \Phi^{\dagger-1}\Psi'\eta & 0 \\ 0 & \eta\overline{\Psi}\Phi^{-1} \end{bmatrix}$$
$$+\begin{bmatrix} \Phi^{\dagger-1} & 0 \\ 0 & \Phi'^{-1} \end{bmatrix}\begin{bmatrix} \eta\Psi^{\dagger}\Phi^{\dagger-1} & 1 \\ -1 & \Phi^{-1}\Psi\eta \end{bmatrix}^{-1}\begin{bmatrix} \overline{\Phi}^{-1} & 0 \\ 0 & \Phi^{-1} \end{bmatrix}$$

The existence of two inverses in the right hand sides in the above equalities is assured by $||\Phi^{-1}\Psi||<1$ and Lemma 4.3. It is easy to see that these inverses restricted to $\mathcal{H}\oplus\mathcal{H}$ are identical. What is more, in calculation of Bf for $f\in\mathcal{H}\oplus\mathcal{H}$, P's in the definition of B act as 1. Thus, for any $f\in\mathcal{H}\oplus\mathcal{H}$, $Bf=Cf$ holds, and so, by Lemma 4.4, $[B^{(k)}]_{ij}=[C^{(k)}]_{ij}$ for $i,j\in\boldsymbol{N}$. But $[C^{(k)}]_{ij}$ for $i,j\in\boldsymbol{N}$ are standard complex numbers. On the other hand $A_{ij}^{(k)}=[B^{(k)}]_{ij}$. So $A_{ij}^{(k)}$ with $i,j\in\boldsymbol{N}$ are standard complex numbers. This completes the proof.

Propositton 5.7. *Let Ψ be a Hilbert Schmidt operator and assume that Φ and Ψ commute with η. Then the operator U is an isometric operator on $\mathcal{G}\{e_i\}$ with respect to the inner product* $(\cdot,\cdot)$.

Proof. Let T be the operator defined by (4.7). Then we have

$$\begin{aligned}\det[T] &= \det(1+P\eta\Psi^{\dagger-1}\Phi^{\dagger-1}P\eta\Phi^{-1}\Psi P) \\ &= \det(1+(P\Phi^{-1}\Psi P)^{\dagger}(P\Phi^{-1}\Psi P))>0\,,\end{aligned}$$

where we used the relation (4.5). This shows that the existence of an inverse $[T]^{-1}$ of $[T]$ is proved without assuming $||\Phi^{-1}\Psi||<1$. $U(\bar{z},w)$ of (4.1) satisfies the condition

$$U(\bar{z},\eta w)=U(\eta\bar{z},w)\,.$$

Therefore we have, for $f\in\mathcal{G}\{e_i\}$,

$$\begin{aligned}(U\Lambda f)(\bar{z}) &= \int U(\bar{z},w)f(\eta\overline{w})\,e^{-w\overline{w}}\prod_{i=1}^{n}dw_i\,d\overline{w}_i \\ &= \int U(\bar{z},\eta u)f(\bar{u})\,e^{-u\bar{u}}\prod_{i=1}^{n}du_i\,d\bar{u}_i=(\Lambda Uf)(\bar{z})\,,\end{aligned}$$

where we used the change of variables:

$$u_i = \sum_{j=1}^{n} [\eta]_{ij} w_j , \quad \bar{u}_j = \sum_{j=1}^{n} [\eta]_{ij} \bar{w}_j .$$

The relation (4.3) shows that the operator U is isometric on $\mathcal{Q}\{e_i\}$.

6. Intertwining property

In this section we show that the operator U defined in §4 impliments the linear canonical transformation (1.6) for $f=f_i$, $i \in \boldsymbol{N}$ (Proposition 6.2). We begin with the following lemma.

Lemma 6.1. *Let A, B and C be operators on F such that*

$$B' A' f = C' f$$

for all $f \in \mathcal{H}$, then

$$\sum_{k=1}^{n} [A]_{ik} [B]_{kj} = [C]_{ij}$$

for $i \in \boldsymbol{N}$, $1 \leq j \leq n$.

Proof. Since $f_i \in \mathcal{H}$ for $i \in \boldsymbol{N}$,

$$\begin{aligned}[C]_{ij} &= \langle f_i, Cf_j \rangle = \langle C' f_i, f_j \rangle = \langle B' A' f_i, f_j \rangle = \langle A' f_i, Bf_j \rangle \\ &= \sum_{k=1}^{n} \langle A' f_i, f_k \rangle \langle f_k, Bf_j \rangle = \sum_{k=1}^{n} \langle f_i, Af_k \rangle \langle f_k, Bf_j \rangle = \sum_{k=1}^{n} [A]_{ik} [B]_{kj} .\end{aligned}$$

As in §3 we set $a_i = a(f_i)$ and $a_i^{(\Lambda)} = a^{(\Lambda)}(f_i)$ and further we set

$$\begin{aligned} b_i &= b(f_i) = a(P\Phi' Pf_i) + a^{(\Lambda)}(P\Psi' Pf_i) \\ b_i^{(\Lambda)} &= b^{(\Lambda)}(f) = a(P\Psi^\dagger Pf_i) + a^{(\Lambda)}(P\Phi^\dagger Pf_i) \end{aligned}$$

in accordance with (1.6). Since a_i corresponds to the left differentiation $\partial/\partial\bar{z}_i$ and $a_i^{(\Lambda)} = a^\dagger(P\eta\, Pf_i)$ is the left multiplication by $\sum_{j=1}^{n} [\eta]_{ij} \bar{z}_j$, we see that b_i and $b_i^{(\Lambda)}$ correspond to

$$[\Phi]_{ij} \frac{\partial}{\partial \bar{z}_j} + [\Psi]_{ik} [\eta]_{kj} \bar{z}_j$$

and

$$[\bar{\Psi}]_{ij} \frac{\partial}{\partial \bar{z}_j} + [\bar{\Phi}]_{ik} [\eta]_{kj} \bar{z}_j ,$$

where we used the Einstein's asummation convention. Let $U(\bar{z}, w)$ be the kernel of U defined by (4.1). Then, by (2.3) and (2.4) we calculate the kernels corresponding to the operators $b_i U$ and $b_i^{(\Lambda)}$ for $i \in \boldsymbol{N}$.

$$(6.1)\quad \begin{aligned} b_i U &\leftrightarrow \{[\Psi]_{ij}[\eta]_{jk}\bar{z}_k-[\Phi]_{ij}[\Phi^{-1}]_{jk}w_k-[\Phi]_{ij}[\Phi^{-1}\Psi\eta]_{jk}\bar{z}_k\}\ U(\bar{z},w) \\ &= -w_i U(\bar{z},w) = -U(\bar{z},w)\,w_i \leftrightarrow Ua_i \end{aligned}$$

$$(6.2)\quad \begin{aligned} b_i^{(\Lambda)} U &\leftrightarrow \{-[\overline{\Psi}]_{ij}[\Phi^{-1}\Psi\eta]_{jk}\bar{z}_k-[\overline{\Psi}]_{ij}[\Phi^{-1}]_{jk}w_k+[\overline{\Phi}]_{ij}[\eta]_{jk}\bar{z}_k\}\ U(\bar{z},w) \\ &= \{-[\overline{\Psi}]_{ij}[\Phi^{-1}]_{jk}w_k+\bar{z}_k[\Phi'^{-1}]_{kj}[\eta]_{ij}\}\ U(\bar{z},w) \\ &= U(\bar{z},w)\,[\eta]_{ij}\frac{\partial}{\partial w_j} \leftrightarrow Ua_i^{(\Lambda)}\,. \end{aligned}$$

Here $\frac{\partial}{\partial w_j}$ denotes right differentiation and we used Lemma 6.1 and the relation

$$-\overline{\Psi}\Phi^{-1}\Psi\eta+\overline{\Phi}\eta = \eta\Phi'^{-1}$$

which follows from (1.11). Thus we have the following proposition.

Proposition 6.2. *For $i\in \boldsymbol{N}$, we have*

$$b_i U = Ua_i$$
$$b_i^{(\Lambda)} U = Ua_i^{(\Lambda)}\,.$$

Proof. It is obvious from (6.1) and (6.2).

7. Standard Theorems

In this section we define a unitary operator U_1 on the standard Fock space $\mathcal{F}(\mathcal{H})$ by using the internal operator U on $\mathcal{G}(F)$ defined in §4, and prove the main theorem (Theorem 7.7). At the end of this section we give two examples.

For $h\in\mathcal{G}\{e_i\}$, Uh is, by Proposition 5.6, a near standard point of $^*\mathcal{F}(\mathcal{H})$. So we define an operator $U_{\{e_i\}}: \mathcal{G}\{e_i\}\to\mathcal{F}(\mathcal{H})$ by

$$U_{\{e_i\}}\,h = {}^\circ(Uh)\quad \text{(the standard part of } Uh)$$

Taking for each $\{e_i\}$, an internal real orthonormal basis $\{f_i\}$ through Theorem 3.1 satisfying the condition

(C) For some $l\in{}^*\boldsymbol{N}\backslash\boldsymbol{N}$ $(l\leq n)$, $f_i=e_i$ for $i=1, 2, \cdots, l$,

we can form $U_{\{e_i\}}$ for each complete real orthonormal basis $\{e_i\}$.

As a special case of Proposition 7.2 which will be stated later, we will see that, for a fixed $\{e_i\}$, $\Psi_0(\bar{z})\,e^{-\phi(\xi,\bar{z})}$ is invariant under the change of $\{f_i\}$ satisfying the condition (C), the operator $U_{\{e_i\}}$ depends only on $\{e_i\}$ and not on the choice of $\{f_i\}$. So, the notation $U_{\{e_i\}}$ is justified.

As we see in §3, if we fix an internal real orthonormal basis $\{f_j\}$ of F then there exists a natural correspondence between $\mathcal{G}(F)$ and $H_a^\infty(X_n)$. We assume that the variables $\{\bar{z}_j\}$ $(\{\xi_j\})$ are changed by $\bar{z}_i=\Sigma_j a_{ij}\bar{z}'_j$ $(\xi_i=\Sigma_j a_{ij}\xi'_j)$ in accordance with the change of real orthonormal basis $f_i=\Sigma_j a_{ij}f'_j$. Then we have the following lemma.

Lemma 7.1. *Let A be an operator on F. Then $\bar{z}[A]\bar{z}$ is invariant under the change of real orthonormal basis of F, that is, let $[A]'_{ij}=\langle f'_j, Af'_j\rangle$ then $\bar{z}[A]\bar{z}=\bar{z}'[A]'\bar{z}'$.*

The lemma is readily verified and we omit the proof.

Proposition 7.2. *$\Psi_0(\bar{z})\, e^{-\phi(\xi,\bar{z})}$ of (5.6) is invariant under the change of the real orthonormal basis of F.*

Proof. From Lemma 7.1, $\Psi_0(\bar{z})$ is invariant. Similarly $e^{-\phi(\xi,\bar{z})}$ is also invariant.

Lemma 7.3. *Let $\{e_i\}$, $\{e'_i\}$ be two complete real orthonormal bases of $\mathcal{H}$ and $h\in\mathcal{G}\{e_i\}\cap\mathcal{G}\{e'_i\}$. Then*

$$U_{\{e_i\}}\, h = U_{\{e'_i\}}\, h\,.$$

Proof. Let $h=f_{i_1\wedge}\cdots{}_\wedge f_{i_k}=\Sigma\, a_{i_1j_1}\cdots a_{i_kj_k} f'_{j_1\wedge}\cdots{}_\wedge f'_{j_k}$. Then we have

$$\begin{aligned}(Uh)(\bar{z}') &= \Sigma\, a_{i_1j_1}\cdots a_{i_kj_k}(\partial/\partial\xi'_{j_1})\cdots(\partial/\partial\xi'_{j_k})\,\Psi_0(\bar{z}')\, e^{-\phi(\xi',\bar{z}')}{}_{|\xi'=0}\\ &= (\partial/\partial\xi_{i_1})\cdots(\partial/\partial\xi_{i_k})\,\Psi_0(\bar{z})\, e^{-\phi(\xi,\bar{z})}{}_{|\xi=0} = (Uh)(\bar{z})\,,\end{aligned}$$

where we used Proposition 7.2 and the relation $\partial/\partial\xi_i=\Sigma_j\, a_{ij}\,\partial/\partial\xi'_j$ which follows from the chain rule.

DEFINITION 7.4. Let h be any element of $\mathcal{G}(\mathcal{H})$. Then there exists a $\mathcal{G}\{e_i\}$ containing h. We define

$$U_1\, h = U_{\{e_i\}}\, h\,.$$

The above lemma assures that the operator U_1 on $\mathcal{G}(\mathcal{H})$ is well defined.

Proposition 7.5. *The operator U_1 of Definition 7.4 satisfies the condition:*

$$(\Lambda h, g) = (\Lambda U_1\, h,\, U_1\, g) \tag{7.1}$$

for $h, g\in\mathcal{G}(\mathcal{H})$.

Proof. Let $h=h_{1\wedge}\cdots{}_\wedge h_j$, $g=g_{1\wedge}\cdots{}_\wedge g_k$. We can choose the generators $\{e_i\}$ such that h_p, $\eta g_q (1\leq p\leq j,\ 1\leq q\leq k)$ belong to $\mathcal{G}\{e_i\}$. Since $h, \Lambda g\in\mathcal{G}\{e_i\}$, Proposition 4.2 shows that

$$(h, \Lambda g) = (\Lambda Uh,\, Ug)\,.$$

Hence (7.1) holds.

Proposition 7.6. *The operator U_1 of Definition 7.4 satisfies the condition*

$$\begin{aligned} b(f)\, U_1\, h &= U_1\, a(f)\, h\\ b^{(\Lambda)}(f)\, U_1\, h &= U_1\, a^{(\Lambda)}(f)\, h\end{aligned} \tag{7.2}$$

for $f \in \mathcal{H}$ *and* $h \in \mathcal{Q}(\mathcal{H})$.

Proof. If we choose the basis $\{e_i\}$ such that f, ηf, $\Phi' f$, $\eta\Psi' f$, $\Psi^\dagger f$, $\eta\bar{\Phi} f$ are finite linear combinations of $\{e_i\}$ and h belongs to $\mathcal{Q}\{e_i\}$, then the relation

$$b(f)\,Uh = Ua(f)\,h$$
$$b^{(\Lambda)}(f)\,Uh = Ua^{(\Lambda)}(f)\,h$$

follows from Proposition 6.2.

Theorem 7.7. *The linear canonical transformation* (1.6) *is weakly* Λ-*unitarily impiementable* (*Definition* 1.2) *if* Ψ *is a Hilbert Schmidt operator and* $\|\Phi^{-1}\Psi\|<1$.

Proof. The Theorem follows from the Propositions 7.5 and 7.6.

REMARK 7.8. The assumption $\|\Phi^{-1}\Psi\|<1$ in Theorem 7.7 is used only to assure the existence of $[T]^{-1}$ in Proposition 4.2.

Theorem 7.9. *The linear canonical transformation* (1.6) *is implementable by a unitary and* Λ-*unitary operator* (*Definition* 1.1) *if* Ψ *is a Hilbert Schmidt operator and* η *commutes with* Φ *and* Ψ.

Proof. By Proposition 5.7, we can define an isometric operator U_1, without assuming the condition $\|\Phi^{-1}\Psi\|<1$. In this case, as we mentioned in the proof of Proposition 5.7, the existence of $[T]^{-1}$ is assured without the assumption $\|\Phi^{-1}\Psi\|<1$, and (7.1) and (7.2) follow (see Remark 7.8). The unitarity of U_1 follows from the existence of isometric operator V_1 which implements the inverse canonical transformation of (1.6) and satisfies $U_1 V_1=1$.

REMARK 7.10. Let Φ be a unitary operator and $\Psi=0$. Then (1.6) is a linear canonical transformation if and only if η and Φ commute. This special form of linear canonical transformation appeared in Nagamachi, S. and N. Mugibayashi [7], and it is unitarily and Λ-unitarily implementable by Theorem 7.9.

Now, we give examples.

EXAMPLE 7.11. Let l^2 be the Hilbert space of sequences $x=(x_1, x_2, \cdots, x_k, \cdots)$ of complex numbers with $\sum_{k=1}^{\infty}|x_k|^2<\infty$ and consider the complete orthonormal system $e_1=(1, 0, 0, \cdots)$, $e_2=(0, 1, 0, \cdots)$, $\cdots$ of l^2. Let ϕ and ψ be the bounded operators on l^2 defined by

$$\phi\colon e_k \to (1+1/k^2)^{1/2}\, e_k\,, \quad \psi\colon e_k \to k^{-1}\, e_k\,.$$

Let $\mathcal{H}=l^2\oplus l^2$. Define operators η, Φ and Ψ on $\mathcal{H}$ as follows: for $f=(g, h)\in\mathcal{H}$,

$$\eta f=(g, -h)\,, \quad \Phi f=(\phi g, \phi h)\,, \quad \Psi f=(\psi h, \psi g)\,.$$

It is readily seen that (Φ, Ψ) defines a canonical transformation, i.e., $\Phi\eta\Phi^{\dagger}+\Psi\eta\Psi^{\dagger}=\eta$, $\Phi\eta\Psi'+\Psi\eta\Phi'=0$.

Let $f_{2k-1}=2^{-1/2}(e_k, -e_k)$, $f_{2k}=2^{-1/2}(e_k, e_k)$, $k=1, 2, \cdots$. Then $\{f_j\}$ is a complete orthonormal system of $\mathcal{H}$ satisfying

$$\Phi f_{2k-1} = (1+1/k^2)^{1/2} f_{2k-1}\,, \quad \Phi f_{2k} = (1+1/k^2)^{1/2} f_{2k}\,,$$
$$\Psi f_{2k-1} = (-1/k) f_{2k-1}\,, \quad \Psi f_{2k} = (1/k) f_{2k}\,.$$

Ψ is a Hilbert-Schmidt operator and $\|\Phi^{-1}\Psi\|<1$ since eigenvalues of $\Phi^{-1}\Psi$ are $\pm(1+k^2)^{-1/2}$. Thus, all the assumptions of Theorem 7.7 are satisfied and there exists a weakly Λ-unitary operator U which implements the canonical transformation (1.6). Let

$$\Psi_0 = [\prod_{k=1}^{\infty}(1-\frac{1}{1+k^2})]^{1/4}\exp\{-\sum_{k=1}^{\infty}(1+k^2)^{-1/2} f_{2k}\wedge f_{2k-1}\}\,.$$

Then from Proposition 5.6, (5.5), (5.6) and (5.7), we have

$$U f_{2j_1}\wedge\cdots\wedge f_{2j_k} = \{\prod_{i=1}^{k}\frac{j_i}{(1+j_i^2)^{1/2}}\} f_{2j_1}\wedge\cdots\wedge f_{2j_k}\wedge\Psi_0\,.$$

This coincides with

$$b^{(\Lambda)}(\eta f_{j_1})\cdots b^{(\Lambda)}(\eta f_{j_k})\,\Psi_0$$

showing the intertwining property. In fact we have

$$\begin{aligned} b^{(\Lambda)}(\eta f_{2j})\,\Psi_0 &= [a^{(\Lambda)}(\Phi^{\dagger}\eta f_{2j})+a(\Psi^{\dagger}\eta f_{2j})]\,\Psi_0 \\ &= ((1+\frac{1}{j^2})^{1/2}-\frac{1}{j(1+j^2)^{1/2}}) f_{2j}\wedge\Psi_0 = \frac{j}{(1+j^2)^{1/2}} f_{2j}\wedge\Psi_0\,. \end{aligned}$$

The following example shows that the unbounded operator is necessary to implement a certain canonical transformation.

EXAMPLE 7.12. Let L be a Hilbert space. Define a Hilbert space $\mathcal{H}=L\oplus L$ and an operator η on $\mathcal{H}$ such that $\eta=1$ on $L\oplus\{0\}$ and $\eta=-1$ on $\{0\}\oplus L$. Let Φ be an operator on $\mathcal{H}$ such that

$$\Phi\colon (g, h)\to((1+i)g+h, g+(1-i)h)\,,$$

then $\Phi^{\dagger}\eta\Phi=\eta$. This shows that the operator Φ defines a canonical transformation:

$$b(f) = a(\Phi' f)\,, \quad b^{(\Lambda)}(f) = a^{(\Lambda)}(\Phi^{\dagger} f)\,.$$

This canonical transformation is implementable by a weakly Λ-unitary operator U, since all the assumptions of Theorem 7.7 are satisfied. In fact, let $\{f_i\}$ be an orthonormal basis of $\mathscr{H}$. Then we have

$$Uf_{i_1\wedge}\cdots{}_\wedge f_{i_k}=\eta\Phi^\dagger\eta f_{i_1\wedge}\cdots{}_\wedge\eta\Phi^\dagger\eta f_{i_k}$$

by (5.6) and (5.7). This shows that U maps $h_{1\wedge}\cdots{}_\wedge h_j$ to $\eta\Phi^\dagger\eta h_{1\wedge}\cdots{}_\wedge\eta\Phi^\dagger\eta h_k$ for $h_j\in\mathscr{H}$. It is easily seen that

$$Ua(f)\,h=b(f)\,Uh\,,\quad Ua^{(\Lambda)}(f)\,h=b^{(\Lambda)}(f)\,Uh\,,$$
$$(Uh,\,\Lambda Uh')=(h,\,\Lambda h')$$

for $h=h_{1\wedge}\cdots{}_\wedge h_k$, $h'=h'_{1\wedge}\cdots{}_\wedge h'_k$, i.e., U is Λ-isometric and implements the canonical transformation. The eigenvalues of $\eta\Phi\Phi^\dagger\eta$ are $3+2\cdot 2^{1/2}$ and $3-2\cdot 2^{1/2}$. Let h_j be the eigenvectors of $\eta\Phi\Phi^\dagger\eta$ whose eigenvalues are $3+2\cdot 2^{1/2}$. Then we have

$$\begin{aligned}(Uh,\,Uh)&=\det((h_i,\,\eta\Phi\Phi^\dagger\eta h_j))\\&=(3+2\cdot 2^{1/2})^k\det((h_i,\,h_j))=(3+2\cdot 2^{1/2})^k(h,\,h)\,.\end{aligned}$$

This shows that U is an unbounded operator. Thus, U is a weakly Λ-unitary operator.

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