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Osaka J. Math.
29 (1992), 471–481

CONVERGENCE TO EQUILIBRIA FOR A CLASS OF REACTION-DIFFUSION SYSTEMS

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(Received June 1, 1991)

1. Introduction

An important question in the study of reaction-diffusion systems is the identification of all steady states and the classification of their stability. In this note, we give a contribution to this question for a class of systems with two diffusing and reacting components.

$$\partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2) \qquad (i=1,2) \tag{1.1}$$

in some cylindrical time-space domain $\Omega \times (0, \infty) \subset \boldsymbol{R}^{N+1}$, together with zero Neumann boundary data on the lateral boundary $\partial\Omega \times (0, \infty)$ and suitable initial data on $\Omega \times \{0\}$. Then all equilibria of the associated system of ordinary differential equations

$$\dot{y}_i = f_i(y_1, y_2) \qquad (i=1,2) \tag{1.2}$$

are also spatially constant steady states of (1.1). Consider now the special case

$$f_i(u_1, u_2) = u_i(a_{i0} - a_{i1}u_1 - a_{i2}u_2) \tag{1.3}$$

with $a_{ij} > 0$ for all i,j. In this case, it turns out that the unique positive equilibrium of (1.2) (if it exists) is globally asymptotically stable for (1.1) if and only if it is so for (1.2). This will be the case if and only if

$$\frac{a_{11}}{a_{21}} > \frac{a_{10}}{a_{20}} > \frac{a_{12}}{a_{22}}\ ; \tag{1.4}$$

see [12] for a detailed study and for results on other possible combinations of coefficients.

Note that in this case $\dfrac{\partial f_i}{\partial u_j} \leq 0$ for $i \neq j$; such a vector field is called *competitive.* On the other hand, it was shown in [11] that for competitive vector

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* Supported by a grant from the National Science Foundation and by the Deutsche Forschungsgemeinschaft (SFB 256).

field with two zeroes that are asymptotically stable for (1.2), non-constant stable steady states of (1.1) can exist, if the domain Ω and the diffusion coefficients d_1, d_2 are chosen appropriately. This "pattern formation" phenomenon cannot occur if Ω is convex ([8]) or if the diffusion coefficients are large ([2]).

The purpose of this note is a study of a certain borderline case in which the zero set Z of the vector field $\vec{f}$ in the first quadrant

$$(1.5) \qquad Z = \{(u_1, u_2) \mid u_1, u_2 \geq 0,\ f_i(u_1, u_2) = 0 \quad \text{for} \quad i = 1, 2\}$$

is a continuum. An example is the vector field (1.3) with all inequalities in (1.4) replaced by equality. In this case, the zero set Z consists of the coordinate axes and a straight line that divides the positive quadrant $\boldsymbol{R}_+^2 = [0, \infty) \times [0, \infty)$ into two components, one of which is bounded, and all these zeroes are stable equilibira of (1.2).

We shall discuss two possible generalizations oi this example and give sufficient conditions that imply that every solution of (1.1) converges to a unique spatially constant solution with values in Z. Our first result uses the maximum principle and is given in section 2; our second result uses Lyapunov functions and is given in section 3. It will turn out that the vector field $\vec{f}$ does not have to be competitive for either generalization, but the competitive case is included in both results. In section 4, we discuss the stability of these steady states and give comments on extensions to systems with more than two components and on relations to recent work on order preserving semiflows. For other results on the asymptotic behavior of solutions of reaction-diffusion systems, the reader is referred to the bibliographies in [4], [10], [18], [19]. For some related work on predator-prey systems of the form (1.1) with vector fields of the form (1.3), see [15]. For other uses of Lyapunov techniques in the study of parabolic systems, see e.g. [1] and [14].

To conclude this introduction, we list the main assumptions and give a basic existence and uniqueness result that serves as the framework of what follows. The set $\Omega \subset \boldsymbol{R}^N$ is an open and bounded domain with $C^{2+\alpha}$-boundary $\partial\Omega$ and unit exterior normal vector field n. The constants d_i are positive; after rescaling, we may assume that $d_1 = 1$ and $d_2 = d > 0$. Problem (1.1) is considered together with zero Neumann boundary conditions

$$(1.6) \qquad \partial_n u_i = 0 \qquad (i = 1, 2)$$

on the lateral boundary $\partial\Omega \times (0, \infty)$ and initial conditions

$$(1.7) \qquad u_i(\cdot, 0) = u_i^0 \qquad (i = 1, 2)$$

on Ω. The initial data are always assumed to be non-negative and not identically zero. The vector field $\vec{f}$: $\boldsymbol{R}_+^2 \to \boldsymbol{R}_+^2$ is of class C^1. The zero set Z is assumed to be of the form $Z = (\{0\} \times (0, \infty)) \cup ((0, \infty) \times \{0\}) \cup Z_1$ with

$$Z_1 = \{(u_1, u_2) \mid u_i > 0,\ f_i(u_1, u_2) = 0 \quad (i = 1, 2)\}. \tag{1.8}$$

We also assume that there exists a number $M > 0$ such that $f_i(u_1, u_2) < 0$ if $u_1 \geq M$ or $u_2 \geq M$.

Lemma 1. *For $(u_1^0, u_2^0) \in L^\infty(\Omega, \boldsymbol{R}_+^2)$ there exists a unique solution $u = u(\cdot\,; u^0)$: $\bar{\Omega} \times [0, \infty) \to \boldsymbol{R}_+^2$ of (1.1), (1.6), (1.7) for which $\partial_{x_i}\partial_{x_j}u$ and $\partial_t u$ are Hölder continuous on $\bar{\Omega} \times (0, \infty)$. Let $u(\cdot, t; u^0)$ denote the restriction of $u(\cdot\,; u^0)$ to $\Omega \times \{t\}$, then $[0, \infty) \in t \to u(\cdot, t; u^0)$ is a continuous curve in $L^p(\Omega, \boldsymbol{R}^2)$ for any $p < \infty$. The components u_i are strictly positive on $\bar{\Omega} \times (0, \infty)$ and essentially uniformly bounded on this set.*

Sketch of proof. Choose $p > n$ and consider (1.1), (1.6), (1.7) as an abstract semilinear evolution equation in the Banach space $X = L^p(\Omega, \boldsymbol{R}^2)$, after continuing the vector field $\vec{f}$ on all $\boldsymbol{R}^2$ in a C^1-fashion. By standard arguments (see [6]), the problem has a unique local mild solution which is Hölder continuous for positive times. Applying the regularity thoery for parabolic equations again, it follows that also the time derivative and all second spatial derivatives of the solution are Hölder continuous up to the boundary of Ω for all positive times (see [9]). The positivity of both components follows from the strong maximum principle (see [3]). Using a large invariant rectangle, a uniform bound for the solution components can be given which implies global existence of the solution (see [19]). Q.E.D.

2. Convergence to Constant Equilibria I

In this section, we discuss (1.1), (1.6), (1.7) under the following assumption:

There exist $\beta > 0$ and γ: $[0, \beta] \to [0, \infty)$, continuous, strictly decreasing, with $\gamma(\beta) = 0$, such that the set Z_1 is given by

$$Z_1 = \{(u_1, u_2) \mid u_1, u_2 > 0,\ 0 < u_1 < \beta,\ u_2 = \gamma(u_1)\}. \tag{2.1}$$

Both f_i are negative on $\{(u_1, u_2) \mid u_i > 0,\ u_2 > \gamma(u_1)\}$ and positive on the complement of this set in the first quadrant.

We still assume throughout that both components of the initial data u^0 are non-negative and not identically zero.

Theorem 2.1. *If (2.1) holds, then every solution $u(\cdot\,; u^0)$ of (1.1), (1.6), (1.7) converges to some spatially constant vector (ρ_1, ρ_2) as $t \to \infty$, uniformly on $\bar{\Omega}$, together with its first and second spatial derivatives. Here either $\rho_1 = 0$ and $\rho_2 \geq \gamma(0)$,* *or* $0<\rho_1<\beta$ *and* $\rho_2=\beta(\rho_1)$, *or* $\rho_1\geq\beta$ *and* $\rho_2=0$.

Proof. Define the usual ω-limit set

$$\omega(u^0) = \{\phi\in L^p(\Omega, \boldsymbol{R}^2)\,|\,u(\cdot, t_k; u^0)\to\phi \text{ in } L^p \text{ as } t_k\to\infty \text{ for some sequence } t_k\}.$$

Then $\omega(u^0)$ is a compact non-empty subset of $C^2(\bar{\Omega})$ and independent of p, and the convergence is actually in $C^2(\bar{\Omega})$ (see [16]). We must show that $\omega(u^0)$ is a singleton with two constant components. Thus let $v=(v_1, v_2)\in\omega(u^0)$ and set $\alpha=\max v_2$, $\delta=\max v_1$.

Case 1: $\alpha>\gamma(0)$. We claim that in this case $\omega(u^0)=\{v^*\}=\{(0, \alpha)\}$ on $\bar{\Omega}$. To show this claim, suppose first that v_2 is not identically equal to α. By the strong maximum principle, the second component of $u(\cdot\,; v^*)$ is strictly less than α for all positive t. Since $u(\cdot\,; v^*)\in\omega(u^0)$ for all t, the solution $u(\cdot, t; u^0)$ must therefore have its range in some rectangle $[0, M_1]\times[0, \alpha-\varepsilon]$ for some positive ε and for some positive t. Such a rectangle is invariant, if M_1 is large, and therefore $v^*\notin\omega(u^0)$, contradicting the assumption. Thus $v_2=\alpha$. The same argument shows that $v_1=0$. It remains to show that v^* is the only element of $\omega(u^0)$. To see this, let $w^*=(w_1, w_2)\in\omega(u^0)$, and set $\tilde{\alpha}=\max w_2$. Then $\tilde{\alpha}=\alpha$, since otherwise an invariant rectangle could be found that contains one of the two solutions but not the other. By the previous argument, $w^*=(0, \alpha)=v^*$, which proves everything in this case.

Case 2: $\delta>\beta$. In this case, $v^*=(\delta, 0)$ and $\omega(u^0)=\{v^*\}$. The proof is as in case 1.

Case 3: $0\leq\alpha\leq\gamma(0)$, $0\leq\delta\leq\beta$. This is the remaining case. Let $[A, B]\times[C, D]$ be the smallest rectangle with sides parallel to the coordinate axes and corners (A, D) and (B, C) on the graph of γ that still contains the range of v^*. Then $\delta\leq B\leq\beta$ and $\alpha\leq D\leq\gamma(0)$. All such rectangles are invariant. If this rectangle were not a single point, then by the strong maximum principle the function $u(\cdot, t; v^*)$ would have its range in a strictly smaller rectangle with the same properties, as soon as $t>0$. Again, this implies that $v^*\notin\omega(u^0)$. Thus all elements of $\omega(u^0)$ must be constants on the graph of γ. Let v^* and w^* be two such elements of $\omega(u^0)$. If they were not the same, then one could find a small invariant rectangle about v^* that does not contain w^*. Thus $\omega(u^0)$ must reduce to a single such point. This proves the theorem completely. Q.E.D.

REMARK 1. In the above result, it cannot be excluded that the limit (ρ_1, ρ_2) lies on one of the coordinate axes. An example is given by a vector field that has the form

$$f_i(u_1, u_2) = -u_1(u_1+u_2-1) \tag{2.2}$$

for $i=1, 2$, near the point $P=(0, 1)$. For solutions of (1.2) with initial data (u_1^0, u_2^0) in a neighborhood of P, u_1-u_2 remains constant along solutions, and thus the limit will be $(\rho_1, \rho_2)=(0, u_2^0-u_1^0)$ if $u_2^0>u_1^0+1$ and $(\rho_1, \rho_2)=\left(\frac{1-c}{2}, \frac{1+c}{2}\right)$, $c=u_2^0-u_1^0$, if $u_2^0\leq u_1^0+1$. Obviously, solutions of (1.2) can be viewed as solutions of (1.1), (1.6), (1.7).

REMARK 2. A crucial assumption in the arguments above is that γ is strictly decreasing. More precisely, it was used to establish that in case 3, $\omega(u^0)$ contains only one element. All other arguments are still valid if γ is only non-increasing; in particular, all elements of $\omega(u^0)$ must still be componentwise constants with their range on the graph of γ. We suspect that the general result is still true in this case, but we can only prove it under additional assumptions such as

$$f\in C^2, \quad \frac{\partial f_2}{\partial u_2}\neq 0 \qquad ((u_1, u_2)\in Z). \tag{2.3}$$

By the implicit function theorem, γ is a C^2-curve in this situation, and writing $f_{ij}=\frac{\partial f_i}{\partial u_j}$, we must have $f_{i1}+\gamma' f_{i2}=0$ on Z_1. Note that necessarily $f_{22}<0$ on Z_1 due to (2.1) and (2.3). Suppose now that only $\gamma'\leq 0$. To show that $\omega(u^0)$ is a singleton, we first can exclude the possibility that $\omega(u^0)$ contains an element $v^*=(v_1, v_2)=(v_1, \gamma(v_1))$ with $\gamma'(v_1)<0$, since in that case arbitrarily small invariant rectangles about v^* can again be found. Since $\omega(u^0)$ is a continuum, it must therefore have the form $\omega(u^0)=[v_1, w_1]\times\{\gamma(v_1)\}$, with γ' vanishing on $[v_1, w_1]$. We now refer to the results in [5], in particular to the proof of Theorem 3.4 in that paper, which implies that $\omega(u^0)$ will be a singleton if the spectrum of the linearization of (1.1), (1.6), (1.7) at each equilibrium point contains 0 as an algebraic simple eigenvalue and if the remainder of the spectrum is bounded away from the imaginary axis. By the above arguments, we only have to show this at points $(v_1, \gamma(v_1))\in Z_1$ for which $\gamma'(v_1)=0$. Let $\mu_1=0<\mu_2\leq\mu_3\leq\cdots$ be the sequence of eigenvalues of $-\Delta$ with zero Neumann boundary values, then the linearization of (1.1), (1.6), (1.7) about any such point on Z_1 has the spectrum

$$\sigma=\{z\in C\,|\, z=-\mu_i \quad \text{or} \quad z=-d\mu_1+f_{22}(v_1, \gamma(v_1)) \quad \text{for some} \quad i>0\} \tag{2,.4}$$

as a computation shows. The required condition on the spectrum therefore holds in this situation, and $\omega(u^0)$ must be a singleton.

3. Convergence to Constant Equilibria. II

In this section, we present a convergence result for systems of the form (1.1) that have a zero set Z_1 wtith a more complicated structure. We assume in this section that

(3.1) $$f_i(u_1, u_2) = h_i(u_i)g(u_1, u_2) \qquad (i = 1, 2, u_i \geq 0).$$

The functions h_i and g are assumed to be continuously differentiable, and

(3.2) $$h_i(0) = 0, \quad h_i(v) > 0 \qquad (v > 0).$$

We assume further that there exist continuous functions k_1, k_2: $[0, \infty) \to \boldsymbol{R}$ such that

(3.3) $$Z_1 = \{(u_1, u_2) \mid u_i > 0, k_1(u_1) + k_2(u_2) = 0\}.$$

The following additional asumptions have to hold:

(3.4) $g(u_1, u_2)(k_1(u_1) + k_2(u_2)) \leq 0 \quad (u_i \geq 0)$, with equality only if $g(u_1, u_2) = 0$.

(3.5) $k_i(0) < 0 \quad (i = 1, 2)$, $k_1(v_1) + k_2(v_2) > 0$ for large v_1, v_2.

(3.6) The functions $z \to \dfrac{k_i(z)}{h_i(z)}$ are non-decreasing on $(0, \infty)$.

(3.7) Each k_i has exactly one zero.

Let $I \subset (0, \infty)$ be any open interval and x: $I \to (0, \infty)$ be a solution of the ordinary differential equation $h_1(s)\dot{x}(s) = h_2(x(s))$ on I, then

(3.8) $\{s \in I \mid k_1(s) + k_2(x(s)) = 0\}$ has empty interior.

A few remarks are in order.

1. The functions k_i need not be differentiable.

2. In general, the functions k_i are not uniquely defined. Clearly, both k_i can be modified for large arguments. Also, one can replace, e.g., $k_1(\cdot)$ by $k_1(\cdot) + \delta$ and $k_2(\cdot)$ by $k_2(\cdot) - \delta$, as long as (3.6) still holds. Thus condition (3.7) is not too restrictive.

3. Consitions (3.4) and (3.5) imply that g is positive near the origin and negative for large arguments, in agreement with the general assumptions in section 1.

4. Suppose now that the h_i are linear functions, $h_1(z) = z$ and $h_2(z) = \alpha z$ and that Z_1 is the graph of a function γ as in the previous section which we assume to be continuously differentiable. We want to choose

$$k_1(v) = \begin{cases} \varepsilon - \gamma(v) & 0 \leq v \leq \beta \\ \dfrac{\varepsilon}{\beta} v & v < \beta \end{cases}$$

and $k_2(v)=v-\varepsilon$ for some suitable ε. Then (3.6) and (3.7) will hold for all small ε if $\gamma'(\beta)<0$ and $\gamma(z)-z\gamma'(z)>0$ on $[0, \beta]$. In this case, (3.8) holds if every parabola of the form $u_2=Cu_1^\alpha$ with $C>0$ intersects Z_1 only at isolated points. However, for linear h_i, Z_1 need not be graph; for instance it can be the intersection of the boundary of any disk containing the origin with the first quadrant.

5. Using nonlinear functions h_i, it is not hard to construct examples in which Z_1 has more than one connected component.

6. The transversality condition (3.8) is a weak form of requiring that every equilibrium of the system (1.2) with the right hand side (3.1) has a one-dimensional center manifold.

As before we discuss (1.1), (1.6), (1.7) for non-negative initial data u^0 for which neither component vanishes identically.

Theorem 3.1. *If* (3.1)-(3.8) *hold, then for any* $u^0\in L^\infty(\Omega, \boldsymbol{R}_+^2)$ *the solution of* (1.1) (1.6), (1.7) *satisfies*

$$u(\cdot\ t; u^0) \to (\rho_1, \rho_2) \tag{3.9}$$

as $t\to\infty$, *uniformly in* $\bar{\Omega}$ *together with its first and second derivatives. Here* ρ_1 *and* ρ_2 *are positive constants and* $g(\rho_1, \rho_2)=0$.

Proof. We define the ω-limit set $\omega(u^0)$ as in the proof of Theorem 2.1 and must show that it is a singleton with two constant components $(\rho_1, \rho_2)\in Z_1$. Set

$$V_0(u_1, u_2) = \int_1^{u_1} \frac{k_1(r)}{h_1(r)}\, dr + \int_1^{u_2} \frac{k_2(r)}{h_2(r)}\, dr$$

for $u_1, u_2>0$. Then V_0 is continuously differentiable, $V_0\to+\infty$ if $u_1\downarrow 0$ or $u_2\downarrow 0$ (due to (3.2), (3.5)), and

$$\nabla V_0(u_1, u_2)\cdot\vec{f}(u_1, u_2)\le 0$$

for all $u_i>0$, with equality only if $g(u_1, u_2)=0$. We then define

$$V(\phi) = \int_\Omega V_0(\phi(x))\, dx$$

for any continuous function $\phi\colon \Omega\to(0, \infty)^2$. We want to use V as a Lyapunov functional for solutions of (1.1), (1.6), (1.7). To do this, we compute for $t>0$, writing $u(\cdot, t; u^0)=(u_1, u_2)$:

$$\frac{d}{dt}V(\cdot, t; u^0)) = \int_\Omega \Delta u_1 \frac{k_1(u_1)}{h_1(u_1)}\, dx + d\int_\Omega \Delta u_2 \frac{k_2(u_2)}{h_2(u_2)}\, dx + \int_\Omega \nabla V_0(u_1, u_2)\cdot \vec{f}(u_1, u_2)\, dx\,.$$

If $\frac{k_1}{h_1}$ and $\frac{k_2}{h_2}$ are smooth, then the first two integrals are non-positive, as an integration by parts shows. An approximation argument implies that this is also the case if these functions are only continuous and non-decreasing. The last integral is negative, unless (u_1, u_2) has all its values in the zero set Z_1. Thus V never increases along orbits with initial data u^0. In particular, $V(v^*)<\infty$ for any $v^*\in\omega(u^0)$, and thus v^* cannot have an identically vanishing component. By the results in [17], V is constant on $\omega(u^0)$, and this set is invariant under the solution flow of (1.1), (1.6), (1.7). This implies that

$$\nabla V_0(v_1, v_2)\cdot \vec{f}(v_1, v_2) = 0 \quad \text{on} \quad \bar{\Omega}$$

for any $v^*=(v_1, v_2)\in\omega(u^0)$. Thus any function in $\omega(u^0)$ must have values on Z_1. The backwards invariance of $\omega(u^0)$ now implies that such functions must be constants which cannot be zero.

It remains to show that $\omega(u^0)$ is a singleton. Thus let $v^*=(v_1, v_2)$ and $w^*=(w_1, w_2)$ be two different elements of $\omega(u^0)$. Without loss of generality, $v_1<w_2$. Then there exists a continuum of points $(\xi, \zeta)\in\omega(u^0)$ that connects v^* to w^*. Since V is constant on $\omega(u^0)$, this implies that

$$\int_1^\xi \frac{k_1(r)}{h_1(r)}\, dr + \int_1^\zeta \frac{k_2(r)}{h_2(r)}\, dr = \text{const}\,. \tag{3.10}$$

and

$$k_1(\xi)+k_2(\zeta) = 0 \tag{3.11}$$

for $v_2\le\zeta\le w_2$ and the corresponding ξ. Without loss of generality, $k_2(\zeta)\neq 0$ for $v_2\le\zeta\le w_2$. Then the solution set of (3.10) is locally the graph of a C^1-function p, and we obtain the differential equation

$$\frac{k_1(\xi)}{h_1(\xi)} + p'(\xi)\frac{k_2(p(\xi))}{h_2(p(\xi))} = 0 \tag{3.12}$$

on some ξ-interval. Using (3.11), it follows that

$$h_1(\xi)p'(\xi) = h_2(p(\xi)) \tag{3.13}$$

on some ξ-interval and that the solution p of this differential equation lies entirely in Z_1. This contradicts assumption (3.8). Therefore $\omega(u^0)$ must be a singleton, and the theorem is completely proved. Q.E.D

There is a systematic way to get the function V_0: Choose a C^1-function χ that has the same sign behavior as $-g$ and solve the first order partial differential equation

$$\frac{\partial V_0}{\partial u_1}\cdot h_1+\frac{\partial V_0}{\partial u_2}\cdot h_2=\chi\,. \tag{3.14}$$

If for such a solution the matrix

$$\mathrm{diag}(1,\mathrm{d})\cdot\nabla^2 V_2 \tag{3.15}$$

is positive semidefinite, then V_0 can serve as a Lyapunov functional. In the argument above, we took both χ and V_0 as sums of functions of u_1 and u_2 alone. In the special linear case $h_1(u_1)=u_1$, $h_2(u_2)=\alpha u_2$ with $\alpha>0$, the general solution of (3.14) is given by

$$V_0(u_1,u_2)=\phi\left(\frac{u_2}{u_1^\alpha}\right)+\int_{1/u_1}^{1}\frac{\chi(ru_1,r^\alpha u_2)}{r}\,dr\,, \tag{3.16}$$

where ϕ is an arbitrary C^1-function. It would be interesting to have more general conditions on ϕ and χ that imply that the matrix in (3.15) is positive semidefinite.

4. Additional Remarks

1. The results in section 2 remain true if the operator diag$(-\Delta,\,-d\Delta)$ is replaced by any diagonal second order elliptic operator that allows the use of the maximum principle. The results in section 3 remain true if this operator is replaced by a diagonal elliptic operator of divergence form.

2. The techniques of section 3 permit an obvious extension to systems with $M>2$ components and coupling terms

$$f_i(\vec{u})=h_i(u_i)g(\vec{u})\,,\quad (i=1,\cdots,M)\,.$$

The zero set of g should be given in the form

$$Z_1=\{\vec{u}>0\,|\,g(\vec{u})=0\}=\{\vec{u}\,|\,\sum_{i=1}^{M}k_i(u_i)=0\}\,,$$

and conditions (3.4)-(3.8) have to be generalized accordingly. It is not clear how the results of section 2 can be generalized to the case of $M>2$ components. The principal difficulty is that such a system will have very few invariant sets, if all diffusion coefficients are different and the coupling terms have a common zero set Z_1 that is, say, an $(M-1)$-dimensional hypersurface. We note that indeed few general results are known for competitive systems with more than two species, even in the absence of diffusion.

3. Although we showed that $\omega(u^0)$ is a subset of Z_1 and a singleton, not all points on Z_1 will be stable. Unstable points on Z_1 can occur in particular in the situation of section 3, if Z_1 is locally a graph and g is negative below and positive above Z_1. Such a behavior is possible, if Z_1 is globally *not* a graph. On the other hand, if Z_1 is locally a decreasing graph and g is negative above and positive below Z_1, then all points on this portion of Z_1 are stable for (1.1), (1.6), (1.7), as was shown implicitly in section 2. A more subtle problem occurs if Z_1 is locally an increasing graph and g is locally positive below and negative above this segment of Z_1. Such a segment will again be unstable if the slope of Z_1 is larger than the slope of the characteristics of the vector field f (given in (3.8)). In the opposite case, it can be shown to be stable under suitable restrictions on the diffusion coefficient d, using the arguments in [5].

4. As indicated in section 1, there is some overlap between the class of competitive vector fields and the classes that are considered in section 2 and 3. In particular, if a competitive vector field $\vec{f}$ has the property that f_1 and f_2 vanish on the same set Z which is in addition the graph of a function γ, then γ must be non-increasing, and the results of section 2 (possibly modified by the remarks after the proof of the main result) imply that $\omega(u^0)$ is a singleton for all initial data u^0. On the other hand, it is well-known that competitive vector fields together with diffusion generate strongly orderpreserving semiflows which will converge for "almost all" u^0 to some equilibrium; see [7], [13], [20], [21], [22]. Thus the question arises whether there are general properties for semiflows (not necessarily the property of preserving order) that could extend the results of section 2 and 3.

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