

J-GROUPS OF THE SUSPENSIONS OF THE STUNTED LENS SPACES MOD $2p$

Dedicated to Professor Michikazu Fujii on his 60th birthday

AKIE TAMAMURA

(Received May 15, 1992)

1. Introduction

Let $L^n(q) = S^{2n+1}/\mathbf{Z}_q$ be the $(2n+1)$ -dimensional standard lens space mod q . As defined in [10], we set

$$(1.1) \quad \begin{aligned} L_q^{2n+1} &= L^n(q), \\ L_q^{2n} &= \{[z_0, \dots, z_n] \in L^n(q) \mid z_n \text{ is real} \geq 0\}. \end{aligned}$$

By the several papers, we determined the KO -groups $\widetilde{KO}(S^j(L_q^m/L_q^n))$ of the suspensions of the stunted lens spaces L_q^m/L_q^n for the cases $j \equiv 1 \pmod{2}$ [25], $q=2$ [12], $q=4$ [20] and $q=8$ [21]. Moreover we determined the J -groups $\widetilde{J}(S^j(L_q^m/L_q^n))$ for the cases odd primes q [19], $q=2$ [18], $q=4$ [20] and $q=8$ [21]. The purpose of this paper is to determine the KO -groups $\widetilde{KO}(S^j(L_{2p}^m/L_{2p}^n))$ and J -groups $\widetilde{J}(S^j(L_{2p}^m/L_{2p}^n))$ for odd primes p .

This paper is organized as follows. In section 2 we state the main theorems: Theorem 2 gives a direct sum decomposition of $\widetilde{KO}(S^j(L_{2r_q}^m/L_{2r_q}^n))$ for $j \equiv 0 \pmod{2}$, Theorem 3 gives a direct sum decomposition of $\widetilde{J}(S^j(L_{2r_q}^m/L_{2r_q}^n))$ for $j \equiv 0 \pmod{2}$ and $n+j+1 \equiv 0 \pmod{4}$, Theorem 4 gives the structure of $\widetilde{J}(S^j(L_{2p}^m/L_{2p}^n))$ for $j \equiv 0 \pmod{2}$ and $n+j+1 \equiv 0 \pmod{4}$ and the necessary conditions for L_{2p}^m/L_{2p}^n and $L_{2p}^{m+t}/L_{2p}^{n+t}$ to be of the same stable homotopy type are given by Theorem 5 which is an application of Theorems 3 and 4. In section 3 we prepare some lemmas and recall known results in [12], [19] and [25]. The proofs of Theorem 2 and Theorem 3 are given in section 4. The proof of Theorem 4 is given in section 5. In the final section we give the proof of Theorem 5.

The author would like to express her gratitude to Mr. Susumu Kôno for helpful suggestions.

2. Statement of results

Let $\nu_p(s)$ denote the exponent of the prime p in the prime power decomposition of s , and $m(s)$ the function defined on positive integers as follows (cf.

[3]):

$$\nu_p(\mathbf{m}(s)) = \begin{cases} 0 & (p \neq 2 \text{ and } s \not\equiv 0 \pmod{p-1}) \\ 1 + \nu_p(s) & (p \neq 2 \text{ and } s \equiv 0 \pmod{p-1}) \\ 1 & (p = 2 \text{ and } s \not\equiv 0 \pmod{2}) \\ 2 + \nu_2(s) & (p = 2 \text{ and } s \equiv 0 \pmod{2}). \end{cases}$$

Let \mathbf{Z}/k denote the cyclic group $\mathbf{Z}/k\mathbf{Z}$ of order k . For the case $j \equiv 1 \pmod{2}$, the following result is known.

Theorem 1. *Let q, j, m and n be non-negative integers with $m > n, j \equiv 1 \pmod{2}$ and $q > 1$.*

(1) ([20] and [25]) *If $q \equiv 0 \pmod{2}$, then we have*

$$\widetilde{KO}(S^j(L_q^m/L_q^n)) \cong \widetilde{KO}(S^j(RP(m)/RP(n)))$$

and

$$\widetilde{J}(S^j(L_q^m/L_q^n)) \cong \widetilde{J}(S^j(RP(m)/RP(n))).$$

(2) ([19] and [25]) *If $q \equiv 1 \pmod{2}$, then we have*

$$\widetilde{KO}(S^j(L_q^m/L_q^n)) \cong B(j, m) \oplus B(j+1, n)$$

and

$$\widetilde{J}(S^j(L_q^m/L_q^n)) \cong C(j, m) \oplus C(j+1, n),$$

where

$$B(j, m) = \begin{cases} \widetilde{KO}(S^{j+m}) & (m \equiv 1 \pmod{2}) \\ 0 & (m \equiv 0 \pmod{2}) \end{cases}$$

and

$$C(j, m) = \begin{cases} \widetilde{J}(S^{j+m}) & (m \equiv 1 \pmod{2}) \\ 0 & (m \equiv 0 \pmod{2}). \end{cases}$$

REMARK. (1) The groups $\widetilde{KO}(S^j(RP(m)/RP(n)))$ and $\widetilde{J}(S^j(RP(m)/RP(n)))$ are determined in [12] and [18] respectively.

(2) J -groups of the spheres are well known (cf. [3]).

Theorem 2. *Let j, m, n, q and r be non-negative integers with $m > n, j \equiv 0 \pmod{2}, q \equiv 1 \pmod{2}, q > 1$ and $r > 0$.*

(1) *If $n+j+1 \not\equiv 0 \pmod{4}$, then we have*

$$\widetilde{KO}(S^j(L_{2^r q}^m/L_{2^r q}^n)) \cong \widetilde{KO}(S^j(L_{2^r}^m/L_{2^r}^n)) \oplus \widetilde{KO}(S^j(L_q^{2\lceil m/2 \rceil}/L_q^{2\lceil n/2 \rceil})).$$

(2) *If $j \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$ or $j \equiv 2 \pmod{4}$ and $n+j+1 \equiv 4 \pmod{8}$, then we have*

$$\widetilde{KO}(S^j(L_{2^r q}^m/L_{2^r q}^n)) \cong \mathbf{Z} \oplus \widetilde{KO}(S^j(L_{2^r}^m/L_{2^r}^{n+1})) \oplus \widetilde{KO}(S^j(L_q^{2\lceil m/2 \rceil}/L_q^{n+1})).$$

(3) If $j \equiv 2 \pmod{4}$ and $n+j+1 \equiv 0 \pmod{8}$, then we have

$$\widetilde{KO}(S^j(L_{2r,q}^m/L_{2r,q}^n)) \cong \mathbf{Z} \oplus (\widetilde{KO}(S^j(L_{2r}^m/L_{2r}^{n+1}))/G) \oplus \widetilde{KO}(S^j(L_q^{2[m/2]}/L_q^{n+1})),$$

where G denotes the kernel of the homomorphism

$$(\mathfrak{p}_{m,n})^1: \widetilde{KO}(S^j(L_{2r}^m/L_{2r}^{n+1})) \rightarrow \widetilde{KO}(S^j(L_{2r}^m/L_{2r}^n))$$

and $\text{ord } G$ is equal to 2.

$$(4) \quad \widetilde{KO}(S^j(L_q^{2[m/2]}/L_q^{2[n/2]})) \cong \bigoplus_{\mathfrak{p}} \widetilde{KO}(S^j(L_{\mathfrak{p}^{v_{\mathfrak{p}}(q)}}^{2[m/2]}/L_{\mathfrak{p}^{v_{\mathfrak{p}}(q)}}^{2[n/2]})),$$

where \mathfrak{p} runs over all prime divisors of q .

REMARK. The partial results for the case $j=n=0$ of this theorem have been obtained (cf. [10]).

Theorem 3. Let j, m, n, q and r be non-negative integers with $m > n, j \equiv 0 \pmod{2}, q \equiv 1 \pmod{2}, q > 1$ and $r > 0$.

(1) If $n+j+1 \not\equiv 0 \pmod{4}$, then we have

$$\mathcal{J}(S^j(L_{2r,q}^m/L_{2r,q}^n)) \cong \mathcal{J}(S^j(L_{2r}^m/L_{2r}^n)) \oplus \mathcal{J}(S^j(L_q^{2[m/2]}/L_q^{2[n/2]}).$$

$$(2) \quad \mathcal{J}(S^j(L_q^{2[m/2]}/L_q^{2[n/2]})) \cong \bigoplus_{\mathfrak{p}} \mathcal{J}(S^j(L_{\mathfrak{p}^{v_{\mathfrak{p}}(q)}}^{2[m/2]}/L_{\mathfrak{p}^{v_{\mathfrak{p}}(q)}}^{2[n/2]})),$$

where \mathfrak{p} runs over all prime divisors of q .

REMARK. (1) In the cases $r=1, 2$ and 3 , the groups $\mathcal{J}(S^j(L_{2r}^m/L_{2r}^n))$ are determined in [18], [20] and [21] respectively.

(2) For odd primes \mathfrak{p} , $\mathcal{J}(S^j(L_{\mathfrak{p}}^m/L_{\mathfrak{p}}^n))$ are determined in [19].

(3) The partial results for the case $j=n=0$ of this theorem have been obtained (cf. [10]).

For an integer n , $A(n)$ denotes the group defined by

$$(2.1) \quad A(n) = \begin{cases} \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (n \equiv 0 \pmod{8}) \\ \mathbf{Z}/2 & (n \equiv 1 \text{ or } 7 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

As defined in [1], we denote by $\varphi(m, n)$ the number of integers s with $n < s \leq m$ and $s \equiv 0, 1, 2$ or $4 \pmod{8}$. Set

$$(2.2) \quad \tilde{\varphi}(m, n) = \begin{cases} \varphi(m, n) & (n \not\equiv 3 \pmod{4}) \\ \varphi(m, n+1) & (n \equiv 3 \pmod{4}). \end{cases}$$

In order to state next theorem, we set $M = \mathfrak{m}((n+j+1)/2)$,

$$(2.3) \quad a(j, m, n) = \begin{cases} \tilde{\varphi}(m, n) & (j = 0) \\ \min \{v_2(j)+1, \tilde{\varphi}(m+j, n+j)\} & (j > 0) \end{cases}$$

and

$$(2.4) \quad b(j, m, n) = \begin{cases} b_0(m, n) & (j = 0) \\ \min\{\nu_p(j)+1, b_0(m+j, n+j)\} & (j > 0), \end{cases}$$

where $b_0(m, n) = [m/2(p-1)] - [(n+1)/2(p-1)]$.

Theorem 4. *Let j, m, n be non-negative integers with $m > n$, and p be an odd prime.*

(1) *If $j \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$, then we have*

$$\mathcal{J}(S^j(L_{2p}^m/L_{2p}^n)) \cong \mathbf{Z}/M \cdot 2^{a(j,m,n)-i_2} \cdot p^{b(j,m,n)-i_p} \oplus \mathbf{Z}/p^{i_p} \oplus \mathbf{Z}/2^{i_2},$$

where $i_2 = \min\{a(j, m, n), \nu_2(n+1)\}$ and $i_p = \min\{b(j, m, n), \nu_p(n+1), \nu_p(M)\}$.

(2) *If $j \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{4}$ and $m > n+2$, Then we have*

$$\mathcal{J}(S^j(L_{2p}^m/L_{2p}^n)) \cong \mathbf{Z}/M \cdot p^{b(j,m,n)-i_p} \oplus \mathbf{Z}/p^{i_p} \oplus A(m+j-1),$$

where $i_p = \min\{b(j, m, n), \nu_p(n+1), \nu_p(M)\}$ and $A(m+j-1)$ is the group defined by (2.1).

REMARK. In the case $m = n+1$, $S^j(L_q^{n+1}/L_q^n)$ is homeomorphic to the sphere S^{n+j+1} . Moreover in the case $m = n+2$, we have the homotopy equivalences

$$S^j(L_{2p}^{n+2}/L_{2p}^n) \simeq \begin{cases} S^{j+n+2} \vee S^{j+n+1} & (n: \text{odd}) \\ S^{j+n}(L_{2p}^2) & (n: \text{even}). \end{cases}$$

Hence, J -groups $\mathcal{J}(S^j(L_{2p}^m/L_{2p}^n))$ are determined completely.

Finally we consider the application of the above results. A space X is said to be stably homotopy equivalent to a space Y if there are non-negative integers u and v such that the u -fold suspension $S^u X$ of X is homotopy equivalent to the v -fold suspension $S^v Y$ of Y . In order to state next theorem, we set

$$(2.5) \quad \bar{\varphi}(m, n) = \max\{\varphi(m, n), \bar{\varphi}(-n-2, -m-2)\},$$

where $\bar{\varphi}$ is the function defined by (2.2).

Theorem 5. *Let m, n and t be non-negative integers with $m > n+2$, and p be an odd prime. If L_{2p}^m/L_{2p}^n is stably homotopy equivalent to $L_{2p}^{m+t}/L_{2p}^{n+t}$, then*

$$\nu_2(t) \geq \begin{cases} \varphi(m-n-1, 0) & (m \leq n+9 \text{ or } \max\{\nu_2(n+1), \nu_2(m+1)\} \geq \varphi(m-n-1, 1)) \\ \bar{\varphi}(m, n) - 1 & (\text{otherwise}) \end{cases}$$

and

$$\nu_p(t) \geq \begin{cases} b(m, n) & (n+1 \equiv 0 \pmod{2p^{b(m,n)}} \text{ or } m+1 \equiv 0 \pmod{2p^{b(m,n)}}) \\ b(m, n+2) & (\text{otherwise}), \end{cases}$$

where $b(m, n) = \frac{[m/2] - [(n+1)/2]}{p-1}$.

REMARK. Theorem 5 shows that the necessary conditions for $q=2p$ coincide with the product of those for $q=2$ and $q=p$ (cf. [8] and [14]).

In order to state the final theorem, we prepare function α defined by

$$(2.6) \quad \alpha(k, n) = \begin{cases} 1 & (n \equiv 0 \pmod{2} \text{ and } k \equiv 1 \pmod{8}) \\ & \text{or } n \equiv 1 \pmod{2} \text{ and } k = 0 \\ 0 & \text{(otherwise).} \end{cases}$$

Theorem 6. *If $k = m - 2 \lfloor (n+1)/2 \rfloor \geq 2$ and $t \equiv 0 \pmod{2^{\varphi(k,0)} - \alpha(k,n)} p^{\lfloor k/2(p-1) \rfloor}$, then L_{2p}^m / L_{2p}^n and $L_{2p}^{m+t} / L_{2p}^{n+t}$ are of the same stable homotopy type.*

3. Preliminaries

In this section we recall known results and set up some lemmas needed later.

We begin by setting some notation. Let $\alpha_i(u, v)$ ($1 \leq i \leq 8$) be the integers defined by

$$(3.1) \quad \left\{ \begin{array}{l} (1) \quad \alpha_1(u, v) = \binom{2u}{u-v} (-1)^{u-v}, \\ (2) \quad \alpha_2(u, v) = \binom{u+v}{u-v} + \binom{u+v-1}{u-v-1}, \\ (3) \quad \alpha_3(u, v) = \left(\binom{2u-1}{u-v} - \binom{2u-1}{u-v-1} \right) (-1)^{u-v}, \\ (4) \quad \alpha_4(u, v) = \binom{u+v-1}{u-v}, \\ (5) \quad \alpha_5(u, v) = \binom{v}{u-v} + \binom{v-1}{u-v-1}, \\ (6) \quad \alpha_6(u, v) = \binom{2u-v-1}{u-v} (-1)^{u-v}, \\ (7) \quad \alpha_7(u, v) = \binom{v-1}{u-v}, \\ (8) \quad \alpha_8(u, v) = \left(\binom{2u-v-2}{u-v} - \binom{2u-v-2}{u-v-1} \right) (-1)^{u-v}. \end{array} \right.$$

Then we have following lemma.

Lemma 3.2. *We have the following equalities:*

- (1) $\alpha_1(u+1, v) = \alpha_1(u, v+1) - 2\alpha_1(u, v) + \alpha_1(u, v-1)$,
- (2) $\alpha_2(u+1, v) = \alpha_2(u, v-1) + 2\alpha_2(u, v) - \alpha_2(u-1, v)$,

- (3) $\alpha_3(u+1, v) = \alpha_3(u, v+1) - 2\alpha_3(u, v) + \alpha_3(u, v-1)$,
 (4) $\alpha_4(u+1, v) = \alpha_4(u, v-1) + 2\alpha_4(u, v) - \alpha_4(u-1, v)$,
 (5) $\alpha_5(u+1, v) = \alpha_5(u, v-1) + \alpha_5(u-1, v-1)$,
 (6) $\alpha_6(u+1, v) = \alpha_6(u, v-1) - \alpha_6(u+1, v+1)$,
 (7) $\alpha_7(u+1, v) = \alpha_7(u, v-1) + \alpha_7(u-1, v-1)$,
 (8) $\alpha_8(u+1, v) = \alpha_8(u, v-1) - \alpha_8(u+1, v+1)$.

Proof. By the definition (3.1), we have

$$\begin{aligned} \alpha_1(u+1, v) &= \binom{2u+2}{u-v+1} (-1)^{u+1-v} \\ &= \left(\binom{2u}{u-v+1} + 2 \binom{2u}{u-v} + \binom{2u}{u-v-1} \right) (-1)^{u+1-v} \\ &= \alpha_1(u, v-1) - 2\alpha_1(u, v) + \alpha_1(u, v+1), \\ \alpha_4(u+1, v) &= \binom{u+v}{u-v+1} \\ &= \binom{u+v-2}{u-v+1} + 2 \binom{u+v-2}{u-v} + \binom{u+v-2}{u-v-1} \\ &= \binom{u+v-2}{u-v+1} + 2 \binom{u+v-1}{u-v} - \binom{u+v-2}{u-v-1} \\ &= \alpha_4(u, v-1) + 2\alpha_4(u, v) - \alpha_4(u-1, v), \\ \alpha_6(u+1, v) &= \binom{2u-v+1}{u-v+1} (-1)^{u-v+1} \\ &= \left(\binom{2u-v}{u-v+1} + \binom{2u-v}{u-v} \right) (-1)^{u+1-v} \\ &= \alpha_6(u, v-1) - \alpha_6(u+1, v+1), \\ \alpha_7(u+1, v) &= \binom{v-1}{u-v+1} \\ &= \binom{v-2}{u-v+1} + \binom{v-2}{u-v} \\ &= \alpha_7(u, v-1) + \alpha_7(u-1, v-1). \end{aligned}$$

Thus the equalities (1), (4), (6) and (7) are established.

By making use of the equalities

$$\begin{aligned} \alpha_2(u, v) &= \alpha_4(u+1, v+1) - \alpha_4(u-1, v+1), \\ \alpha_3(u, v) &= \alpha_1(u-1, v-1) - \alpha_1(u-1, v+1), \\ \alpha_5(u, v) &= \alpha_7(u+1, v+1) + \alpha_7(u-1, v) \end{aligned}$$

and $\alpha_8(u, v) = \alpha_6(u-1, v-1) + \alpha_6(u, v+1)$, (2), (3), (5) and (8) follows from (4), (1), (7) and (6) respectively. q.e.d.

Lemma 3.3. *In the polynomial ring $\mathbf{Z}[x]$, the following equalities hold, where i denotes a positive integer.*

- (1) $\sum_{u=1}^i \alpha_2(i, u) \sum_{k=1}^u \alpha_1(u, k) x^k = x^i.$
- (2) $\sum_{u=1}^i \alpha_4(i, u) \sum_{k=1}^u \alpha_3(u, k) x^k = x^i.$
- (3) $\sum_{u=1}^i \alpha_6(i, u) \sum_{k=1}^u \alpha_5(u, k) x^k = x^i.$
- (4) $\sum_{u=1}^i \alpha_8(i, u) \sum_{k=1}^u \alpha_7(u, k) x^k = x^i.$

Proof. (1) Since $\alpha_1(1, 1)=\alpha_2(1, 1)=\alpha_1(2, 2)=\alpha_2(2, 2)=1$ and $-\alpha_1(2, 1) = \alpha_2(2, 1)=4$, the equality holds for $1 \leq i \leq 2$. We argue by induction over i ; let us assume that $i \geq 2$ and the result is true for i and $i-1$. Using Lemma 3.2 and the inductive hypothesis, we have

$$\begin{aligned} & \sum_{u=1}^{i+1} \alpha_2(i+1, u) \sum_{k=1}^u \alpha_1(u, k) x^k \\ &= \sum_{u=1}^{i+1} (\alpha_2(i, u-1) + 2\alpha_2(i, u) - \alpha_2(i-1, u)) \sum_{k=1}^u \alpha_1(u, k) x^k \\ &= \sum_{u=1}^{i+1} \alpha_2(i, u-1) \sum_{k=1}^u \alpha_1(u, k) x^k + 2 \sum_{u=1}^i \alpha_2(i, u) \sum_{k=1}^u \alpha_1(u, k) x^k \\ & \quad - \sum_{u=1}^i \alpha_2(i-1, u) \sum_{k=1}^u \alpha_1(u, k) x^k \\ &= \sum_{u=0}^i \alpha_2(i, u) \sum_{k=1}^{u+1} \alpha_1(u+1, k) x^k + 2x^i - x^{i-1} \\ &= \sum_{u=0}^i \alpha_2(i, u) \sum_{k=1}^{u+1} (\alpha_1(u, k+1) - 2\alpha_1(u, k) + \alpha_1(u, k-1)) x^k + 2x^i - x^{i-1} \\ &= \sum_{u=0}^i \alpha_2(i, u) \sum_{k=1}^{u+1} \alpha_1(u, k+1) x^k - 2 \sum_{u=1}^i \alpha_2(i, u) \sum_{k=1}^u \alpha_1(u, k) x^k \\ & \quad + \sum_{u=0}^i \alpha_2(i, u) \sum_{k=1}^{u+1} \alpha_1(u, k-1) x^k + 2x^i - x^{i-1} \\ &= \sum_{u=1}^i \alpha_2(i, u) \sum_{k=2}^u \alpha_1(u, k) x^{k-1} - 2x^i + x^{i+1} + \sum_{u=0}^i \alpha_2(i, u) \alpha_1(u, 0) x \\ & \quad + 2x^i - x^{i-1} \\ &= x^{i-1} - \sum_{u=1}^i \alpha_2(i, u) \alpha_1(u, 1) + x^{i+1} + \sum_{u=0}^i \alpha_2(i, u) \alpha_1(u, 0) x - x^{i-1} \\ &= x^{i+1} + \sum_{u=0}^i \alpha_2(i, u) (\alpha_1(u+1, 1) - \alpha_1(u, 2) + 2\alpha_1(u, 1)) x \\ &= x^{i+1} + \sum_{u=1}^{i+1} \alpha_2(i, u-1) \alpha_1(u, 1) x - 0^{i-2} x \\ &= x^{i+1} + \sum_{u=1}^{i+1} (\alpha_2(i+1, u) + \alpha_2(i-1, u) - 2\alpha_2(i, u)) \alpha_1(u, 1) x - 0^{i-2} x \\ &= x^{i+1} + \sum_{u=1}^{i+1} \alpha_2(i-1, u) \alpha_1(u, 1) x - 0^{i-2} x = x^{i+1}. \end{aligned}$$

This completes the induction.

The proof of (2) is similar to that of (1).

(3) Let $\beta_3(i, k)$ be the integer defined by

$$\beta_3(i, k) = \sum_{u=k}^i \alpha_6(i, u) \alpha_5(u, k).$$

It suffices to prove

$$(*) \quad \beta_3(i, k) = \begin{cases} 1 & (i = k \geq 0) \\ 0 & (i > k \geq 0). \end{cases}$$

Since $\alpha_5(i, i)=\alpha_6(i, i)=1$ and $\alpha_5(k+1, k)=-\alpha_6(k+1, k)=k+1$, (*) holds for $k \leq i \leq k+1$. Assume that $i \geq k+2 \geq 2$. Then we have

$$\begin{aligned} \beta_3(i, k) &= \sum_{u=k}^i \alpha_6(i, u) \alpha_5(u, k) \\ &= \binom{2i-k-1}{i-k} (-1)^{i-k} + \sum_{u=k+1}^i \binom{2i-u-1}{i-u} (-1)^{i-u} (u/(u-k)) \binom{k-1}{u-k-1} \\ &= \binom{2i-k-1}{i-k} (-1)^{i-k} + \sum_{u=k+1}^i \binom{2i-u-1}{i-u} (-1)^{i-u} ((2i-u)/(i-k)) \binom{k-1}{u-k-1} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{u=k+1}^i \binom{2i-u-1}{i-u} (-1)^{i-u} ((u/(u-k)) - ((2i-u)/(i-k))) \binom{k-1}{u-k-1} \\
 = & \binom{2i-k-1}{i-k} (-1)^{i-k} - \sum_{u=k}^{i-1} \binom{2i-u-2}{i-u-1} (-1)^{i-u} ((2i-u-1)/(i-k)) \binom{k-1}{u-k} \\
 & + \sum_{u=k+1}^{i-1} \binom{2i-u-1}{i-u} (-1)^{i-u} ((2k-u)(i-u)/(u-k)(i-k)) \binom{k-1}{u-k-1} \\
 = & - \sum_{u=k+1}^{i-1} \binom{2i-u-1}{i-u} (-1)^{i-u} ((i-u)/(i-k)) \binom{k-1}{u-k} \\
 & + \sum_{u=k+1}^{i-1} \binom{2i-u-1}{i-u} (-1)^{i-u} ((i-u)/(i-k)) \binom{k-1}{u-k} \\
 = & 0.
 \end{aligned}$$

This completes the proof of (3).

(4) Let $\beta_4(i, k)$ be the integer defined by

$$\beta_4(i, k) = \sum_{u=k}^i \alpha_8(i, u) \alpha_7(u, k).$$

It suffices to prove

$$(**) \quad \beta_4(i, k) = \begin{cases} 1 & (i = k \geq 1) \\ 0 & (i > k \geq 1). \end{cases}$$

Since $\alpha_7(i, i) = \alpha_8(i, i) = 1$ and $\alpha_7(k+1, k) = -\alpha_8(k+1, k) = k-1$, $(**)$ holds for $k \leq i \leq k+1$. Assume that $i \geq k+2 \geq 3$. Then we have

$$\begin{aligned}
 \beta_4(i, k) & = \sum_{u=k}^i \alpha_8(i, u) \alpha_7(u, k) \\
 & = \sum_{u=k}^i \binom{2i-u-2}{i-u} ((u-1)/(i-1)) (-1)^{i-u} \binom{k-1}{u-k} \\
 & = \sum_{u=k-1}^{i-1} \binom{2i-u-3}{i-u-1} (u/(i-1)) (-1)^{i-u-1} \binom{k-1}{u-k+1} \\
 & = \sum_{u=k-1}^{i-1} \binom{2i-u-3}{i-u-1} (-1)^{i-u-1} ((k-1+u-k+1)/(i-1)) \binom{k-1}{u-k+1} \\
 & = ((k-1)/(i-1)) \sum_{u=k-1}^{i-1} \binom{2i-u-3}{i-u-1} (-1)^{i-u-1} \left(\binom{k-1}{u-k+1} + \binom{k-2}{u-k} \right) \\
 & = ((k-1)/(i-1)) \sum_{u=k-1}^{i-1} \alpha_6(i-1, u) \alpha_5(u, k-1) \\
 & = ((k-1)/(i-1)) \beta_3(i-1, k-1) = 0.
 \end{aligned}$$

This completes the proof of (4).

q.e.d.

In the rest of this section j denotes non-negative integer with $j \equiv 0 \pmod{2}$. Considering the (\mathbf{Z}/q) -action on $S^{2n+1} \times \mathbf{C}$ given by

$$\exp(2\pi\sqrt{-1}/q)(z, u) = (z \cdot \exp(2\pi\sqrt{-1}/q), u \cdot \exp(2\pi\sqrt{-1}/q))$$

for $(z, u) \in S^{2n+1} \times \mathbf{C}$, we have a complex line bundle

$$\eta_q: (S^{2n+1} \times \mathbf{C})/(\mathbf{Z}/q) \rightarrow L_q^{2n+1}.$$

Set

$$(3.4) \quad \sigma_q = \eta_q - 1 \in \tilde{K}(L_q^{2n+1}).$$

We also denote by σ_q the restriction of σ_q to L_q^{2n} . Considering the $(\mathbf{Z}/2q)$ -action on $S^{2n+1} \times \mathbf{R}$ given by

$$\exp(2\pi\sqrt{-1}/2q)(z, u) = (z \cdot \exp(2\pi\sqrt{-1}/2q), -u)$$

for $(z, u) \in S^{2n+1} \times \mathbf{R}$, we have a real line bundle

$$\nu_{2q}: (S^{2n+1} \times \mathbf{R})/(\mathbf{Z}/2q) \rightarrow L_{2q}^{2n+1}.$$

Set

$$(3.5) \quad \kappa_{2q} = \nu_{2q} - 1 \in \tilde{K}\tilde{O}(L_{2q}^{2n+1}).$$

We also denote by κ_{2q} the restriction of κ_{2q} to L_{2q}^{2n} .

For each integer n with $0 \leq n < m$, we denote the inclusion map of L_q^n into L_q^m by i_n^m , and the kernel of homomorphism

$$(i_n^m)^!: \tilde{K}\tilde{O}(S^j L_q^m) \rightarrow \tilde{K}\tilde{O}(S^j L_q^n)$$

by $VO_{m,n}^j(q)$. We set

$$(3.6) \quad UO_{m,n}^j(q) = \sum_k \cap_* k^i (\psi^k - 1) VO_{m,n}^j(q).$$

Let $a_i(q)$, $b_i(q)$ and $c_i(q)$ ($i > 0$) be elements of $\tilde{K}\tilde{O}(S^j L_q^m)$ defined by

$$(3.7) \quad \begin{cases} a_i(q) = r(I^{j/2}(\eta_q^i - 1)) \\ b_i(q) = \begin{cases} \sum_{u=1}^i \alpha_1(i, u) a_u(q) & (j \equiv 0 \pmod{4}) \\ \sum_{u=1}^i \alpha_3(i, u) a_u(q) & (j \equiv 2 \pmod{4}) \end{cases} \\ c_i(q) = r(I^{j/2}(\sigma_q^i)), \end{cases}$$

where $r: K \rightarrow KO$ denotes the real restriction and $I: \tilde{K}(X) \rightarrow \tilde{K}(S^2 X)$ is the Bott periodicity isomorphism.

We define the function

$$(3.8) \quad \mu_q: \mathbf{Z} \rightarrow \mathbf{Z}$$

by setting $\mu_q(k)$ to be the remainder of k divided by q for every $k \in \mathbf{Z}$.

Lemma 3.9. *The elements $a_i(q)$, $b_i(q)$ and $c_i(q)$ satisfy following relations.*

- (1) $a_1(q) = b_1(q) = c_1(q)$.
- (2) $a_i(q) = \begin{cases} \sum_{u=1}^i \alpha_2(i, u) b_u(q) & (j \equiv 0 \pmod{4}) \\ \sum_{u=1}^i \alpha_4(i, u) b_u(q) & (j \equiv 2 \pmod{4}). \end{cases}$
- (3) $a_i(q) = \sum_{u=1}^i \binom{i}{u} c_u(q)$.

$$(4) \quad c_i(q) = \sum_{u=1}^i \binom{i}{u} (-1)^{i-u} a_u(q).$$

$$(5) \quad c_i(q) = \begin{cases} \sum_{u=1}^i \alpha_5(i, u) b_u(q) & (j \equiv 0 \pmod{4}) \\ \sum_{u=1}^i \alpha_7(i, u) b_u(q) & (j \equiv 2 \pmod{4}). \end{cases}$$

$$(6) \quad b_i(q) = \begin{cases} \sum_{u=1}^i \alpha_6(i, u) c_u(q) & (j \equiv 0 \pmod{4}) \\ \sum_{u=1}^i \alpha_8(i, u) c_u(q) & (j \equiv 2 \pmod{4}). \end{cases}$$

$$(7) \quad a_i(q) = a_{\mu_q(i)}(q) = \begin{cases} a_{q-\mu_q(i)}(q) & (j \equiv 0 \pmod{4}) \\ -a_{q-\mu_q(i)}(q) & (j \equiv 2 \pmod{4}). \end{cases}$$

$$(8) \quad \text{For the Adams operation } \psi^k, \text{ we have } \psi^k(a_i(q)) = k^{j/2} a_{ki}(q).$$

Proof. (1), (3) and (4) are evident from the definition (3.7).

(2) Suppose that $j \equiv 0 \pmod{4}$. It follows from the definition (3.7) that we have

$$\sum_{u=1}^i \alpha_2(i, u) b_u(q) = \sum_{u=1}^i \alpha_2(i, u) \sum_{k=1}^u \alpha_1(u, k) a_k(q) = a_i(q)$$

by (1) of Lemma 3.3. The proof of the case $j \equiv 2 \pmod{4}$ is similar by making use of (2) of Lemma 3.3.

(5) Suppose that $j \equiv 0 \pmod{4}$. By (4) and (2) we have

$$\begin{aligned} c_i(q) &= \sum_{k=1}^i \binom{i}{k} (-1)^{i-k} a_k(q) \\ &= \sum_{k=1}^i \binom{i}{k} (-1)^{i-k} \sum_{u=1}^k \alpha_2(k, u) b_u(q) \\ &= \sum_{u=1}^i \sum_{k=u}^i \binom{i}{k} (-1)^{i-k} \alpha_2(k, u) b_u(q). \end{aligned}$$

It suffices to prove

$$(*) \quad \sum_{k=u}^i \binom{i}{k} (-1)^{i-k} \alpha_2(k, u) = \alpha_5(i, u) \quad (i \geq u \geq 1).$$

Since we have

$$\begin{aligned} \sum_{k=u}^i \binom{i}{k} (-1)^{i-k} \alpha_2(k, u) &= \sum_{k=u}^i \binom{i}{k} (-1)^{i-k} (2k/(k+u)) \binom{k+u}{k-u} \\ &= \sum_{j=0}^{i-u} \binom{i}{u+j} (-1)^{i-u-j} (2(u+j)/(2u+j)) \binom{2u+j}{j} \\ &= (2(i!)/((i-u)!(2u)!)) \sum_{j=0}^{i-u} \binom{i-u}{j} (-1)^{i-u-j} (2u+j-1) \cdots (u+j) \\ &= \begin{cases} 0 & (i > 2u) \\ 2 & (i = 2u) \end{cases} \end{aligned}$$

by [22, Lemma 3.7], (*) holds for $i \geq 2u$. Since we have

$$\sum_{k=u}^u \binom{u}{k} (-1)^{u-k} \alpha_2(k, u) = 1$$

and

$$\begin{aligned} \sum_{k=u}^{u+1} \binom{u+1}{k} (-1)^{u-k+1} \alpha_2(k, u) &= -(u+1) \alpha_2(u, u) + \alpha_2(u+1, u) \\ &= -(u+1) + 2u + 1 + 1 = u + 1 = \alpha_5(u+1, u), \end{aligned}$$

(*) holds for $u \leq i \leq u+1$. In particular, (*) holds for $u=1$. We argue by induction over $i-u$ and u ; let us assume that $i \geq u+1 \geq 3$ and the result is true for $(i, u-1)$ and $(i-1, u-1)$. Using Lemma 3.2 and the inductive hypothesis, we have

$$\begin{aligned} &\sum_{k=u}^{i+1} \binom{i+1}{k} (-1)^{i-k+1} \alpha_2(k, u) \\ &= \sum_{k=u}^{i+1} \left(\binom{i-1}{k} + 2 \binom{i-1}{k-1} + \binom{i-1}{k-2} \right) (-1)^{i-k+1} \alpha_2(k, u) \\ &= \sum_{k=u}^{i-1} \binom{i-1}{k} (-1)^{i-k+1} \alpha_2(k, u) + 2 \sum_{k=u}^{i-1} \binom{i-1}{k-1} (-1)^{i-k+1} \alpha_2(k, u) \\ &\quad + \sum_{k=u}^{i+1} \binom{i-1}{k-2} (-1)^{i-k+1} \alpha_2(k, u) \\ &= \sum_{k=u}^{i-1} \binom{i-1}{k} (-1)^{i-k+1} \alpha_2(k, u) + 2 \sum_{k=u-1}^{i-1} \binom{i-1}{k} (-1)^{i-k} \alpha_2(k+1, u) \\ &\quad + \sum_{k=u-2}^{i-1} \binom{i-1}{k} (-1)^{i-k-1} \alpha_2(k+2, u) \\ &= \sum_{k=u-2}^{i-1} \binom{i-1}{k} (-1)^{i-k-1} (\alpha_2(k, u) - 2\alpha_2(k+1, u) + \alpha_2(k+2, u)) \\ &= \sum_{k=u-2}^{i-1} \binom{i-1}{k} (-1)^{i-k-1} \alpha_2(k+1, u-1) \\ &= \sum_{k=u-1}^i \binom{i-1}{k-1} (-1)^{i-k} \alpha_2(k, u-1) \\ &= \sum_{k=u-1}^i \left(\binom{i}{k} - \binom{i-1}{k} \right) (-1)^{i-k} \alpha_2(k, u-1) \\ &= \sum_{k=u-1}^i \binom{i}{k} (-1)^{i-k} \alpha_2(k, u-1) + \sum_{k=u-1}^{i-1} \binom{i-1}{k} (-1)^{i-1-k} \alpha_2(k, u-1) \\ &= \alpha_5(i, u-1) + \alpha_5(i-1, u-1) = \alpha_5(i+1, u). \end{aligned}$$

This completes the induction.

Suppose that $j \equiv 2 \pmod{4}$. By (4) and (2) we have

$$\begin{aligned} c_i(q) &= \sum_{k=1}^i \binom{i}{k} (-1)^{i-k} a_k(q) \\ &= \sum_{k=1}^i \binom{i}{k} (-1)^{i-k} \sum_{u=1}^k \alpha_4(k, u) b_u(q) \end{aligned}$$

$$= \sum_{u=1}^i \sum_{k=u}^i \binom{i}{k} (-1)^{i-k} \alpha_4(k, u) b_u(q).$$

It suffices to prove

$$(**) \quad \sum_{k=u}^i \binom{i}{k} (-1)^{i-k} \alpha_4(k, u) = \alpha_7(i, u) \quad (i \geq u \geq 1).$$

Since we have

$$\begin{aligned} \sum_{k=u}^i \binom{i}{k} (-1)^{i-k} \alpha_4(k, u) &= \sum_{k=u}^i \binom{i}{k} (-1)^{i-k} \binom{k+u-1}{k-u} \\ &= \sum_{j=0}^{i-u} \binom{i}{u+j} (-1)^{i-u-j} \binom{2u+j-1}{j} \\ &= ((i!)/((i-u)! (2u-1)!)) \sum_{j=0}^{i-u} \binom{i-u}{j} (-1)^{i-u-j} (2u+j-1) \cdots (u+j+1) \\ &= \begin{cases} 0 & (i \geq 2u) \\ 1 & (i = 2u-1) \end{cases} \end{aligned}$$

by [22, Lemma 3.7], (**) holds for $i \geq 2u-1$. Since we have

$$\sum_{k=u}^u \binom{u}{k} (-1)^{u-k} \alpha_4(k, u) = 1$$

and

$$\begin{aligned} \sum_{k=u}^{u+1} \binom{u+1}{k} (-1)^{u-k+1} \alpha_4(k, u) &= -(u+1) \alpha_4(u, u) + \alpha_4(u+1, u) \\ &= -(u+1) + 2u = u-1 = \alpha_7(u+1, u), \end{aligned}$$

(**) holds for $u \leq i \leq u+1$. In particular, (**) holds for $u=1$. The rest of the proof is similar to that for the case $j \equiv 2 \pmod{4}$.

(6) Suppose that $j \equiv 0 \pmod{4}$. It follows from (5) that we have

$$\sum_{u=1}^i \alpha_6(i, u) c_u(q) = \sum_{u=1}^i \alpha_6(i, u) \sum_{k=1}^u \alpha_5(u, k) b_k(q) = b_i(q)$$

by (3) of Lemma 3.3. The proof of the case $j \equiv 2 \pmod{4}$ is similar by making use of (4) of Lemma 3.3.

(7) is obtained by the properties $\eta_q^q = 1$ and $r \circ t = r$, where $t: K \rightarrow K$ denotes the complex conjugation.

(8) is immediately obtained by [1] and [4]. q.e.d.

Now we prepare some notations. Set

$$\begin{aligned} (1) \quad A(d, u, i) &= \sum_{k=0}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} \alpha_2(d+k-u+1, i). \\ (3.10) \quad (2) \quad \beta_{u,i} &= (-1)^i \binom{2u-1}{i} + 2 \sum_{v=0}^{i-1} (-1)^v \binom{2u-1}{v}. \\ (3) \quad B(d, u, i) &= \sum_{k=0}^{2u-1} \beta_{u,k} \alpha_4(d+k-u+1, i). \end{aligned}$$

Then we have the following lemma.

Lemma 3.11. *Let u be a positive integer. Then we have*

- (1) $A(d, u, i) = \alpha_5(2d+1, d+i-u+1)$.
- (2) $\beta_{u,i} = 0$ ($i \geq 2u$ or $i < 0$).
- (3) $\beta_{u,2u-1-i} = \beta_{u,i}$.
- (4) $\beta_{u,i+1} - \beta_{u,i} = \alpha_3(u, u-i-1)$.
- (5) $B(d, u, i) = A(d, u, i)$.

Proof. (1) If $u=1$, then we have

$$\begin{aligned} A(d, u, i) &= \sum_{k=0}^1 (-1)^{1-k} \binom{1}{k} \alpha_2(d+k, i) \\ &= \alpha_2(d+1, i) - \alpha_2(d, i) \\ &= \binom{d+1+i}{d+1-i} - \binom{d-1+i}{d-1-i} \\ &= \binom{d+i}{d+1-i} + \binom{d+i-1}{d-i} \\ &= \alpha_5(2d+1, d+i). \end{aligned}$$

This implies (1) for the case $u=1$. If $u+1 > 1$, then we have

$$\begin{aligned} A(d, u+1, i) &= \sum_{k=0}^{2u+1} (-1)^{k+1} \binom{2u+1}{k} \alpha_2(d+k-u, i) \\ &= \sum_{k=0}^{2u+1} (-1)^{k+1} \left(\binom{2u-1}{k} + 2 \binom{2u-1}{k-1} + \binom{2u-1}{k-2} \right) \alpha_2(d+k-u, i) \\ &= \sum_{k=0}^{2u-1} (-1)^{k+1} \binom{2u-1}{k} \alpha_2(d+k-u, i) \\ &\quad - 2 \sum_{k=0}^{2u-1} (-1)^k \binom{2u-1}{k} \alpha_2(d+k-u+1, i) + \sum_{k=0}^{2u-1} (-1)^{k+1} \binom{2u-1}{k} \\ &\quad \alpha_2(d+k-u+2, i) \\ &= \sum_{k=0}^{2u-1} (-1)^{k+1} \binom{2u-1}{k} (\alpha_2(d+k-u, i) - 2\alpha_2(d+k-u+1, i) \\ &\quad + \alpha_2(d+k-u+2, i)) \\ &= \sum_{k=0}^{2u-1} (-1)^{k+1} \binom{2u-1}{k} \alpha_2(d+k-u+1, i-1) = A(d, u, i-1). \end{aligned}$$

Thus (1) is proved by the induction with respect to u .

(2) is evident from the definition (3.10) (2).

(3) Suppose $0 \leq i \leq 2u-1$. Then we have

$$\begin{aligned} \beta_{u,i} - \beta_{u,2u-1-i} &= 2(-1)^i \binom{2u-1}{i} + 2 \sum_{v=0}^{i-1} (-1)^v \binom{2u-1}{v} - 2 \sum_{v=0}^{2u-i-2} (-1)^v \binom{2u-1}{v} \end{aligned}$$

$$\begin{aligned}
 &= 2\sum_{v=0}^i (-1)^v \binom{2u-1}{v} + 2\sum_{v=i+1}^{2u-1} (-1)^v \binom{2u-1}{v} \\
 &= 2\sum_{v=0}^{2u-1} (-1)^v \binom{2u-1}{v} = 0.
 \end{aligned}$$

(4) Suppose $-1 \leq i \leq 2u-1$. Then we have

$$\begin{aligned}
 \beta_{u,i+1} - \beta_{u,i} &= (-1)^{i+1} \binom{2u-1}{i+1} + 2(-1)^i \binom{2u-1}{i} - (-1)^i \binom{2u-1}{i} \\
 &= \alpha_3(u, u-i-1).
 \end{aligned}$$

(5) $B(d, u, i) = \sum_{k=0}^{2u-1} \beta_{u,k} \alpha_4(d+k-u+1, i)$

$$\begin{aligned}
 &= \sum_{k=0}^{2u-1} \beta_{u,k} (\alpha_4(d+k-u+2, i+1) - 2\alpha_4(d+k-u+1, i+1) \\
 &\quad + \alpha_4(d+k-u, i+1)) \\
 &= \sum_{k=0}^{2u-1} \beta_{u,k} (\alpha_4(d+k-u+2, i+1) - \alpha_4(d+k-u+1, i+1)) \\
 &\quad - \sum_{k=0}^{2u-1} \beta_{u,k} (\alpha_4(d+k-u+1, i+1) - \alpha_4(d+k-u, i+1)) \\
 &= \sum_{k=0}^{2u-1} \beta_{u,k} (\alpha_4(d+k-u+2, i+1) - \alpha_4(d+k-u+1, i+1)) \\
 &\quad - \sum_{k=-1}^{2u-2} \beta_{u,k+1} (\alpha_4(d+k-u+2, i+1) - \alpha_4(d+k-u+1, i+1)) \\
 &= \sum_{k=-1}^{2u-1} (\beta_{u,k} - \beta_{u,k+1}) (\alpha_4(d+k-u+2, i+1) - \alpha_4(d+k-u+1, i+1)) \\
 &= \sum_{k=-1}^{2u-1} -\alpha_3(u, u-k-1) (\alpha_4(d+k-u+2, i+1) - \alpha_4(d+k-u+1, i+1)) \\
 &= \sum_{k=-1}^{2u-1} (-1)^{k+1} \left(\binom{2u-1}{k} - \binom{2u-1}{k+1} \right) (\alpha_4(d+k-u+2, i+1) \\
 &\quad - \alpha_4(d+k-u+1, i+1)) \\
 &= \sum_{k=0}^{2u-1} (-1)^{k+1} \binom{2u-1}{k} (\alpha_4(d+k-u+2, i+1) - \alpha_4(d+k-u+1, i+1)) \\
 &\quad + \sum_{k=-1}^{2u-2} (-1)^{k+2} \binom{2u-1}{k+1} (\alpha_4(d+k-u+2, i+1) \\
 &\quad - \alpha_4(d+k-u+1, i+1)) \\
 &= \sum_{k=0}^{2u-1} (-1)^{k+1} \binom{2u-1}{k} (\alpha_4(d+k-u+2, i+1) - \alpha_4(d+k-u+1, i+1)) \\
 &\quad + \sum_{k=0}^{2u-1} (-1)^{k+1} \binom{2u-1}{k} (\alpha_4(d+k-u+1, i+1) - \alpha_4(d+k-u, i+1)) \\
 &= \sum_{k=0}^{2u-1} (-1)^{k+1} \binom{2u-1}{k} (\alpha_4(d+k-u+2, i+1) - \alpha_4(d+k-u, i+1)) \\
 &= \sum_{k=0}^{2u-1} (-1)^{k+1} \binom{2u-1}{k} \alpha_2(d+k-u+1, i) \\
 &= A(d, u, i). \tag{q.e.d.}
 \end{aligned}$$

Lemma 3.12. *Let $q \geq 3$ be an odd integer and $d = (q-1)/2$. Then we have*

$$b_{d+u}(q) = -\sum_{i=1}^d \alpha_5(q, d+i) b_{i+u-1}(q),$$

where u is a positive integer.

Proof. Suppose $j \equiv 0 \pmod{4}$. Then by Lemma 3.9, we have

$$a_{q-i}(q) = a_i(q) \quad (0 \leq i \leq q).$$

If $0 < u \leq d+1$, then we have

$$\begin{aligned} 0 &= \sum_{k=0}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} a_{d+k-u+1}(q) \\ &= \sum_{k=0}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} \left(\sum_{i=1}^{d+k-u+1} \alpha_2(d+k-u+1, i) b_i(q) \right) \\ &= \sum_{i=1}^{d+u} \left(\sum_{k=0}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} \alpha_2(d+k-u+1, i) b_i(q) \right) \\ &= \sum_{i=1}^{d+u} A(d, u, i) b_i(q) \\ &= \sum_{i=u}^{d+u} \alpha_5(2d+1, d+i-u+1) b_i(q). \end{aligned}$$

If $u > d+1$, then we have

$$\begin{aligned} 0 &= \sum_{k=u-d-1}^{u+d} (-1)^{2u-1-k} \binom{2u-1}{k} a_{d+k-u+1}(q) \\ &= -\sum_{k=u+d+1}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} a_{-d+k-u}(q) \\ &\quad + \sum_{k=u-d}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} a_{d+k-u+1}(q) \\ &= -\sum_{k=u+d+1}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} \left(\sum_{i=1}^{d+k-u} \alpha_2(k-d-u, i) b_i(q) \right) \\ &\quad + \sum_{k=u-d}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} \left(\sum_{i=1}^{d+k-u+1} \alpha_2(d+k-u+1, i) b_i(q) \right) \\ &= \sum_{i=1}^{u-d-1} \left(\sum_{k=d+u+1}^{2u-1} (-1)^{2u-k} \binom{2u-1}{k} \alpha_2(k-d-u, i) \right) b_i(q) \\ &\quad + \sum_{i=1}^{d+u} \left(\sum_{k=u-d}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} \alpha_2(d+k-u+1, i) \right) b_i(q) \\ &= \sum_{i=1}^{u-d-1} -A(-d-1, u, i) b_i(q) + \sum_{i=1}^{d+u} A(d, u, i) b_i(q) \\ &= \sum_{i=1}^{u-d-1} -\alpha_5(-2d-1, -d+i-u) b_i(q) + \sum_{i=1}^{d+u} \alpha_5(2d+1, d+i-u+1) b_i(q) \\ &= \sum_{i=u}^{d+u} \alpha_5(2d+1, d+i-u+1) b_i(q). \end{aligned}$$

Suppose $j \equiv 2 \pmod{4}$. Then by Lemma 3.9, we have

$$a_{q-i}(q) = -a_i(q) \quad (0 \leq i \leq q).$$

If $0 < u \leq d+1$, then we have

$$\begin{aligned} 0 &= \sum_{k=0}^{2u-1} \beta_{u,k} a_{d+k-u+1}(q) \\ &= \sum_{k=0}^{2u-1} \beta_{u,k} \left(\sum_{i=1}^{d+k-u+1} \alpha_4(d+k-u+1, i) b_i(q) \right) \\ &= \sum_{i=1}^{d+u} \left(\sum_{k=0}^{2u-1} \beta_{u,k} \alpha_4(d+k-u+1, i) \right) b_i(q) \\ &= \sum_{i=1}^{d+u} B(d, u, i) b_i(q) \\ &= \sum_{i=1}^{d+u} \alpha_5(2d+1, d+i-u+1) b_i(q) \\ &= \sum_{i=u}^{d+u} \alpha_5(2d+1, d+i-u+1) b_i(q). \end{aligned}$$

If $u > d+1$, then we have

$$\begin{aligned} 0 &= \sum_{k=u-d-1}^{u+d} \beta_{u,k} a_{d+k-u+1}(q) \\ &= -\sum_{k=d+u+1}^{2u-1} \beta_{u,k} a_{-d+k-u}(q) + \sum_{k=u-d}^{2u-1} \beta_{u,k} a_{d+k-u+1}(q) \end{aligned}$$

$$\begin{aligned}
 &= -\sum_{k=d+u+1}^{2u-1} \beta_{u,k} (\sum_{i=1}^{-d+k-u} \alpha_4(-d+k-u, i) b_i(q)) \\
 &\quad + \sum_{k=u-d}^{2u-1} \beta_{u,k} (\sum_{i=1}^{d+k-u+1} \alpha_4(d+k-u+1, i) b_i(q)) \\
 &= -\sum_{i=1}^{u-d-1} (\sum_{k=d+u+1}^{2u-1} \beta_{u,k} \alpha_4(-d+k-u, i)) b_i(q) \\
 &\quad + \sum_{i=1}^{d+u} (\sum_{k=u-d}^{2u-1} \beta_{u,k} \alpha_4(d+k-u+1, i)) b_i(q) \\
 &= \sum_{i=1}^{u-d-1} -B(-d-1, u, i) b_i(q) + \sum_{i=1}^{d+u} B(d, u, i) b_i(q) \\
 &= \sum_{i=1}^{u-d-1} -\alpha_5(-2d-1, -d+i-u) b_i(q) + \sum_{i=1}^{d+u} \alpha_5(2d+1, d+i-u+1) b_i(q) \\
 &= \sum_{i=1}^{d+u} \alpha_5(2d+1, d+i-u+1) b_i(q).
 \end{aligned}$$

Thus we have

$$0 = \sum_{i=u}^{d+u} \alpha_5(2d+1, d+i-u+1) b_i(q) = \sum_{i=1}^{d+1} \alpha_5(q, d+i) b_{i+u-1}(q).$$

This completes the proof of lemma 3.12.

q.e.d.

Lemma 3.13. *Let p be an odd prime, $d=(p-1)/2, u=sd+j (1 \leq j \leq d)$ and $i \geq 0$. Then we have*

$$b_{u+i}(p) \equiv (-p)^s b_{j+i}(p)$$

modulo the subgroup

$$\langle \{p^{s+1} b_{1+i}(p), \dots, p^{s+1} b_{j+i}(p), p^s b_{j+1+i}(p), \dots, p^s b_{d+i}(p)\} \rangle.$$

Proof. We choose inductively integers $B_{u,k} (u \geq 1$ and $1 \leq k \leq d)$ such that

$$(*) \quad b_{u+i}(p) = \sum_{k=1}^d B_{u,k} b_{k+i}(p)$$

with

$$B_{u,k} = \begin{cases} 0 & (\text{mod } p^{s+1}) \quad (k < j) \\ (-p)^s & (\text{mod } p^{s+1}) \quad (k = j) \\ 0 & (\text{mod } p^s) \quad (k > j). \end{cases}$$

If $1 \leq u \leq d+1$, we put

$$B_{u,k} = \begin{cases} 0 & (1 \leq k \leq d \text{ and } k \neq u \leq d) \\ 1 & (1 \leq k = u \leq d) \\ -\alpha_5(p, d+k) & (1 \leq k \leq d \text{ and } u = d+1). \end{cases}$$

It follows from Lemma 3.12 that $B_{u,k} (1 \leq k \leq d)$ satisfy $(*) (1 \leq u \leq d+1)$. Assume that $u \geq d+1$ and $B_{u,k} (1 \leq k \leq d)$ have been chosen to satisfy the condition $(*)$. Put $B_{u+1,1} = B_{u,d} B_{d+1,1}$ and $B_{u+1,k} = B_{u,k-1} + B_{u,d} B_{d+1,k} (2 \leq k \leq d)$. Then we have

$$\begin{aligned}
 b_{u+1+i}(p) &= \sum_{k=1}^d B_{u,k} b_{k+1+i}(p) = \sum_{k=2}^{d+1} B_{u,k-1} b_{k+i}(p) \\
 &= \sum_{k=2}^d B_{u,k-1} b_{k+i}(p) + B_{u,d} b_{d+1+i}(p) \\
 &= \sum_{k=2}^d B_{u,k-1} b_{k+i}(p) + B_{u,d} \sum_{k=1}^d B_{d+1,k} b_{k+i}(p) \\
 &= B_{u,d} B_{d+1,1} b_{1+i}(p) + \sum_{k=2}^d (B_{u,k-1} + B_{u,d} B_{d+1,k}) b_{k+i}(p)
 \end{aligned}$$

$$= \sum_{k=1}^d B_{u+1,k} b_{k+i}(p)$$

and

$$B_{u+1,k} \equiv \begin{cases} 0 & (\text{mod } p^{r+1}) \quad (k < l) \\ (-p)^r & (\text{mod } p^{r+1}) \quad (k = l) \\ 0 & (\text{mod } p^r) \quad (k > l), \end{cases}$$

where $u+1=rd+l(1 \leq l \leq d)$. The lemma is a direct consequence of the condition (*). q.e.d.

The part (1) of the following proposition is obtained by making use of Lemmas 3.9 and 3.13.

Proposition 3.14. (1) *Let p be an odd prime. Then the group $\widetilde{KO}(S^j(L_p^{2[m/2]}/L_p^{2[n/2]}))$ is isomorphic to $VO_{[m/2],2[n/2]}^j(p)$, which is isomorphic to the direct sum of cyclic groups of order $p^{b_0(m+j-4i,j)-b_0(n+j-4i,j)}$ generated by $p^{b_0(n+j-4i,j)+1} b_i(p)$ ($1 \leq i \leq (p-1)/2$), where b_0 is the function defined in (2.4).*

(2) ([12]) *Assume that $j \equiv 0 \pmod{4}$ and $n \not\equiv 3 \pmod{4}$. Then the group $\widetilde{KO}(S^j(L_2^m/L_2^n))$ is isomorphic to $VO_{m,n}^j(2)$, and*

$$VO_{m,n}^j(2) \cong \begin{cases} \langle 2^{\varphi(n,0)} I_R^{j/8}(\kappa_2) \rangle / \langle 2^{\varphi(m,0)} I_R^{j/8}(\kappa_2) \rangle & (j \equiv 0 \pmod{8}) \\ \langle 2^{\varphi(n+4,4)} c_1(2) \rangle / \langle 2^{\varphi(m+4,4)} c_1(2) \rangle & (j \equiv 4 \pmod{8}). \end{cases}$$

REMARK. The partial result for the case $j=n=0$ of Proposition 3.14. (1) has been obtained in [13].

We define the function $h(q, k)$ by setting

$$(3.15) \quad h(q, k) = \text{ord } \langle J(r(\sigma_q)) \rangle,$$

where $J(r(\sigma_q))$ is the image of $r(\sigma_q) \in \widetilde{KO}(L_q^k)$ by the J -homomorphism $J: \widetilde{KO}(L_q^k) \rightarrow \mathcal{J}(L_q^k)$.

REMARK. The function $h(q, k)$ have been determined completely by K. Fujii (cf. [9], [11] and [10]).

We recall the following lemma from [17] for the proof of Theorems 5 and 6.

Lemma 3.16. *Suppose that $k=2[m/2]+1-2[(n+1)/2] \geq 3$, $N \equiv 0 \pmod{2h(q, k)}$ and $N > m+1$. Then the S -dual of L_q^m/L_q^n is L_q^{N-n-2}/L_q^{N-m-2} .*

From [6, Propositions (2.6) and (2.9)] and Lemma 3.16, we have

(3.17) (1) *If $k=m-2[(n+1)/2] \geq 2$ and $t \equiv 0 \pmod{2h(q, k)}$, then L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} are of the same stable homotopy type.*

(2) *If $k=m-2[(n+1)/2] \geq 2$ and $n+1 \equiv 0 \pmod{2h(q, k)}$, then $t \equiv 0 \pmod{2h}$*

- (q, k) if and only if L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} have the same stable homotopy type.
 (3) If $l=2[m/2]-n \geq 2$ and $t \equiv 0 \pmod{2h(q, l)}$, then L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} are of the same stable homotopy type.
 (4) If $l=2[m/2]-n \geq 2$ and $n+1 \equiv 0 \pmod{2h(q, l)}$, then $t \equiv 0 \pmod{2h(q, l)}$ if and only if L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} have the same stable homotopy type.

From [17] we have the following.

(3.18) Suppose that $q \equiv 0 \pmod{2}$ and $m \geq n+2$. Then $v_2(t) \geq [\log_2 2(m-n-1)]$ if L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} are of the same stable homotopy type.

Proposition 3.19 ([18]). Suppose that $j \equiv 0 \pmod{4}$.

- (1) If $n \not\equiv 3 \pmod{4}$, then we have

$$\mathcal{J}(S^j(RP(m)|RP(n))) \cong \mathbf{Z}/2^{a(j, m, n)},$$

where $a(j, m, n)$ is the integer defined by (2.3).

- (2) If $n \equiv 3 \pmod{4}$, then we have

$$\mathcal{J}(S^j(RP(m)|RP(n))) \cong \mathbf{Z}/m((n+j+1)/2) \cdot 2^{a(j, m, n)-i_2} \oplus \mathbf{Z}/2^{i_2},$$

where $a(j, m, n)$ is the integer defined by (2.3) and

$$i_2 = \min \{a(j, m, n), v_2(n+1)\}.$$

Proposition 3.20 ([19]). Let p be an odd prime, and suppose that $j \equiv 0 \pmod{2}$.

- (1) If $n \equiv 0 \pmod{2}$, then we have

$$\mathcal{J}(S^j(L_p^{2[m/2]}/L_p^n)) \cong \mathbf{Z}/p^{b(j, m, n)},$$

where $b(j, m, n)$ is the integer defined by (2.4).

- (2) If $n \equiv 1 \pmod{2}$, then we have

$$\mathcal{J}(S^j(L_p^{2[m/2]}/L_p^n)) \cong \mathbf{Z}/m((n+j+1)/2) \cdot p^{b(j, m, n)-i_p} \oplus \mathbf{Z}/p^{i_p},$$

where $b(j, m, n)$ is the integer defined by (2.4) and

$$i_p = \min \{b(j, m, n), v_p(n+1), m((n+j+1)/2)\}.$$

4. Proof of Theorems 2 and 3

We denote the projection map of $L_{2^r}^m$ (resp. $L_q^{2[m/2]}$) into $L_{2^r q}^m$ by π_2 (resp. π_q). Then we have

Lemma 4.1. Let j be a positive integer with $j \equiv 2 \pmod{4}$. Then we have

- (1) The induced homomorphism $(\pi_q)^1: \widetilde{K}\widetilde{O}(S^j L_{2^r q}^m) \rightarrow \widetilde{K}\widetilde{O}(S^j L_q^{2[m/2]})$ is an epimorphism.

- (2) The induced homomorphism $(\pi_2)^1: \widetilde{KO}(S^j L_{2^r q}^m) \rightarrow \widetilde{KO}(S^j L_{2^r}^m)$ is an epimorphism.
- (3) If $m+j+1 \not\equiv 0 \pmod{4}$, the induced homomorphism

$$(\pi_2)^1: \widetilde{KO}(S^{j+1} L_{2^r q}^m) \rightarrow \widetilde{KO}(S^{j+1} L_{2^r}^m)$$

is an isomorphism.

Proof. (1) In the commutative diagram

$$\begin{CD} \widetilde{KO}(S^j L_{2^r q}^m) @>(\pi_q)^1>> \widetilde{KO}(S^j L_q^{2[m/2]}) \\ @AAr'AA @AArA \\ \widetilde{K}(S^j L_{2^r q}^m) @>\pi_{q,c}>> \widetilde{K}(S^j L_q^{2[m/2]}) \\ @AAI^{j/2}AA @AAI^{j/2}A \\ \widetilde{K}(L_{2^r q}^m) @>\pi_{q,c}>> \widetilde{K}(L_q^{2[m/2]}), \end{CD}$$

r is an epimorphism [19, Lemma 3.1] and $I^{j/2}$ is an isomorphism. There exist an element $\sigma_{2^r q} \in \widetilde{K}(L_{2^r q}^m)$ which maps to a generator $\sigma_q \in \widetilde{K}(L_q^{2[m/2]})$ by $\pi_{q,c}$. This implies that $\pi_{q,c}$ is an epimorphism. Thus, $(\pi_q)^1$ is an epimorphism. This completes the proof of (1).

- (3) If $m+j+1 \equiv 5, 6$ or $7 \pmod{8}$, then we have

$$\widetilde{KO}(S^{j+1} L_{2^r q}^m) \cong \widetilde{KO}(S^{j+1} L_{2^r}^m) \cong 0.$$

If $m+j+1 \equiv 2 \pmod{8}$, then in the commutative diagram

$$\begin{CD} \widetilde{KO}(S^{m+j+1}) \oplus \widetilde{KO}(S^{m+j}) @>(\hat{p}_{m-2}^m)^1>> \widetilde{KO}(S^{j+1}(L_{2^r q}^m/L_{2^r q}^{m-2})) @>(\hat{p}_{m-2}^m)^1>> \widetilde{KO}(S^{j+1} L_{2^r q}^m) \\ @VVg^1 \oplus 1V @VVV @VV(\pi_2)^1V \\ \widetilde{KO}(S^{m+j+1}) \oplus \widetilde{KO}(S^{m+j}) @>(\hat{p}_{m-2}^m)^1>> \widetilde{KO}(S^{j+1}(L_{2^r}^m/L_{2^r}^{m-2})) @>(\hat{p}_{m-2}^m)^1>> \widetilde{KO}(S^{j+1} L_{2^r}^m), \end{CD}$$

$\deg g = q$ and both $(\hat{p}_{m-2}^m)^1$ are isomorphisms [25, Remark of (3.3)]. Since $q \equiv 1 \pmod{2}$, g^1 is an isomorphism. Hence $(\pi_2)^1$ is an isomorphism.

Next consider the commutative diagram

$$\begin{CD} \widetilde{KO}(S^{m+j+1}) @>(\hat{p}_{m-1}^m)^1>> \widetilde{KO}(S^{j+1} L_{2^r q}^m) @>(i_{m-1}^m)^1>> \widetilde{KO}(S^{j+1} L_{2^r q}^{m-1}) \\ @VVg^1V @VV(\pi_2)^1V @VV(\pi_2')^1V \\ \widetilde{KO}(S^{m+j+1}) @>(\hat{p}_{m-1}^m)^1>> \widetilde{KO}(S^{j+1} L_{2^r}^m) @>(i_{m-1}^m)^1>> \widetilde{KO}(S^{j+1} L_{2^r}^{m-1}), \end{CD}$$

where the rows are exact and

$$\deg g = \begin{cases} 1 & (m \equiv 0 \pmod{2}) \\ q & (m \equiv 1 \pmod{2}). \end{cases}$$

If $m+j+1 \equiv 1 \pmod{8}$, then we have $\widetilde{KO}(S^{j+1} L_{2^r q}^{m-1}) \cong \widetilde{KO}(S^{j+1} L_{2^r}^{m-1}) \cong \mathbf{Z}$ and $\widetilde{KO}(S^{j+1} L_{2^r q}^m) \cong \widetilde{KO}(S^{j+1} L_{2^r}^m) \cong \mathbf{Z}/2$. Hence both $(\hat{p}_{m-1}^m)^1$ are epimor-

phisms. Since $\widetilde{KO}(S^{m+j+1}) \cong \mathbf{Z}/2$, g^1 and both $(p_{m-1}^m)^1$ are isomorphisms. Thus $(\pi_2)^1$ is an isomorphism.

If $m+j+1 \equiv 3 \pmod{8}$, then in the above diagram we have $\widetilde{KO}(S^{m+j+1}) \cong 0$. Hence upper $(i_{m-1}^m)^1$ is a monomorphism. By the proof in the case $m+j+1 \equiv 2 \pmod{8}$, $(\pi_2)^1$ is an isomorphism. Hence $(\pi_2)^1$ is a monomorphism. Since $\text{ord } \widetilde{KO}(S^{j+1}L_{2^r}^m) = \text{ord } \widetilde{KO}(S^{j+1}L_{2^r}^m) = 2$, $(\pi_2)^1$ is an isomorphism. This completes the proof of (3).

(2) We consider the commutative diagram

$$\begin{CD} \tilde{K}(S^{j+2}L_{2^r}^m) @> rI^{-1} >> \tilde{KO}(S^jL_{2^r}^m) @> \delta >> \tilde{KO}(S^{j+1}L_{2^r}^m) @> c >> \tilde{K}(S^{j+1}L_{2^r}^m) \\ @V \pi_{\frac{j+2}{2}, c}^1 VV @V (\pi_2)^1 VV @V (\pi_{\frac{j+1}{2}}^1) VV @V \pi_{\frac{j+1}{2}, c}^1 VV \\ \tilde{K}(S^{j+2}L_{2^r}^m) @> rI^{-1} >> \tilde{KO}(S^jL_{2^r}^m) @> \delta >> \tilde{KO}(S^{j+1}L_{2^r}^m) @> c >> \tilde{K}(S^{j+1}L_{2^r}^m) \end{CD}$$

in which the rows are exact ([5] and [7, (12.2)]), where $c: \widetilde{KO}(X) \rightarrow \tilde{K}(X)$ is the complexification and δ is the homomorphism defined by the exterior product with the generator of $\widetilde{KO}(S^1)$.

If $m+j+1 \equiv 1, 2$ and $3 \pmod{8}$, then $\widetilde{KO}(S^{j+1}L_{2^r}^m)$ is a free group and $\text{ord } \widetilde{KO}(S^jL_{2^r}^m)$ is finite. Hence δ is a zero-map. Since lower rI^{-1} and $\pi_{\frac{j+2}{2}, c}^1$ are epimorphisms, $(\pi_2)^1$ is also an epimorphism.

If $m+j+1 \equiv 1, 2$ or $3 \pmod{8}$, then $(\pi_{\frac{j+1}{2}}^1)^1$ is an isomorphism by (3), $\pi_{\frac{j+2}{2}, c}^1$ is an epimorphism and $\pi_{\frac{j+1}{2}, c}^1$ is a monomorphism. Thus $(\pi_2)^1$ is an epimorphism from 4-lemma. This completes the proof of (2). q.e.d.

Now we define the homomorphism

$$f_1: \widetilde{KO}(S^jL_{2^r}^m) \rightarrow \widetilde{KO}(S^jL_{2^r}^m) \oplus \widetilde{KO}(S^jL_q^{2[m/2]})$$

by $f_1(x) = ((\pi_2)^1(x), (\pi_q)^1(x))$ for $x \in \widetilde{KO}(S^jL_{2^r}^m)$.

Lemma 4.2. *Let j be a positive integer with $j \equiv 2 \pmod{4}$. Then f_1 is an isomorphism.*

Proof. By [25, Theorems 1 and 2]

$$(4.3) \quad \begin{aligned} \text{ord } \widetilde{KO}(S^jL_{2^r}^m) &= 2^{h(m+j)+1}(2^{r-1}q)^{\lfloor (m+2)/4 \rfloor}, \\ \text{ord } \widetilde{KO}(S^jL_q^{2[m/2]}) &= q^{\lfloor (m+2)/4 \rfloor} \end{aligned}$$

and

$$(4.4) \quad \text{ord } \widetilde{KO}(S^jL_{2^r}^m) = 2^{h(m+j)+1}(2^{r-1})^{\lfloor (m+2)/4 \rfloor},$$

where $h: \mathbf{Z} \rightarrow \mathbf{Z}$ is the function defined by

$$h(s) = \begin{cases} 2 & (s \equiv 1 \pmod{8}) \\ 1 & (s \equiv 0 \text{ or } 2 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

Hence, we have

$$(4.5) \quad \text{ord } \widetilde{KO}(S^j L_{2^r q}^m) = \text{ord } (\widetilde{KO}(S^j L_{2^r}^m) \oplus \widetilde{KO}(S^j L_q^{2[m/2]})) .$$

By Lemma 4.1, $(\pi_2)^1$ and $(\pi_q)^1$ are epimorphisms. For each element $(x, y) \in \widetilde{KO}(S^j L_{2^r}^m) \oplus \widetilde{KO}(S^j L_q^{2[m/2]})$, there exist elements v and $w \in \widetilde{KO}(S^j L_{2^r q}^m)$ such that $(\pi_2)^1(v) = x$ and $(\pi_q)^1(w) = y$. Now we put $h(m+j)+1+(r-1)[(m+2)/4] = s$ and $[(m+2)/4] = t$ for the sake of simplicity. Since 2^s is relatively prime to q^t , we can choose integers a and b such that

$$(4.6) \quad a2^s + bq^t = 1 .$$

Set $z = bq^t v + a2^s w$. Then by (4.3), (4.4) and (4.6) we have

$$\begin{aligned} f_1(z) &= bq^t f_1(v) + a2^s f_1(w) \\ &= (bq^t(\pi_2)^1(v) + a2^s(\pi_2)^1(w), bq^t(\pi_q)^1(v) + a2^s(\pi_q)^1(w)) \\ &= ((1-a2^s)(\pi_2)^1(v), (1-bq^t)(\pi_q)^1(w)) \\ &= ((\pi_2)^1(v), (\pi_q)^1(w)) \\ &= (x, y) . \end{aligned}$$

Thus f_1 is an epimorphism. By (4.5), f_1 is an isomorphism. q.e.d.

We have the homomorphisms

$$\begin{aligned} f_2: \widetilde{KO}(S^j L_{2^r q}^n) &\rightarrow \widetilde{KO}(S^j L_{2^r}^n) \oplus \widetilde{KO}(S^j L_q^{2[n/2]}), \\ f_3: \widetilde{KO}(S^{j+1} L_{2^r q}^n) &\rightarrow \widetilde{KO}(S^{j+1} L_{2^r}^n) \oplus \widetilde{KO}(S^{j+1} L_q^{2[n/2]}) \end{aligned}$$

and

$$f: \widetilde{KO}(S^j(L_{2^r q}^m/L_{2^r q}^n)) \rightarrow \widetilde{KO}(S^j(L_{2^r}^m/L_{2^r}^n)) \oplus \widetilde{KO}(S^j(L_q^{2[m/2]}/L_q^{2[n/2]}))$$

defined similarly as f_1 . In the following commutative diagram

$$\begin{array}{ccc} \widetilde{KO}(S^{j+1} L_{2^r q}^n) & \xrightarrow{f_3} & \widetilde{KO}(S^{j+1} L_{2^r}^n) \oplus \widetilde{KO}(S^{j+1} L_q^{2[n/2]}) \\ \downarrow & & \downarrow \\ \widetilde{KO}(S^j(L_{2^r q}^m/L_{2^r q}^n)) & \xrightarrow{f} & \widetilde{KO}(S^j(L_{2^r}^m/L_{2^r}^n)) \oplus \widetilde{KO}(S^j(L_q^{2[m/2]}/L_q^{2[n/2]})) \\ \downarrow & & \downarrow \\ \widetilde{KO}(S^j L_{2^r q}^m) & \xrightarrow{f_1} & \widetilde{KO}(S^j L_{2^r}^m) \oplus \widetilde{KO}(S^j L_q^{2[m/2]}) \\ \downarrow & & \downarrow \\ \widetilde{KO}(S^j L_{2^r q}^n) & \xrightarrow{f_2} & \widetilde{KO}(S^j L_{2^r}^n) \oplus \widetilde{KO}(S^j L_q^{2[n/2]}) \end{array}$$

the columns are exact.

If $j \equiv 2 \pmod{4}$, $n+j+1 \not\equiv 0 \pmod{4}$ and $m \geq n+3$, then f_1, f_2 and f_3 are isomorphisms by Lemmas 4.1 and 4.2. From 4-lemma, f is an epimorphism.

By [25, Theorems 1 and 2]

$$\begin{aligned} & \text{ord}(\widetilde{KO}(S^j(L_{2^r q}^m/L_{2^r q}^n))) \\ &= \text{ord}(\widetilde{KO}(S^j(L_{2^r}^m/L_{2^r}^n)) \oplus \widetilde{KO}(S^j(L_4^{2[m/2]} / L_4^{2[n/2]}))) . \end{aligned}$$

Thus f is an isomorphism. This completes the proof for the case $j \equiv 2 \pmod{4}$ of the part (1) of Theorem 2. The corresponding proof for the case $j \equiv 0 \pmod{4}$ is quite similar to that of the above case.

Combining the part (1) and [25, Theorem 2], we obtain the parts (2) and (3) of Theorem 2.

The proof of the part (4) of Theorem 2 is similar to that of the part (1).

Since the isomorphisms of the parts (1) and (4) of Theorem 2 are ψ -maps, Theorem 3 is an easy consequence of Theorem 2.

5. Proof of Theorem 4

Assume that $j \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$. It follows from [25] and Theorem 2 that we have the commutative diagram

$$(5.1) \quad \begin{array}{ccccccc} 0 & \rightarrow & VO_{m,n+1}^j(2) \oplus VO_{2[m/2],n+1}^j(p) & \xrightarrow{f_1} & \widetilde{KO}(S^j(L_{2p}^m/L_{2p}^n)) & \xrightarrow{f_2} & \widetilde{KO}(S^{j+n+1}) \rightarrow 0 \\ & & \parallel & & \downarrow f_3 & & \\ 0 & \rightarrow & VO_{m,n+1}^j(2) \oplus VO_{2[m/2],n+1}^j(p) & \rightarrow & \widetilde{KO}(S^j L_2^m) \oplus \widetilde{KO}(S^j L_p^{2[m/2]}) & & \end{array}$$

in which the rows are exact. For each i prime to p (resp. 2), $N_p(i)$ (resp. $N_2(i)$) denote the integer chosen to satisfy the property

$$(5.2) \quad iN_p(i) \equiv 1 \pmod{p^m} \quad (\text{resp. } iN_2(i) \equiv 1 \pmod{2^m}) .$$

As defined in [19], let w be the remainder of $j/2$ divided by $p-1$ and set $v = p-1-w$, $N_v = N(\sum_{i=1}^v \binom{v}{i} (-1)^{v-i} N_p(i^{2j/l^2}))$ and $C_l(p) = c_l(p) - \sum_{i=1}^l \binom{l}{i} (-1)^{l-i} N_p(i^{2j/l^2}) N_v c_v(p)$ ($1 \leq l \leq p-1$). In order to state the next lemma, we set

$$(5.3) \quad \begin{aligned} (1) \quad u_2 &= \begin{cases} 2^{(n-1)/2} c_1(2) & (n+j+1 \equiv 4 \pmod{8}) \\ 2^{(n-3)/2} c_1(2) & (n+j+1 \equiv 0 \pmod{8}) . \end{cases} \\ (2) \quad u_p &= \begin{cases} (-p)^s c_v(p) & (n+j+1 \equiv 4 \pmod{8} \text{ and } l = v) \\ (-p)^s C_l(p) & (n+j+1 \equiv 4 \pmod{8} \text{ and } l \neq v) \\ N_p(2) (-p)^s c_v(p) & (n+j+1 \equiv 0 \pmod{8} \text{ and } l = v) \\ N_p(2) (-p)^s C_l(p) & (n+j+1 \equiv 0 \pmod{8} \text{ and } l \neq v) , \end{cases} \end{aligned}$$

where $s = [n/2(p-1)]$ and $l = (n+1)/2 - s(p-1)$.

According to Lemmas 3.9, 3.13 and 3.14, we have the following lemma.

Lemma 5.4. *If $j \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$, then $\widetilde{KO}(S^j(L_{2p}^m/L_{2p}^n))$ has*

an element x , which satisfies the following conditions:

- (1) $f_2(x)$ generates the group $\widetilde{KO}(S^{j+n+1})$,
- (2) $f_3(x) = (u_2, u_p)$.

In the diagram (5.1), since $\widetilde{KO}(S^{j+n+1})$ is isomorphic to \mathbf{Z} , we have a direct decomposition

$$\widetilde{KO}(S^j(L_{2p}^m/L_{2p}^n)) \cong f_1(VO_{m,n+1}^j(2) \oplus VO_{\frac{1}{2}[m/2],n+1}^j(p)) \oplus \mathbf{Z} \{x\}$$

where $\mathbf{Z} \{x\}$ means the infinite cyclic group generated by x .

For the Adams operation, we have the following lemma.

Lemma 5.5. *If $j \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$, then the Adams operation ψ^k is given as follows.*

$$\psi^k(x) = k^{(n+j+1)/2} x + f_1(b_2, b_p)$$

where $b_2 \in VO_{m,n+1}^j(2)$, $b_p \in VO_{\frac{1}{2}[m/2],n+1}^j(p)$,

$$b_p \equiv \begin{cases} 0 & (k \not\equiv 0 \pmod{p} \text{ and } (n+j+1)/2 \not\equiv 0 \pmod{p-1}) \\ -(((k^{(n+j+1)/2} - 1) + (j/2)(k^{p-1} - 1))/p) (pu_p) & \\ (k \not\equiv 0 \pmod{p} \text{ and } (n+j+1)/2 \equiv 0 \pmod{p-1}) & \end{cases}$$

(mod $UO_{\frac{1}{2}[m/2],n+1}^j(p)$) and

$$b_2 = \begin{cases} -(k^{(n+j+1)/2}/2) (2u_2) & (k \equiv 0 \pmod{2}) \\ -((k^{(n+j+1)/2} - k^{j/2})/2) (2u_2) & (k \equiv 1 \pmod{2}). \end{cases}$$

Proof. We necessarily have

$$\psi^k(x) = \alpha x + f_1(b_2, b_p)$$

for some integer α and an element

$$(b_2, b_p) \in VO_{m,n+1}^j(2) \oplus VO_{\frac{1}{2}[m/2],n+1}^j(p).$$

By using the ψ -map f_2 , we see that $\alpha = k^{(n+j+1)/2}$. Under $f_3, f_1(b_2, b_p)$ maps (b_2, b_p) and x maps into $f_3(x)$, and by above Lemma we see that

$$\psi^k(u_2, u_p) = k^{(n+j+1)/2}(u_2, u_p) + (b_2, b_p).$$

It follows from [18, Lemma 2.3] and [19, Lemma 2.13] that

$$\psi^k(u_2) = \begin{cases} 0 & (k \equiv 0 \pmod{2}) \\ k^{j/2} u_2 & (k \equiv 1 \pmod{2}) \end{cases}$$

and

$$\psi^k(u_p) \equiv \begin{cases} k^{(n+j+1)/2} u_p & (n+j+1)/2 \not\equiv 0 \pmod{p-1} \\ ((1 + (j/2)(1 - k^{p-1})) u_p) & (n+j+1)/2 \equiv 0 \pmod{p-1} \end{cases}$$

(mod $VO_{2[m/2],n+1}^i$) ($k \not\equiv 0 \pmod{p}$). Therefore,

$$b_2 = \begin{cases} -k^{(n+j+1)/2} u_2 & (k \equiv 0 \pmod{2}) \\ (k^{j/2} - k^{(n+j+1)/2}) u_2 & (k \equiv 1 \pmod{2}) \end{cases}$$

and

$$b_p \equiv \begin{cases} 0 & ((n+j+1)/2 \not\equiv 0 \pmod{p-1}) \\ (1+(j/2)(1-k^{p-1}) - k^{(n+j+1)/2}) u_p & ((n+j+1)/2 \equiv 0 \pmod{p-1}) \end{cases}$$

(mod $VO_{2[m/2],n+1}^i(p)$) ($k \not\equiv 0 \pmod{p}$). q.e.d.

We now recall some definition in [3], set $Y = \widetilde{KO}(S^j(L_{2p}^n/L_{2p}^n))$ and let f be a function which assigns to each integer k a non-negative integer $f(k)$. Given such a function f , we define Y_f to be the subgroup of Y generated by

$$\{k^{f(k)}(\psi^k - 1)(y) \mid k \in \mathbf{Z}, y \in Y\};$$

that is

$$Y_f = \langle \{k^{f(k)}(\psi^k - 1)(y) \mid k \in \mathbf{Z}, y \in Y\} \rangle.$$

Then the kernel of the homomorphism $J'' : Y \rightarrow J''(Y)$ coincides with $\bigcap_f Y_f$, where the intersection runs over all functions f .

Suppose that f satisfies

(5.6) $f(k) \geq m + \max\{\nu_p(m((n+j+1)/2)) \mid p \text{ is a prime divisor of } k\}$ for every $k \in \mathbf{Z}$.

In the following calculation we put $(n+j+1)/2 = u$ and

$$U_{n+1} = VO_{m,n+1}^i(2) \oplus VO_{2[m/2],n+1}^i(p)$$

for the sake of simplicity.

Now we consider the case $(n+j+1)/2 \equiv 0 \pmod{p-1}$. From Lemma 5.5, we have

$$\begin{aligned} & k^{f(k)}(\psi^k - 1)(x) \\ & \equiv k^{f(k)}(k^u - 1)x - k^{f(k)}(((k^u - 1) + (j/2)(k^{p-1} - 1))/p) f_1(pu_p) \\ & \quad - k^{f(k)}((k^u - k^{j/2})/2) f_1(2u_2) \pmod{f_1(U_{n+1})} \\ & \equiv k^{f(k)}(k^u - 1)x \\ & \quad - k^{f(k)} N_p(u/p^{\nu_p(u)}) ((u(k^u - 1) - (j/2)(k^u - 1))/p^{\nu_p(u)+1}) f_1(pu_p) \\ & \quad - k^{f(k)} N_2(u/2^{\nu_2(u)}) ((u(k^u - 1) - (j/2)(k^u - 1))/2^{\nu_2(u)+1}) f_1(2u_2) \\ & \quad \pmod{f_1(U_{n+1})} \\ & = (k^{f(k)}(k^u - 1)/p^{\nu_p(u)+1} 2^{\nu_2(u)+1}) (2^{\nu_2(u)+1} p^{\nu_p(u)+1} x \\ & \quad - N_p(u/p^{\nu_p(u)}) ((n+1)/2) 2^{\nu_2(u)+1} f_1(pu_p) \\ & \quad - N_2(u/2^{\nu_2(u)}) ((n+1)/2) p^{\nu_p(u)+1} f_1(2u_2)). \end{aligned}$$

By virtue of [3, Theorem (2.7) and Lemma (2.12), we have

$$\begin{aligned} & \langle f_1(U_{n+1}) \cup \{k^{f(k)}(\psi^k - 1)(x) \mid k \in \mathbf{Z}\} \rangle \\ &= \langle f_1(U_{n+1}) \cup \{(\mathfrak{m}(u)/p^{\nu_p(u)+1} 2^{\nu_2(u)+1}) (2^{\nu_2(u)+1} p^{\nu_p(u)+1} x \\ & \quad - N_p(u/p^{\nu_p(u)}) ((n+1)/2) 2^{\nu_2(u)+1} f_1(pu_p) \\ & \quad - N_2(u/2^{\nu_2(u)}) ((n+1)/2) p^{\nu_p(u)+1} f_1(2u_2)\} \rangle . \end{aligned}$$

Therefore,

$$Y_f = \langle f_1(U_{n+1}) \cup \{\mathfrak{m}(u) x - M_p f_1(pu_p) - M_2 f_1(2u_2)\} \rangle ,$$

where $u = (n+j+1)/2$,

$$\begin{aligned} M_p &= (\mathfrak{m}(u)/p^{\nu_p(u)+1}) N_p(u/p^{\nu_p(u)}) ((n+1)/2) , \\ M_2 &= (\mathfrak{m}(u)/2^{\nu_2(u)+2}) N_2(u/2^{\nu_2(u)}) (n+1) . \end{aligned}$$

Since this is true for every function f which satisfies (5.6), we have

$$J''(Y) \cong Y / \langle f_1(U_{n+1}) \cup \{\mathfrak{m}(u) x - M_p f_1(pu_p) - M_2 f_1(2u_2)\} \rangle .$$

Therefore,

$$J''(Y) \cong \langle \{x, u_2, u_p\} \rangle / \langle \{X_1, X_2, X_3\} \rangle$$

where $M_0 = \mathfrak{m}((n+j+1)/2)$, $X_1 = M_0 x - M_2 u_2 - M_p u_p$, $X_2 = 2^{a(j,m,n)} u_2$ and $X_3 = p^{b(j,m,n)} u_p$.

we set

$$i_2 = \min \{a(j, m, n), \nu_2(n+1)\}$$

and

$$i_p = \min \{b(j, m, n), \nu_p(n+1), \nu_p(\mathfrak{m}((n+j+1)/2))\} .$$

Since $\nu_2(M_2) = \nu_2(n+1)$ and $\nu_p(M_p) = \nu_p(n+1)$, the greatest common divisor of $2^{a(j,m,n)}$ and $M_2 p^{b(j,m,n)-i_p}$ is equal to 2^{i_2} , and the greatest common divisor of $p^{b(j,m,n)}$ and $M_p 2^{a(j,m,n)-i_2}$ is equal to p^{i_p} . Choose integers e_1, e_2, e_3 and e_4 with

$$e_1 2^{a(j,m,n)} + e_2 M_2 p^{b(j,m,n)-i_p} = 2^{i_2}$$

and

$$e_3 p^{b(j,m,n)} + e_4 M_p 2^{a(j,m,n)-i_2} = p^{i_p} .$$

For the sake of simplicity, we put $a = a(j, m, n)$ and $b = b(j, m, n)$ in the following calculation. Set

$$A = \begin{pmatrix} 2^{a-i_2} p^{b-i_p} & p^{b-i_p} M_2 / 2^{i_2} & 2^{a-i_2} M_p / p^{i_p} \\ e_2 p^{b-i_p} & -e_1 & e_2 M_p / p^{i_p} \\ e_4 2^{a-i_2} & e_4 M_2 / 2^{i_2} & -e_3 \end{pmatrix} ,$$

then we have

$$A \begin{pmatrix} M_0 x - M_2 u_2 - M_p u_p \\ 2^a u_2 \\ p^b u_p \end{pmatrix} = \begin{pmatrix} 2^{a-i_2} p^{b-i_p} M_0 x \\ 2^{i_2} (e_2 p^{b-i_p} M_0 x / 2^{i_2} - u_2) \\ p^{i_p} (e_4 2^{a-i_2} M_0 x / p^{i_p} - u_p) \end{pmatrix}$$

and $\det A = 1$. This implies that

$$J''(Y) \cong \mathbf{Z}/\mathfrak{m}((n+j+1)/2) \cdot 2^{a(j,m,n)-i_2} \cdot p^{b(j,m,n)-i_p} \oplus \mathbf{Z}/p^{i_p} \oplus \mathbf{Z}/2^{i_2}.$$

Thus the proof of (1) for the case $(n+j+1)/2 \equiv 0 \pmod{(p-1)}$ is completed by [24].

We now turn to the case $u = (n+j+1)/2 \not\equiv 0 \pmod{(p-1)}$. Then we have

$$\begin{aligned} k^{f(k)} (\psi^k - 1)(x) \\ \equiv (k^{f(k)} (k^u - 1) / 2^{\nu_2(u)+1}) (2^{\nu_2(u)+1} x - N_2(u/2^{\nu_2(u)}) ((n+1)/2) f_1(2u_2)) \end{aligned}$$

(mod $f_1(U_{n+1})$). Hence

$$J''(Y) \cong Y / \langle f_1(U_{n+1}) \cup \{m(u)x - M_2 f_1(2u_2)\} \rangle.$$

Therefore,

$$J''(Y) \cong \langle \{x, u_2, u_p\} \rangle / \langle \{X_1, X_2, X_3\} \rangle$$

where $M_0 = \mathfrak{m}((n+j+1)/2)$, $X_1 = M_0 x - M_2 u_2$, $X_2 = 2^{a(j,m,n)} u_2$ and $X_3 = p^{b(j,m,n)} u_p$. We set

$$i_2 = \min \{a(j, m, n), \nu_2(n+1)\}.$$

Since $\nu_2(M_2) = \nu_2(n+1)$ the greatest common divisor of $2^{a(j,m,n)}$ and M_2 is equal to 2^{i_2} . Choose integers e_1 and e_2 with

$$e_1 2^{a(j,m,n)} + e_2 = 2^{i_2}.$$

Set

$$B = \begin{pmatrix} 2^{a-i_2} & M_2/2^{i_2} & 0 \\ e_2 & -e_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then we have

$$B \begin{pmatrix} M_0 x - M_2 u_2 \\ 2^a u_2 \\ p^b u_p \end{pmatrix} = \begin{pmatrix} 2^{a-i_2} M_0 x \\ 2^{i_2} (e_2 M_0 / 2^{i_2} x - u_2) \\ p^b u_p \end{pmatrix}$$

and $\det B = -1$. This implies that

$$\begin{aligned} J''(Y) &\cong \mathbf{Z}/\mathfrak{m}((n+j+1)/2) \cdot 2^{a(j,m,n)-i_2} \oplus \mathbf{Z}/p^{b(j,m,n)} \oplus \mathbf{Z}/2^{i_2} \\ &\cong \mathbf{Z}/\mathfrak{m}((n+j+1)/2) \cdot 2^{a(j,m,n)-i_2} \cdot p^{b(j,m,n)} \oplus \mathbf{Z}/2^{i_2}. \end{aligned}$$

Thus the proof of (1) is completed by [24].

Now we turn to the case $j \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{4}$. In the corresponding diagram of (5.1), $\widetilde{KO}(S^j(L_{2p}^m/L_{2p}^n))$ has an element x , which satisfies the following conditions:

- i) $f_2(x)$ generates the group $\widetilde{KO}(S^{j+n+1})$,
- ii) the 2-component of $f_3(x)$ is equal to 0.

Since the Adams operations are given by $\psi^k = k - 2 [k/2]$ on the 2-component of $\widetilde{KO}(S^j(L_{2p}^m/L_{2p}^n))$, the rest of the proof of (2) is similar to that of (1).

6. Proofs of Theorems 5 and 6

In this section we state proofs of Theorems 5 and 6.

Proof of Theorem 5. Suppose that the spaces L_{2p}^m/L_{2p}^n and $L_{2p}^{m+t}/L_{2p}^{n+t}$ are of the same stable homotopy type with $m > n + 2$. Then there exists a homotopy equivalence

$$f: S^j(L_{2p}^m/L_{2p}^n) \rightarrow S^{j-t}(L_{2p}^{m+t}/L_{2p}^{n+t}),$$

which induces an isomorphism

$$(6.1) \quad J(f^!): \widetilde{J}(S^{j-t}(L_{2p}^{m+t}/L_{2p}^{n+t})) \rightarrow \widetilde{J}(S^j(L_{2p}^m/L_{2p}^n)).$$

We can assume that $\nu_2(j) \geq \max\{3, \varphi(m, n)\}$. By (3.18), $t \equiv 0 \pmod{4}$. It follows from Proposition 3.19, Theorem 3 and Theorem 4, that we have

$$\min\{\nu_2(j) + 1, \varphi(m, n)\} = \min\{\nu_2(j - t) + 1, \varphi(m, n)\}.$$

Thus we have

$$(6.2) \quad \nu_2(t) \geq \varphi(m, n) - 1.$$

If $m \geq n + 9$ and $\nu_2(n + 1) \geq \varphi(m - n - 1, 0) - 1$, then we have the following from Theorem 4:

$$\begin{aligned} & \widetilde{J}(S^{j-t}(L_{2p}^{m+t}/L_{2p}^{n+t})) \\ & \cong \mathbf{Z}/m((n+j+1)/2) \cdot 2^{\varphi(m, n+1) - k_2} \cdot p^{b(j-t, m+t, n+t) - k_p} \oplus \mathbf{Z}/p^{k_p} \oplus \mathbf{Z}/2^{k_2} \end{aligned}$$

and

$$\begin{aligned} & \widetilde{J}(S^j(L_{2p}^m/L_{2p}^n)) \\ & \cong \mathbf{Z}/m((n+j+1)/2) \cdot 2^{\varphi(m, n+1) - i_2} \cdot p^{b(j, m, n) - i_p} \oplus \mathbf{Z}/p^{i_p} \oplus \mathbf{Z}/2^{i_2}, \end{aligned}$$

where

$$\begin{aligned} k_2 &= \min\{\varphi(m, n+1), \nu_2(n+t+1)\}, \\ k_p &= \min\{b(j-t, m+t, n+t), \nu_p(n+t+1), \nu_p(m((n+j+1)/2))\}, \\ i_2 &= \min\{\varphi(m, n+1), \nu_2(n+1)\} \end{aligned}$$

and

$$i_p = \min \{b(j, m, n), v_p(n+1), v_p(\mathfrak{m}((n+j+1)/2))\}.$$

By the isomorphism (6.1), we have $k_2=i_2$. This implies that we have $v_2(n+t+1) \geq \varphi(m, n+1)$ if $v_2(n+1) \geq \varphi(m, n+1)$ and $v_2(n+t+1) = \varphi(m, n+1) - 1$ if $v_2(n+1) = \varphi(m, n+1) - 1$. Since $v_2(n+1) + 1 \geq \varphi(m-n-1, 0) = \varphi(m, n+1)$, we have

(6.3) *If $m \geq n+9$ and $v_2(n+1) \geq \varphi(m-n-1, 0) - 1$, then we have*

$$v_2(t) \geq \varphi(m-n-1, 0).$$

On the other hand, we can assume that

$$j \equiv 0 \pmod{2p^{[(m/2) - [(n+3)/2]/(p-1)]}}$$

and $j/2 \equiv p-2 - [(n+1)/2] \pmod{(p-1)}$. It follows from Proposition 3.20, Theorem 3 and Theorem 4, that we have

$$\begin{aligned} & \min \{v_p(j-t)+1, [(m+j)/2(p-1)] - [(n+j+1)/2(p-1)]\} \\ &= \min \{v_p(j)+1, [(m+j)/2(p-1)] - [(n+j+1)/2(p-1)]\} \\ &= [(m+j)/2(p-1)] - [(n+j+1)/2(p-1)] \\ &= [[(m+j)/2]/(p-1)] - ([[(n+1)/2] - p + 2 + (j/2))/(p-1) \\ &= [([m/2] - [(n+3)/2])/(p-1)] + 1. \end{aligned}$$

This implies

$$(6.4) \quad v_p(t) \geq [([m/2] - [(n+3)/2])/(p-1)].$$

In the case $n+1 \equiv 0 \pmod{2p^{[(m/2) - [(n+1)/2]/(p-1)]}}$, we assume that

$$j \equiv 0 \pmod{2p^{[(m/2) - [(n+1)/2]/(p-1)]}}$$

and $n+j+1 \equiv 0 \pmod{2(p-1)}$. It follows from Theorem 4 that we have

$$\begin{aligned} & \min \{v_p(n+t+1), [(m+j)/2(p-1)] - [(n+j+1)/2(p-1)]\} \\ &= \min \{v_p(n+1), [(m+j)/2(p-1)] - [(n+j+1)/2(p-1)]\} \\ &= \min \{v_p(n+1), [([m/2] - [(n+1)/2])/(p-1)]\} \\ &= [([m/2] - [(n+1)/2])/(p-1)]. \end{aligned}$$

This implies

(6.5) *If $n+1 \equiv 0 \pmod{2p^{[(m/2) - [(n+1)/2]/(p-1)]}}$, we have*

$$v_p(t) \geq [([m/2] - [(n+1)/2])/(p-1)].$$

Combining (6.2), (6.3), (6.4), (6.5), Lemma 3.16 and (3.18), we obtain Theorem 5.

Proof of Theorem 6. According to [10], we have

$$h(2p, k) = 2^{q(k,0)-1} p^{[k/2(b-1)]}.$$

Then Theorem 6 follows from (3.17).

References

- [1] J.F. Adams: *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.
- [2] J.F. Adams: *On the groups $J(X)$ -I*, Topology **2** (1963), 181–195.
- [3] J.F. Adams: *On the groups $J(X)$ -II, -III*, Topology **3** (1965), 137–171, 193–222.
- [4] J.F. Adams and G. Walker: *On complex Stiefel manifolds*, Proc. Camb. Phil. Soc. **61** (1965), 81–103.
- [5] D.W. Anderson: *A new cohomology theory*, Thesis, Univ. of California, Berkeley, 1964.
- [6] M.F. Atiyah: *Thom complexes*, Proc. London Math. Soc. (3) **11** (1961), 291–310.
- [7] R. Bott: *Lectures on $K(X)$* , Benjamin, 1969.
- [8] D.M. Davis and M. Mahowald: *Classification of stable homotopy types of stunted real projective spaces*, Pacific J. Math. **123** (1989), 335–345.
- [9] K. Fujii: *J -groups of lens spaces modulo powers of two*, Hiroshima Math. J. **10** (1980), 659–689.
- [10] K. Fujii, T. Kobayashi and M. Sugawara: *Stable homotopy types of stunted lens spaces*, Mem. Fac. Sci. Kochi Univ. (Math.) **3** (1982), 21–27.
- [11] K. Fujii and M. Sugawara: *The order of the canonical element in the J -groups of lens spaces*, Hiroshima Math. J. **10** (1980), 369–374.
- [12] M. Fujii and T. Yasui: *K_0 -groups of the stunted real projective spaces*, Math. J. Okayama Univ. **16** (1973), 47–54.
- [13] T. Kambe: *The structure of K_A -rings of the lens space and their applications*, J. Math. Soc. Japan **18** (1966), 135–146.
- [14] T. Kobayashi: *Stable homotopy types of stunted lens spaces mod p* , Mem. Fac. Sci. Kochi Univ. (Math.) **5** (1984), 7–14.
- [15] T. Kobayashi: *Stable homotopy types of stunted lens spaces mod 2^r* , Mem. Fac. Sci. Kochi Univ. (Math.) **11** (1990), 17–22.
- [16] T. Kobayashi and M. Sugawara: *On stable homotopy types of stunted lens spaces*, Hiroshima Math. J. **1** (1971), 287–304; *II*, **7** (1977), 689–705.
- [17] S. Kôno: *Stable homotopy types of stunted lens spaces mod 4*, to appear in Osaka J. Math..
- [18] S. Kôno and A. Tamamura: *On J -groups of $S^l(RP(t-l)/RP(n-l))$* , Math. J. Okayama Univ. **24** (1982), 45–51.
- [19] S. Kôno and A. Tamamura: *J -groups of the suspensions of the stunted lens spaces mod p* , Osaka J. Math. **24** (1987), 481–498.
- [20] S. Kôno and A. Tamamura: *J -groups of suspensions of stunted lens spaces mod 4*, Osaka J. Math. **26** (1989), 319–345.
- [21] S. Kôno and A. Tamamura: *J -groups of suspensions of stunted lens spaces mod 8*, to appear in Osaka J. Math..
- [22] S. Kôno, A. Tamamura and M. Fujii: *J -groups of the orbit manifolds $(S^{2m+1} \times S^l)/$*

- D_n by the dihedral group D_n , Math. J. Okayama Univ. **22** (1980), 205–221.
- [23] N. Mahammed: *A propos dela K -théorie des espaces lenticulaires*, C.R. Acad. Sc. Paris **271** (1970), 639–642.
- [24] D. Quillen: *The Adams conjecture*, Topology **10** (1971), 67–80.
- [25] A. Tamamura and S. Kôno: *On the KO -cohomologies of the stunted lens spaces*, Math. J. Okayama Univ. **29** (1987), 233–244.

Department of Applied Mathematics
Okayama University of Science
Ridai, Okayama 700, Japan