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ON THE EXISTENCE OF SOLUTIONS TO WAVE INTEGRODIFFERENTIAL EQUATIONS WITH SUBDIFFERENTIAL OPERATORS

Dedicated to Professor Hiroki Tanabe on his sixtieth birthday

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0. Introduction

In this paper we consider the following integrodifferential equation

$$(0.1)\quad \begin{cases} \dfrac{d^2u}{dt^2}(t)+\partial\psi u(t)+\partial\varphi u(t)+\displaystyle\int_0^t a(t-s)\,\partial\varphi u(s)ds\ni f(t, u(t)) \\ u(0)=a\,, \qquad \dfrac{du}{dt}(0)=b \end{cases}$$

in a real Hilbert space H. Here ψ and φ are lower semicontinuous proper convex functions from H to $[0, \infty]$, and $\partial\psi$ and $\partial\varphi$ are the subdifferentials of ψ and φ respectively. The functions $a(\cdot)$ and $f(\cdot, \cdot)$ are continuous from $[0, T]$ to $(-\infty, \infty)$ and from $[0, T]\times H$ to H.

Our purpose here is to prove the existence of a global solution on $[0, T]$ of the initial value problem (0.1). In the case of $a(t)\equiv 0$ K. Maruo [3] proved the existence of a solution to the above equation under some restrictions. Moreover, we showed that this class of equations contains vibrating string equations with unilateral constraints which were deeply investigated by M. Schatzman [4], A. Bamberger and M. Schatzman [1] and C. Citrini and L. Amerio in [5]. We will extend the result of [3] to the equation containing a delay term which corresponds to vibrating string with not only a unilateral constraint but also a memory (see the example of section 4). In a general situation it seems to be difficult to solve the above initial value problem (0.1). Hence we will seek a solution which satisfies (0.1) in some generalized sense as in [3].

The outline of the present paper is as follows. In section 1 we list the notations and state the assumptions and theorem. In section 2 we obtain an energy estimate to Yosida approximate solutions of the initial value problem (0.1). In section 3 we prove our theorem. In section 4 we show an example.

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1. Assumptions, theorem and notations.

We list some notations which will be used throughout the paper. Let X_1, X_2 and V be real Banach spaces and V^* the dual space of V. We use the same notation (,) as the inner product of H to denote the pairings between X_1, V and their corresponding duals. We denote the norm of a Banach space S by $|\cdot|_S$ and use the usual notations $L_p(0, T; S)$, $C([0, T]; S)$ etc. to denote variable spaces of functions with values in S. By $\partial\varphi_\lambda$ and $\varphi_\lambda(\cdot)$ we denote Yosida approximations of $\partial\varphi$ and $\varphi(\cdot)$ respectively, i.e $\partial\varphi_\lambda x=\lambda^{-1}(x-J_\lambda^\varphi x)$ and $\varphi_\lambda(x)=(2\lambda)^{-1}|x-J_\lambda^\varphi x|_H^2+\varphi(J_\lambda^\varphi x)$ where $J_\lambda^\varphi=(1+\lambda\partial\varphi)^{-1}$. The notations $d^\pm u/dt$ denote the left and right derivatives of u in H.

Next we state the assumptions and theorem.

The Banach spaces V, X_1, H and X_2 hold the following properties.
(A-1) The following inclusion relations hold:

$$V\subset X_1\subset H\subset X_2 \quad \text{and} \quad X_2\subset \{\text{the dual space } X_1\}$$

where each inclusion mapping is continuous. Moreover, X_1 is separable, the imbedding mapping $V\to X_1$ is compact, and V is reflexive and dense in H.

We introudce the assumptions of $\psi(\cdot)$ (see [2]).

A-2) $\psi(\cdot)$ is a lower semicontinuous, convex function from Domain $D(\psi)=V$ to $[0, \infty]$ and the subdifferential $\partial\psi$ of $\psi(\cdot)$ is single valued and bounded from V to V^*. Moreover they satisfy the following conditions.

(1) The function ψ is coercive in the sense that
$\lim_{|x|_V\to\infty}\psi(x)/|x|_V=\infty$.

(2) Suppose we are given a sequence of functions $\{u_n\}\subset W_\infty^1(0, T; H)\cap L_\infty(0, T; V)$ such that
$u_n\to u$ in $C([0, T]; H)$,
$u_n\to u$ in the weak star topology of $L_\infty(0, T; V)$.
Then a subsequence $\{u_{n_k}\}$ can be extracted so that $\partial\psi u_{n_k}\to\partial\psi u$ in the weak star topology of $L_\infty(0, T; V^*)$.

REMARK. In view of the coerciveness condition (1) ψ is lower semicontinuous also in the topology of H.

Next we state the assumptions of φ.

A-3) There exists $z\in V$ scuh that, for any $x\in H$,

$$(\partial\varphi_\lambda x, x-z)\geq C_1|\partial\varphi_\lambda x|_{X_2}-C_2\{\varphi_\lambda(x)+\psi(x)+1\}$$

where C_1 and C_2 are positive constants independent of x and λ.

The function $f(t, x)$ from $[0, T]\times H$ to H satisfies the following conditions.

A-4)

(1) For each $x\in H$ $f(\cdot, x)$ is continuous in H in $[0, T]$.

(2) The following inequalities hold:

$|f(t, x)-f(t, y)|_H\leq C|x-y|_H$,

$|f(t, x)|_H\leq C\{1+|x|_H\}$

for any $x, y\in H$ and any $t\in[0, T]$ where C is a constant independent of x, y and t.

Let $k(t)$ be the solution of the following integral equation

$$k(t) = a^+(t)-\int_0^t a^-(s)k(t-s)ds, \qquad 0\leq t\leq T \tag{1.1}$$

where $a^+(t)=\mathrm{Max}\{a(t), 0\}$ and $a^-(t)=\mathrm{Min}\{a(t), 0\}$. As is easliy seen the solution $k(t)$ exists, is unique and nonnegative.

A-5) The function $a(t)$ is real valued and belongs to $C^1([0, T])$.

Furthermore, we assume the following condition either A-6) or A-7).

A-6) The function $a(\cdot)$ belongs to $C^2([0, T])$ and the following inequalities hold:

$$\operatorname*{Max}_{0\leq t\leq T}\int_0^t \{k(t-s)\int_0^s a^+(\xi)d\xi-a^-(s)\}ds<1 \quad \text{and} \quad \int_0^T k(s)ds<1 .$$

In addition to A-3) we assume that

A-7) For any positive ε there exists a constant C_ε such that

$$|(\partial\psi x, x-y)|\leq\varepsilon\psi(y)+C_\varepsilon(\psi(x)+1) \quad \text{for any } x, y\in H .$$

REMARK. If $\int_0^T |a(s)|ds<1$ and $a(\cdot)\in C^2([0, T])$ then the assumption A-6) is satisfied. Indeed, integrating both sides of (1.1) over $[0, T]$ and noting that $a^+(t)=|a(t)|+a^-(t)$ we have

$$\int_0^T k(s)ds\leq\int_0^T |a(s)|ds+\int_0^T a^-(s)ds-\int_0^T a^-(s)ds\int_0^T k(s)ds ,$$

which implies

$$\int_0^T k(s)ds<1 .$$

Therefore

$$\int_0^t\{k(t-s)\int_0^s a^+(\xi)d\xi-a^-(s)\}ds<\int_0^T |a(\xi)|d\xi<1 .$$

With regard to the type of the initial value problem (0.1) we consider solutions in the following sense.

DEFINITION. We say that a function $u \in C([0, T]; X_1) \cap W^1_\infty(0, T; H)$ is the solution of the initial value problem (0.1) if the following conditions are satisfied:

1) $\varphi(u(t)) + |u(t)|_V$ is bounded in $[0, T]$.
2) There exists a linear functional F on $C([0, T]; X_1)$ such that

$$F(v-u) \leq \int_0^T \varphi(v(s))ds - \int_0^T \varphi(u(s))ds$$

for any $v \in C([0, T]; X_1)$ and

$$F(v(\cdot)) + \int_\cdot^T a(s-\cdot)v(s)ds)$$

$$= \int_0^T (\frac{du}{ds}(s), \frac{dv}{ds}(s))ds + \int_0^T (f(s, u(s)) - \partial\psi u(s), v(s))ds$$

$$+(b, v(0)) - (\frac{d^-}{dt}u(T), v(T))$$

for any $v \in C([0, T]; X_1) \cap L_1(0, T; V) \cap W^1_1(0, T; H)$.
3) The initial conditions are satisfied in the following sense

$$u(0) = a, \quad b - \frac{d^+}{dt}u(0) \in \partial I_K a$$

where K is the closure of the effective domain of φ, I_K is the indicator function of K and ∂I_K is the subdifferantial of I_K.

Now we state our theorem.

Theorem. *Let the initial values a and b be given so that*

$$a \in V \cap D(\varphi) \quad \text{and} \quad b \in H.$$

Then under the assumptions A-1), A-2), A-3), A-4) *and* A-6) *or* A-1), A-2), A-3), A-4), (A-5) *and* A-7) *we have at least one solution to the initial value problem* (0.1).

2. Approximate solutions.

To begin with we prove some lemmas concerning the properties of the subdifferential $\partial\psi$. Throughout this paper we assume the conditions stated in our Theorem.

Lemma 1. *Let g be a continuous mapping from $C([0, T]; H)$ to $L_2(0, T; H)$ such that the following inequality holds:*

$$\int_0^t |g(v)(s) - g(w)(s)|_H^2 \, ds$$
$$\leq C \int_0^t |v(s) - w(s)|_H^2 \, ds$$

for any $v, w \in C([0, T]; H)$ *and* $t \in [0, T]$.
Then there exists a solution $u \in L_\infty(0, T; V) \cap W^1_\infty(0, T; H) \cap W^2_\infty(0, T; V^*)$ *of the following equation*

$$
(2.1) \qquad \begin{cases} \dfrac{d^2u}{dt^2} + \partial\psi u = g(u) & \text{on } [0, T] \times V^*, \\ u(0) = a\,, \quad \dfrac{du}{dt}(0) = b\,. \end{cases}
$$

Moreover the solution satisfies the following energy inequality

$$
(2.2) \qquad \begin{cases} 2^{-1} \left| \dfrac{d^\pm}{dt} u(t) \right|_H^2 + \psi(u(t)) \le 2^{-1} |b|_H^2 + \psi(a) \\ + \displaystyle\int_0^t \left(g(u)(s)\,, \ \frac{du}{ds}(s)\right) ds \qquad \textit{for any} \quad t \in (0, T)\,. \end{cases}
$$

Proof. We consider the following approximate equation to the inital value problem (2.1), for any $\mu > 0$,

$$
(2.3) \qquad \begin{cases} \dfrac{d^2}{dt^2} u_\mu + \partial\psi_\mu u_\mu = g(u_\mu) & \text{on } [0, T] \times V^*, \\ u_\mu(0) = a\,, \quad \dfrac{d}{dt} u_\mu(0) = b\,. \end{cases}
$$

Here ψ_μ is the Yosida approximation of ψ considered as a convex function on H which is lower semicontinuous also in the topology of H (Remark after A-2)). Taking the inner products of both sides of (2.3) with $(d/dt)u_\mu(t)$ and integrating the resultant equality over $[0, t]$, we have

$$
\begin{aligned} 2^{-1} \left| \frac{d}{dt} u_\mu(t) \right|_H^2 + \psi_\mu(u_\mu(t)) &= 2^{-1} |b|_H^2 + \psi_\mu(a) \\ + \int_0^t \left(g(u_\mu)(s), \frac{d}{ds} u_\mu(s)\right) ds & \qquad \text{for any} \quad t \in (0, T)\,. \end{aligned}
$$

Using Gronwall's lemma and the assumptions of the lemma we see that the functions $|(d/dt)u_\mu(t)|_H$ and $\psi_\mu(u_\mu(t))$ are uniformly bounded on $[0, T]$. Then using (1) in A-2) we see that $|J^\psi_\mu u_\mu(t)|_V$ are uniformly bounded on $[0, T]$. From A-1) we know that $\{J^\psi_\mu u_\mu(t)\}_\mu$ is relatively compact in H for each fixed t. Combining the uniform boundedness of $|(d/dt)u_\mu(t)|_H$ and the above result and using Ascoli-Arezela's theorem we obtain that there exists a subsequence of $\{J^\psi_\mu u_\mu(t)\}$ such that

$$
\lim_{j \to \infty} J^\psi_{\mu_j} u_{\mu_j}(t) = u(t) \quad \text{in} \quad C([0, T]; H)\,.
$$

Moreover, since both functions $|(d/dt)u_\mu(t)|_H$ and $|J^\psi_\mu u_\mu(t)|_V$ are uniformly bounded it follows that $u(t)\in W^1_\infty(0, T; H)\cap L_\infty(0, T; V)$. Noting the equation (2.3), (2) in A-2) and the above resultants we know that u belongs to $W^2_\infty(0, T; V^*)$. Thus we complete the proof.

We consider the Yosida approximate equations of (0.1) with φ_λ in place of φ:

$$\text{(2.4)}\qquad \begin{cases} \dfrac{d^2}{dt^2}u_\lambda+\partial\psi u_\lambda+\partial\varphi_\lambda u_\lambda+\displaystyle\int_0^\cdot a(\cdot-s)\partial\varphi_\lambda u_\lambda(s)ds=f(\cdot, u_\lambda)\,, \\ u_\lambda(0)=a\,, \quad \dfrac{d}{dt}u_\lambda(0)=b\,. \end{cases}$$

We set

$$g(u)(t)=-\partial\varphi_\lambda u(t)-\int_0^t a(t-s)\partial\varphi_\lambda u(s)ds+f(t, u(t))\,.$$

From the assumption A-4) and the Lipschitz continuity of $\partial\varphi_\lambda$ it follows that the mapping g satisfies the hypothesis of Lemma 1. Hence we have the following lemma.

Lemma 2. *For each $\lambda>0$ there exists a solution of the equation* (2.4) *in V^*. Moreover the following energy inequality holds:*

$$\text{(2.5)}\qquad \begin{cases} 2^{-1}\left|\dfrac{d^\pm}{dt}u_\lambda(t)\right|^2_H+\psi(u_\lambda(t))+\varphi_\lambda(u_\lambda(t)) \\ \leq 2^{-1}|b|^2_H+\psi(a)+\varphi_\lambda(a)+\displaystyle\int_0^t\left(f(s, u_\lambda(s)), \frac{d}{ds}u_\lambda(s)\right)ds \\ -\displaystyle\int_0^t\int_0^s a(s-\xi)\left(\partial\varphi_\lambda(u_\lambda(\xi)), \frac{d}{ds}u_\lambda(s)\right)d\xi\, ds \end{cases}$$

for any $t\in(0, T)$.

Next we show that the functions $|(d^\pm/dt)u_\lambda(t)|_H$, $\varphi_\lambda(u_\lambda(t))$ and $\psi(u_\lambda(t))$ are uniformly bounded in t and λ.

For a while we assume the assumtion A-6).

We set

$$w^\pm(t)=\int_0^t a^\pm(t-s)(\partial\varphi_\lambda u_\lambda(s), u_\lambda(t)-u(s))ds\,,$$

$$\tilde{w}^\pm(t)=\int_0^t -(\dot{a})^\mp(t-s)(\partial\varphi_\lambda u_\lambda(s), u_\lambda(t)-u_\lambda(s))ds$$

where $(\dot{a})(t)=da(t)/dt$.

Lemma 3. *There exists a constant M such that*

$$(2.6)\qquad w^+(t)\geq -M\cdot\int_0^t\{|\frac{d}{ds}u_\lambda(s)|_H^2+1+\psi(u_\lambda(s))+\varphi_\lambda(u_\lambda(s))\}ds$$
$$-\int_0^t k(s)ds\cdot\psi(u_\lambda(t))$$
$$-\int_0^t k(t-s)\int_0^s a^+(\xi)d\xi\, ds\cdot\varphi_\lambda(u_\lambda(t))$$

where $k(t)$ is the function in the assumption A-6). *Furthermore*

$$(2.7)\qquad w^-(t)\geq\int_0^t a^-(t-s)\{\varphi_\lambda(u_\lambda(t))-\varphi_\lambda(u_\lambda(s))\}ds\,.$$

Proof. Inductively we define functions $h_n(t)$ as follows:

$$(2.8)\qquad h_1(t)=a^+(t)\,,\quad h_{n+1}(t)=\int_0^t -a^-(s)h_n(t-s)ds$$

where $n=1, 2, 3, \cdots$. Then we have

$$(2.9)\qquad 0\leq h_n(t)\leq M^n\cdot t^{n-1}/(n-1)\,!$$

where $M=\operatorname*{Max}_{0\leq s\leq T}|a(s)|$.
From (2.8) we know

$$\sum_{n=1}^\infty h_n(t)=a^+(t)+\int_0^t -a^-(s)\sum_{n=1}^\infty h_n(t-s)ds\,.$$

In view of the uniqueness of the solution of the integral equation (1.1) we have $k(t)=\sum_{n=1}^\infty h_n(t)$. Moreover from (2.8) we see that the functions $k(t)$ and $(d/dt)k(t-s)$ are uniformly bounded in $0\leq t\leq T$.
For any natural number n we set

$$(2.10)\qquad w_n(t)=\int_0^t h_n(t-s)(\partial\varphi_\lambda u_\lambda(s),\, u_\lambda(t)-u_\lambda(s))ds,$$
$$f_1^n(t)=\int_0^t h_n(t-s)(-\frac{d^2}{ds^2}u_\lambda(s),\, u_\lambda(t)-u_\lambda(s))ds,$$
$$f_2^n(t)=\int_0^t h_n(t-s)(-\partial\psi u_\lambda(s),\, u_\lambda(t)-u_\lambda(s))ds,$$
$$f_3^n(t)=\int_0^t h_n(t-s)(f(s,\, u_\lambda(s)),\, u_\lambda(t)-u_\lambda(s))ds,$$
$$f_4^n(t)=\int_0^t h_n(t-s)(-\int_0^s a^+(s-\mu)(\partial\varphi_\lambda u_\lambda(\mu),\, u_\lambda(t)-u_\lambda(\mu))d\mu ds$$

and

$$f_5^n(t)=\int_0^t h_n(t-s)(-\int_0^s a^-(s-\mu)(\partial\varphi_\lambda u_\lambda(\mu),\, u_\lambda(\mu)-u_\lambda(s))d\mu ds.$$

From (2.4) we see

$$\partial\varphi_\lambda u_\lambda = -\frac{d^2}{dt^2}u_\lambda - \partial\psi u_\lambda + f(\cdot, u_\lambda) - \int_0^\cdot a^+(\cdot-\mu)\partial\varphi_\lambda u_\lambda(\mu)d\mu$$
$$-\int_0^\cdot a^-(\cdot-\mu)\partial\varphi_\lambda u_\lambda(\mu)d\mu .$$

Substituting this in (2.10), noting (2.8) and using Fubini's theorem we get

$$w_n(t) = \sum_{i=1}^5 f_i^n(t) + \int_0^t h_n(t-s)w_1(s)ds + w_{n+1}(t) .$$

In view of (2.9) we see

$$|w_n(t)| \leq M^n T^n/n! \cdot \operatorname*{Max}_{0\leq s\leq t\leq T} |(\partial\varphi_\lambda u_\lambda(s), u_\lambda(t)-u_\lambda(s))| .$$

Thus it follows that

$$w_1(t) = \sum_{i=1}^5 \sum_{n=1}^\infty f_i^n(t) + \int_0^t k(t-s)w_1(s)ds .$$

Set $L_i(t) = \sum_{n=1}^\infty f_i^n(t)$, $i=1, 2, 3, 4, 5$, and $L(t) = \sum_{i=1}^5 L_i(t)$. Solving the above integral equation we get the following equality

$$w_1(t) = L(t) + \int_0^t \mathscr{X}(t-s)L(s)ds , \tag{2.11}$$

where $\mathscr{X}(t)$ is a positive continuous function in $0\leq s\leq t\leq T$. With the aid of an integration by parts we get

$$\begin{aligned} L_1(t) &= k(t)(b, u_\lambda(t)-a) \\ &\quad + \int_0^t (d/ds)k(t-s)((d/ds)u_\lambda(s), u_\lambda(t)-u_\lambda(s))ds \\ &\quad - \int_0^s k(t-s)|(d/ds)u_\lambda(s)|_H^2 ds . \end{aligned}$$

Noting that

$$\begin{aligned} |u_\lambda(t)-a|_H &= \left|\int_0^t (d/ds)u_\lambda(s)ds\right|_H \\ &\leq 2^{-1}\int_0^t (1+|(d/ds)u_\lambda(s)|_H^2)ds \end{aligned}$$

we obtain

$$|L_1(t)| \leq C\int_0^t (|\frac{d}{ds}u_\lambda(s)|_H^2 + |u_\lambda(t)-u_\lambda(s)|_H^2 + 1)ds . \tag{2.12}$$

Using the assumption A-4) and Schwarz's inequality we see

$$|L_3(t)| \leq C\int_0^t (|u_\lambda(t)-u_\lambda(s)|_H^2 + |u_\lambda(s)|_H^2 + 1)ds . \tag{2.13}$$

The definition of the subdifferential yeilds

$$L_2(t)\geq\int_0^t k(t-s)\psi(u_\lambda(s))ds-\int_0^t k(t-s)ds\cdot\psi(u_\lambda(t))\,, \tag{2.14}$$

$$\begin{aligned}L_4(t)\leq&\int_0^t\int_0^s k(t-s)a^+(s-\mu)\varphi_\lambda(u_\lambda(\mu))d\mu ds\\&-\int_0^t\int_0^s k(t-s)a^+(s-\mu)d\mu ds\cdot\varphi_\lambda(u_\lambda(t))\,,\end{aligned} \tag{2.15}$$

$$\begin{aligned}L_5(t)\geq&-\int_0^t\int_0^s k(t-s)a^-(s-\mu)\varphi(u_\lambda(\mu))d\mu ds\\&+\int_0^t\int_0^s k(t-s)a^-(s-\mu)\varphi(u_\lambda(s))d\mu ds\,.\end{aligned} \tag{2.16}$$

Combining (2.11), (2.12), (2.13), (2.14), (2.15), (2.16) and the assumption A-6) we obtain (2.6). The equlity (2.7) is a direct consequence of the difinition of the subdifferential.

Noting Lemma 3 and using the argument of the proof of Lemma 3 we can establish the following lemma, where $\tilde{k}(t)$ is the solution of

$$\tilde{k}(t)=-(\dot{a})^-(t)+\int_0^t -a^-(s)\tilde{k}(t-s)ds\,.$$

Lemma 4. *There exists a constant M such that*

$$\begin{aligned}\tilde{w}^+(t)\geq&-M\cdot\int_0^t\{|\frac{d}{ds}u_\lambda(s)|_H^2+1+\psi(u_\lambda(s))+\varphi_\lambda(u_\lambda(s))\}ds\\&-\int_0^t\tilde{k}(t-s)ds\cdot\psi(u_\lambda(t))\\&-\int_0^t\tilde{k}(t-s)\int_0^s a^+(\xi)d\xi ds\cdot\varphi_\lambda^t(t)\,.\end{aligned}$$

Moreover

$$\tilde{w}^-(t)\geq\int_0^t(\dot{a})^+(t-s)\{\varphi_\lambda(u_\lambda(s)-\varphi_\lambda(u_\lambda(t))\}ds\,.$$

Proposition 5. *Under the assumptions* A-1), A-2), A-3), A-4) *and* A-6) *the funcions $|(d/dt)u_\lambda(t)|_H$, $\varphi_\lambda(u_\lambda(t))$ and $\psi(u_\lambda(t))$ are uniformly bounded in λ and t.*

Proof. Using Fubini's theorem and the integration by parts we see

$$\begin{aligned}&\int_0^t\int_0^s a(s-\xi)((\partial\varphi_\lambda u_\lambda(\xi)),\ \frac{d}{ds}u_\lambda(s))d\xi ds\\&=\int_0^t\int_\xi^t a(s-\xi)(\partial\varphi_\lambda(u_\lambda(\xi)),\ \frac{d}{ds}(u_\lambda(s)-u_\lambda(\xi)))dsd\xi\\&=w^+(t)+w^-(t)+\int_0^t(\tilde{w}^+(s)+\tilde{w}^-(s))ds\,.\end{aligned}$$

Combining the inequality (2.5), Lemma 3, Lemma 4 and the above equality and using the assumption A-6) and Gronwall's inequality we complete the proof.

Proposition 6. *Under the assumptions of Proposition* 5 *there exists a constant M independent of λ and t such that*

$$\int_0^T |\partial\varphi_\lambda u_\lambda(s)|_{X_2} ds \leq M .$$

Proof. We set

$$y(t) = \int_0^t (\partial\varphi_\lambda(u_\lambda(s)), u_\lambda(s)-z)ds$$

where z is the element in the assumption A-3).
In view of (2.4) we get

$$\begin{aligned} y(t) &= \int_0^t (-\frac{d^2}{ds^2}u_\lambda(s)-\partial\psi u_\lambda(s)+f(s, u_\lambda(s)), u_\lambda(s)-z)ds \\ &\quad -\int_0^t\int_0^s a^+(s-\xi)\,(\partial\varphi_\lambda u_\lambda(\xi), u_\lambda(s)-u_\lambda(\xi))\,d\xi ds \\ &\quad -\int_0^t\int_0^s a^-(s-\xi)\,(\partial\varphi_\lambda u_\lambda(\xi), u_\lambda(s)-u_\lambda(\xi))\,d\xi ds \\ &\quad -\int_0^t\int_0^s a^+(s-\xi)\,(\partial\varphi_\lambda u_\lambda(\xi), u_\lambda(\xi)-z)\,d\xi ds \\ &\quad -\int_0^t\int_0^s a^-(s-\xi)\,(\partial\varphi_\lambda u_\lambda(\xi), u_\lambda(\xi)-z)\,d\xi ds \\ &= I_1+I_2+I_3+I_4+I_5 . \end{aligned}$$

Using the integration by parts, the definition of the usbdifferential and the assumption A-4) we get

$$I_1 \leq \int_0^t \{|\frac{d}{ds}u_\lambda(s)|_H^2+1+\psi(u_\lambda(s))+\varphi_\lambda(u_\lambda(s))\}\,ds .$$

From the difinition of the subdifferential it follows

$$I_3+I_4 \leq \int_0^t\int_0^s |a(t-s)|\,\{\varphi_\lambda(u_\lambda(s))+\varphi_\lambda(u_\lambda(\xi))\}\,d\xi ds .$$

From Proposition 5 it follows $I_1+I_3+I_4 \leq$ Constant. Combining $I_2=-\int_0^t w^+(s)ds$, Lemma 3 and Proposition 5 we see $I_2 \leq$ Constant.
Using the integration by parts we see

$$I_5 = -\int_0^t a^-(t-\xi)y(\xi)d\xi ds .$$

Then we get

$$y(t) \leq \text{Const} - \int_0^t a^-(t-s)y(s)ds\,.$$

Combining the assumption A-3) and Proposition 5 we know that $(y(t)+N)$ is a positive function on $[0, T]$ where N is some large positive number. Then using Gronwall's lemma and the above inequality we get

$$(y(t)+N) \leq \text{Const}\,.$$

Using a similar method to the proof of lemma 3 of [3] and combining the above inequality and Proposition 5 we complete the proof.

Next we assume the assumption A-5) and A-7).

We define $w_n(t, \xi)$ by

$$\text{(2.18)} \qquad \begin{aligned} w_1(t, \zeta) &= \int_\zeta^t a(s-\zeta)\frac{d}{ds}u_\lambda(s)ds \quad \text{and} \\ w_{n+1}(t, \zeta) &= \int_\zeta^t -a(s-\zeta)w_n(t, s)ds \quad \text{inductively}\,. \end{aligned}$$

Lemma 7. *We have the following inequalities*

$$|w_n(t, \zeta)|_H \leq A^{n-1}\cdot M_\lambda(t)\cdot(t-\zeta)^{n-1}/(n-1)!$$

where $A = \underset{0\leq t\leq T}{\text{Max}}|a(t)|$, $L = \underset{0<t<T}{\text{Max}}|\frac{d}{dt}a(t)|$ *and*

$$M_\lambda(t) = (2A+TL)\underset{0\leq s\leq t}{\text{Max}}|u_\lambda(s)|_H\,.$$

Proof. With the aid of the integation by parts we see

$$|w_1(t, \xi)| \leq M_\lambda(t)\,.$$

The remaing part can be established by induction.

REMARK. From Lemma 2 we know $w_n(t, \cdot) \in L_\infty(0, T; V)$ for each $t, n = 1, 2, 3, \cdots$.

We set, for $n=1, 2, 3, \cdots$,

$$\begin{aligned} f_{n,1}(t) &= -\int_0^t (w_n(t, \zeta), \frac{d^2}{d\zeta^2}u_\lambda(\zeta))d\zeta\,, \\ f_{n,2}(t) &= -\int_0^t (w_n(t, \zeta), \partial\psi u_\lambda(\zeta))d\zeta \quad \text{and} \\ f_{n,3}(t) &= \int_0^t (w_n(t, \zeta), f(\zeta, u_\lambda(\zeta))d\zeta\,. \end{aligned}$$

Lemma 8. *We get the following equality*

$$\int_0^t \int_0^s a(s-\zeta)\,(\partial\varphi_\lambda u_\lambda(\zeta),\ \frac{d}{ds}u_\lambda(s))d\zeta ds = \sum_{u=1}^{\infty} F_n(t)$$

where $F_n(t) = \{f_{n,1}(t) + f_{n,2}(t) + f_{n,3}(t)\}$.

Proof. Using the equation (2.5) and Fubini's theorem we get

$$\int_0^t (w_n(t,\xi),\ \partial\varphi_\lambda u_\lambda(\xi))d\xi = F_n(t) + \\ + \int_0^t (w_{n+1}(t,\xi),\ \partial\varphi_\lambda u_\lambda(\xi))d\xi\ .$$

Noting that

$$\int_0^t \int_0^s a(s-\zeta)\,(\partial\varphi_\lambda u_\lambda(\zeta),\ \frac{d}{ds}u_\lambda(s))d\zeta ds \\ = \int_0^t (w_1(t,\xi),\ \partial\varphi_\lambda u_\lambda(\xi))d\xi$$

and Lemma 7 we can prove this lemma.

Lemma 9. *There exists a constant C independent of* λ *and t such that*

$$|f_{n,1}(t)| + |f_{n,3}(t)| \le \\ C\{(At)^{n-2}/(n-2)\,!\}\,(\int_0^t |\frac{d}{ds}u_\lambda(s)|^2 ds + 1)$$

for $n=1, 2, 3, \cdots$, *where we set* $(At)^{-1}/(-1)\,!=1$.

Proof. In view of Lemma 7 and the assumption A-4) we find

$$|f_{n,3}(t)| \le (At)^{n-1} M_\lambda(t) \int_0^t C(1+|u_\lambda(s)|_H)ds/(n-1)\,!\ . \tag{2.19}$$

On the other hand there exist a constant K independet of λ and t such that

$$M_\lambda(t),\ C(1+|u_\lambda(t)|_H) \le K(\int_0^t |\frac{d}{ds}u_\lambda(s)|_H + 1)ds\ . \tag{2.20}$$

The desired result on $f_{n,3}(t)$ follows from (2.19) and (2.20). Using the integration by parts and noting $w_1(t, t)=0$ we see

$$|f_{1,1}(t)| \le |w_1(t,0)|_H |b|_H + \int_0^t |a(0)|\ |\frac{d}{ds}u_\lambda(s)|_H^2 ds \\ + \int_0^t \int_\xi^t |\frac{da}{ds}(s-\xi)|\ |\frac{d}{ds}u_\lambda(s)|_H |\frac{d}{d\xi}u_\lambda(\xi)|_H ds d\xi\ .$$

Noting Lemma 7, (2.20) and choosing a constant M so large that $M \ge ((LT + A)T + (K+AT)|b|_H)$ we get the required inequality for $f_{1,1}(t)$. Noting the following equalities

$$\frac{d}{ds}w_n(t, s) = a(0)w_{n-1}(t, s)+\int_s^t \frac{d}{ds}a(\xi-s)w_{n-1}(t, \xi)d\xi .$$

and lemma 7 we have

$$|\frac{d}{ds}w_n(t, s)|_H \leq (A(t-s))^{n-2}M_\lambda(t)/(n-2)!\{A+LT\}$$

where $n \geq 2$.
On the other hand it follows

$$|f_{n,1}(t)| \leq |w_n(t, 0)|_H|b|_H+\int_0^t |\frac{d}{ds}w_n(t, s)|_H|\frac{d}{ds}u_\lambda(s)|_H ds .$$

Then using Lemma 7, the above two inequalities and (2.20) we know

$$\begin{aligned}|f_{n,1}(t)| &\leq (At)^{n-2}M_\lambda(t)/(n-2)!\{AT|b|_H+(A+LT)\int_0^t |\frac{d}{ds}u_\lambda(s)|_H ds\} \\ &\leq M(At)^{n-2}/(n-2)!\{1+\int_0^t |\frac{d}{ds}u_\lambda(s)|_H^2 ds\}\end{aligned}$$

where M is a positive large number independent of λ, t and n. Our required inequalities for $f_{n,1}(t)$ are obtained.

Lemma 10. *For any $\varepsilon>0$, there exists a constant K_ε independent of n and t such that*

$$\begin{aligned}|f_{n,2}(t)| \leq &\{\varepsilon(At)^{n-1}/(n-1)!\}\psi(u_\lambda(t))+ \\ &\{K_\varepsilon(At)^{n-1}/(n-1)!\}\{\int_0^t \psi(u_\lambda(s))ds+1\}.\end{aligned}$$

Proof. From (2.18) we see that the functions $w_n(t, \xi)$ are equal to

$$(-1)^{n-1}\int_\xi^t\int_{\xi_1}^t \cdots \int_{\xi_{n-1}}^t a(\xi_1-\xi)a(\xi_2-\xi_1)\cdots a(\xi_{n-1}-\xi_{n-2})w_1(t, \xi_{n-1})d\mathcal{S}_n$$

where $d\mathcal{S}_n = d\xi_{n-1}d\xi_{n-2}\cdots d\xi_1$ and $n=1, 2, \cdots$.
From (2.18) we have the following equality

$$\begin{aligned}w_1(t, \xi_{n-1}) &= a(t-\xi_{n-1})u_\lambda(t)-a(0)u_\lambda(\xi_{n-1})-\int_{\xi_{n-1}}^t \dot{a}(s-\xi_{n-1})u_\lambda(s)ds \\ &= a(t-\xi_{n-1})(u_\lambda(t)-u_\lambda(\xi_{n-1}))-\int_{\xi_{n-1}}^t \dot{a}(s-\xi_{n-1})(u_\lambda(s)-u_\lambda(\xi_{n-1}))ds\end{aligned}$$

where $\dot{a}=(d/dt)a(t)$.
Using the above two lemmas and the assumption A-7) and noting

$$u_\lambda(\cdot)-u_\lambda(\xi_{n-1}) = (u_\lambda(\cdot)-u_\lambda(\xi))+(u_\lambda(\xi)-u_\lambda(\xi_{n-1}))$$

we obtain the following inequalities

$$
\begin{aligned}
&|(w_n(t,\xi),\partial\psi u_\lambda(\xi))| \\
&\leq A^{n-1}\int_\xi^t\cdots\int_{\xi_{n-2}}^t(\varepsilon\psi(u_\lambda(t))+(C_\varepsilon+1)\psi(u_\lambda(\xi))+C_1\psi(u_\lambda(\xi_{n-1}))+(C_\varepsilon+C_1))d\mathcal{E}_n \\
&+A^{n-2}L\int_\xi^t\cdots\int_{\xi_{n-1}}^t(\psi(u_\lambda(s))+(C_1+1)\psi(u_\lambda(\xi))+C_1\psi(u_\lambda(\xi_{n-1}))+2C_1)dsd\mathcal{E}_n\,.
\end{aligned}
$$

Then it follows

$$
\begin{aligned}
&|f_{n,2}(t)|\leq\varepsilon(At)^{n-1}/(n-1)!\cdot\psi(u_\lambda(t)) \\
&+(C_\varepsilon+C_1+1)(At)^{n-2}A/(n-1)!\cdot\int_0^t\psi(u_\lambda(s))ds+(C_\varepsilon+C_1)(tA)^{n-1}/(n-1)! \\
&+LA^{n-2}t^{n-1}/(n-1)!\cdot 2(C_1+2)\int_0^t\psi(u_\lambda(s))ds\,.
\end{aligned}
$$

Therefore the proof of the lemma is complete.

Combining Lemmas 8, 9 and 10 and the inequality (2.5), choosing ε sufficently small and using Gronwall's lemma we get the following proposition.

Proposition 11. *Under the assumptions* A-1), A-2), A-3), A-4), A-7) *the functions* $|\frac{d}{dt}u_\lambda(t)|_H$, $\varphi_\lambda(u_\lambda(t))$ *and* $\psi(u_\lambda(t))$ *are uniformly bounded in* λ *and* t.

Noting the above proposition and using a similar argment to the proof of Proposition 6 we have the following lemma.

Proposition 12. *Under the assumptions of Proposition* 11 *there exists a constant M independent of* λ *and t such that*

$$
\int_0^T|\partial\varphi_\lambda u_\lambda(s)|_{X_2}ds\leq M\,.
$$

3. Proof of Theorem.

We set

$$
F_\lambda(v)=\int_0^T(\partial\varphi_\lambda u_\lambda s),v(s))ds\qquad\text{for any}\quad v\in C([0,T];X_1)\,.
$$

From the definition of F_λ and Fubini's theorem we get the following lemma.

Lemma 13. *We have the following equality*

$$
\int_0^T\int_0^\zeta a(\zeta-s)(\partial\varphi_\lambda u_\lambda(s),v(\zeta))dsd\zeta=F_\lambda(\int_\cdot^T a(\zeta-\cdot)v(\zeta)d\zeta)\,.
$$

Combining Propositions 5, 6 and lemma 13 or Proposition 11, 12 and Lemma 13 and using the argument of the proof of Theorem in [3] we obtain our theorem.

4. Example. (see the example in [3])

Put $H=L_2(0, 1)$, $X_1=C([0, 1])$, $X_2=L_1(0, 1)$ and $V=\mathring{H}_1(0, 1)$. Then from Sobolev's imbedding theorem the assumption A-1) follows.

We consider the following symmetric sesqulinear form $a(u, v)$ defined on $V\times V$.

$$1)\quad a(u, u)\geq\delta|u|_V^2$$
$$2)\quad |a(u, v)|\leq K|u|_V|v|_V$$

for any $u, v\in V$ where δ and K are some positive constants. We put $\psi(u)=a(u, u)$. Then it is easy to prove that $\psi(\cdot)$ satisfies the assumption A-2). Moreover we know that it satisfies the assumption A-7).

Set

$$K=\{f\in L_2(0, 1); f(x)\geq r(x) \quad a.e\ x\in[0, 1]\}$$

where $r\in C([0, 1])$ and $r(0), r(1)<1$.
Let $\varphi=I_K$ be the indicator function of K. Then we show that the Yosida approximation of $\partial\varphi$ satisfies the assumption A-3). We choose a function $\theta\in C^1([0, 1])$ such that $\theta(0)=\theta(1)=0$ and $\theta(x)-r(x)>\sigma>0$ for any $x\in[0, T]$. In the assumption A-3) we define z, c_1 and c_2 as θ, σ and 0 respectively.
Since

$$\partial\varphi_\lambda f(x)=\begin{cases}0 & \text{if } f(x)\geq r(x)\\ \lambda^{-1}(f(x)-r(x)) & \text{if } f(x)<r(x),\end{cases}$$

and $f(x)<r(x)$ implies

$$\theta(x)-f(x)>\theta(x)-r(x)>\sigma,$$

we have

$$(\partial\varphi_\lambda f, f-\theta)\geq\sigma|\partial\varphi_\lambda f|_{X_2}.$$

Therefore we have the assumption A-3).
Now we can consider the term $\partial\varphi u$ as a unilateral constraint and the integral term in the equation (0.1) as a memory term. Then we can regard the initial value problem (0.1) as the vibrating equations with a unilateral constraint and a memory term.

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