Shibata, K.
Osaka J. Math.
31 (1994), 431–453

UNORIENTED ANALOGUE OF ELLIPTIC GENERA

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

KATSUYUKI SHIBATA

(Received October 12, 1992)

0. Introduction

Serge Ochanine [8] determined the ideal $\mathscr{I}$ of $\Omega^{SO}\otimes\mathbf{Q}$, generated by all the oriented bordism classes $[CP(\xi^{2n})]$ of the total spaces of odd-dimensional complex projective space bundles over closed oriented smooth manifolds. He showed, in particular, that a multiplicative genus

$$\phi\ :\ \Omega^{SO}\otimes\mathbf{Q}\ \rightarrow\ \mathbf{Q}$$

annihilates the ideal $\mathscr{I}$ if and only if its logarithm $g(u)$ is formally given by an elliptic integral

$$g(u)=\int_0^u \frac{du}{\sqrt{1-2\phi([CP_2])u^2+\phi([H_{3,2}])u^4}}\ .$$

In this note we examine an unoriented bordism analogue of the above results. Namely, let $\mathscr{I}^R$ be the ideal of the unoriented bordism ring $\mathfrak{N}_*$, consisting of all the bordism classes $[RP(\zeta^{2m})]$ of the total spaces of odd-dimensional real projective space bundles over closed manifolds. We will prove that a multiplicative genus

$$\phi\ :\ \mathfrak{N}_*\rightarrow\ \mathbf{Z}_2=\mathbf{Z}/(2)$$

annihilates $\mathscr{I}^R$ if and only if its logarithm $g(u)$ is given by

$$\begin{aligned} g(u)&=\int_0^u \frac{du}{\sqrt{1-2\phi([RP_2])u^2+\phi([H^R_{3,2}])u^4}}\\ &=\int_0^u \frac{du}{1+\phi([RP_2])u^2}\ , \end{aligned}$$

which means $\phi([RP_{2i}])=\phi([RP_2])^i$ for $i\geqq 2$.

Let $\mathcal{PR}$ (2) be the subalgebra of $\mathfrak{N}_*$ generated by the bordism classes of real projective spaces of dimension a power of 2; $\{[RP_2], [RP_4], [RP_8], [RP_{16}], \cdots\}$. It is easy to see that the natural composite homomorphism

$$\mathcal{PR}\ (2) \rightarrow \mathfrak{N}_* \rightarrow \mathfrak{N}_*/\mathcal{I}^R$$

is surjective. It will be proved below by computations of the unoriented elliptic genus that the polynomials in $\mathcal{PR}$ (2) consisting of odd number of monomials are not annihilated by the homomorphism above. The investigations on polynomials with even number of terms will be carried out elsewhere by computations of other types of genera (Shibata [11]).

REMARK. I was informed by R. Stong that a result of Cappobianco [3] implies that

$$\mathfrak{N}_*/\mathcal{T}^R \cong \mathbf{Z}_2[[RP_2]]\otimes E_2(\{[RP_{2^s}]-[RP_2]^{2^{s-1}};\ s\geqq 2\}),$$

where $E_2()$ denotes the exterior algebra over $\mathbf{Z}_2$.

1. Ideal $\mathcal{I}_\infty$

Let $H^R_{i,j}$ $(i\leqq j)$ be the Milnor hypersurface of $RP_i\times PR_j=\{([a_0;\cdots;a_i], [b_0;\cdots;b_j])\}$ defined by the equation

$$a_0b_0+a_1b_1+\cdots+a_ib_i=0.$$

The natural projection

$$\pi\colon H^R_{i,j} \rightarrow RP_i$$

is a $(j-1)$-dimensional projective space bundle and hence the bordism class $[H^R_{i,2j}]$ belongs to the ideal $\mathcal{I}^R$ when $i\leqq 2j$.

For $i,j>1$, the Stiefel-Whitney number $S_{(i+j-1)}$ of $H^R_{i,j}$ is $\binom{i+j}{i}_{(2)}$ the mod 2 reduction of the binomial coefficient.

When n is an odd integer with $n+1$ not a power of 2, n is expressed as $2^d(2e+1)-1$ $(d,e$ integer $\geqq 1)$ and

$$S_{(n)}[H^R_{2^d,2^{d+1}e}]=\binom{2^{d+1}e+2^d}{2^d}_{(2)}=1.$$

Thus the class $[H^R_{2^d,2^{d+1}e}]$ can be chosen as an n-dimensional multiplicative generator of $\mathfrak{N}_*$. When n is even which is not a power of 2, n is expressed as $2^f(2g+1)$ $(f,g$ integers$\geqq 1)$ and

$$S_{(n)}[H^R_{2^f+1,2^{f+1}g}]=\binom{2^{f+1}g+2^f+1}{2^f+1}_{(2)}=1,$$

and the class $[H^R_{2^f+1,2^{f+1}g}]$ can be taken as an n-dimensional multiplicative generator.

Here we have used the well-known fact about the mod p reduction of binomial coefficients;

Theorem 1.1. (Lucas [6]). *Let $p\geqq 2$ be a prime integer, and $i=a_kp^k+a_{k-1}p^{k-1}+\cdots+a_0p^0$, and $n=b_kp^k+b_{k-1}p^{k-1}+\cdots+b_0p^0$ be the p-adic expansions of integers i and n. Then it holds that*

$$\binom{i}{n}\equiv\binom{a_k}{b_k}\binom{a_{k-1}}{b_{k-1}}\cdots\binom{a_0}{b_0}\quad \bmod p.$$

Now, for each integer $k\geqq 1$, *let* $\mathscr{I}_{2^k+1}$ be the ideal of $\mathfrak{N}_*$ generated by even-dimensional classes $[H^R_{2^f+1,2^fg}]$ $(k\geqq f\geqq 1,\ g\geqq 2)$ and all the odd-dimenional indecomposable classes $[H^R_{2^d,2^{d+1}e}]$ $(d,e\geqq 1)$.

Then

$$\mathscr{I}_3\subset\mathscr{I}_5\subset\cdots\subset\mathscr{I}_{2^k+1}\subset\cdots\subset\mathscr{I}^R\subset\mathfrak{N}_*,$$

and we define $\mathscr{I}_\infty=\bigcup_{k\geqq 1}\mathscr{I}_{2^k+1}$.

Notice that a system of irredundant multiplicative generators of $\mathfrak{N}_*$ is given by $\{[RP_{2^i}];\ i\geqq 1,\ [H^R_{2^d,2^{d+1}e}];\ d,e\geqq 1,\ [H^R_{2^f+1,2^{f+1}g}];\ f,g\geqq 1\}$.

This implies the following fact.

Proposition 1.2. *Let $\mathscr{PR}(2)=\{[RP_2],[RP_4],[RP_8],[RP_{16}],\cdots\}$ be the subalgebra of $\mathfrak{N}_*$ stated in the introduction. Then the natural composite homomorphism*

$$\mathscr{PR}(2)\to\mathfrak{N}_*\to\mathfrak{N}_*/\mathscr{I}^R$$

is surjective.

2. Dyadic derivatives

Let Λ be a division algebra over $\mathbf{Z}_2=\mathbf{Z}/(2)$, and $\Lambda[[x]]$ be the formal power series ring over Λ. For an element $f(x)=\sum_{i=0}^{\infty}a_ix^i$ of $\Lambda[[x]]$, we define the n-th dyadic derivative $D^{(n)}f(x)$ of $f(x)$ by

$$D^{(n)}f(x)=\sum_{i=n}^{\infty}\binom{i}{n}_{(2)}a_i x^{i-n}\in\Lambda[[x]],$$

where $\binom{i}{n}_{(2)}$ denotes the mod 2 reduction of the binomial coefficient.

For example,

$$D^{(1)}f(x)=\sum_{i=1}^{\infty}\binom{i}{1}_{(2)}a_i x^{i-1}=\sum_{i=0}^{\infty}a_{2i+1}x^{2i},\text{ and}$$

$$D^{(2)}f(x)=\sum_{i=2}^{\infty}\binom{i}{2}_{(2)}a_i x^{i-2}=\sum_{i=0}^{\infty}(a_{4i+2}x^{4i}+a_{4i+3}x^{4i+1}).$$

Thus the first derivative $D^{(1)}f(x)$ does *not* contain enough information to construct the second derivative $D^{(2)}f(x)$. However, the following lemma follows directly from the definition of the dyatic derivatives and Lucas' theorem 1.1 for binomial coefficients.

Lemma 2.1. *For any series $f(x)\in\Lambda[[x]]$ and any integer $j\geqq 1$ with dyadic expansion $j=a_k2^k+a_{k-1}2^{k-1}+\cdots+a_02^0$ and any permutation σ of $\{0,1,2,\cdots,k\}$, it holds that*

$$D^{(j)}f(x)=D^{(a_{\sigma(k)}2^{\sigma(k)})}\cdots\cdots D^{(a_{\sigma(0)}2^{\sigma(0)})}f(x).$$

Here we mean $D^{(0)}$ as the identity map.

Note that $(D^n(f))(0)=a_n$, and so we have the Taylor expansion formula

$f(x)=\sum_{i=0}^{\infty}(D^{(i)}f)(0)x^i$. The n-th dyadic derivative $D^{(n)}f(x)$ corresponds to $\left(\frac{1}{n!}\right)\frac{d^nf}{dx_n}(x)$ in the case of characteristic zero.

One of the important properties of the $D^{(n)}$ is the Leibniz formula.

Proposition 2.2. (Leibniz formula) *For any series $f(x)$, $g(x)\in\Lambda[[x]]$ and for any integer $n\geqq 1$, we have*

$$D^{(n)}(f(x)g(x))=\sum_{0\leqq i\leqq n}D^{(i)}(f(x))D^{(n-i)}(g(x)).$$

Proof. Comparing the coefficients of the binomial expansions of the equation $(x+y)^k=(x+y)^j(x+y)^{k-j}$, we obtain that

$$\binom{\mathrm{k}}{\mathrm{n}}=\sum_{i=0}^{n}\binom{j}{i}\binom{k-j}{n-i}$$

for every j $(0\leqq j\leqq k)$. Using this equality, it is easily verified that the both sides of the equation of the Proposition coinside.

Another important property of the $D^{(n)}$ we need is the derivation rule for composite functions.

Proposition 2.3. *Let* $f(x)$ *and* $g(x)$ *be elements of* $\Lambda[[x]]$ *such that* $g(0)=0$. *Then the composition* $f(g(x))$ *is well-defined in* $\Lambda[[x]]$, *and we have*

(2.3.1) $D^{(1)}(f(g(x)))=(D^{(1)}f)(g(x))D^{(1)}g(x),$ *and*

(2.3.2) $D^{(2)}(f(g(x)))=(D^{(2)}f)(g(x))(D^{(1)}g(x))^2+(D^{(1)}f)(g(x))D^{(2)}g(x).$

Notice that (2.3.2) can *not* be obtained from (2.3.1) and the Leibniz formula since $D^{(2)}\neq D^{(1)}\circ D^{(1)}\equiv 0$.

Proof of Proposition 2.3. When $f(x)=x^i$, we have $f(g(x))=g(x)^i$, and both (2.3.1) and (2.3.2) are obvious consequences of the Leibniz formula. Thus, for each finite sum $\sum_{i=0}^{N}a_ix^i$, the Proposition holds by the linearity of the $D^{(n)}$. For general $f(x)=\sum_{i=0}^{\infty}a_ix^i$, since $g(0)$ is assumed to vanish, the both sides of the equation of the Proposition for the finite sums converge to those for the infinite series $f(x)$ as $N\to\infty$. Q.E.D.

For an element $f(u,v)\in\Lambda[[u,v]]$, we define the n-th dyadic partial derivative $D_v^{(n)}f(u,v)\in\Lambda[[u,v]]$ with respect to the variable v by applying $D^{(n)}$ *to* $f(u,v)$ viewed as an element of $\Lambda[[u]]\,[[v]]$. The n-th partial derivative $D_u^{(n)}f(u,v)$ is defined similarly.

REMARK 2.4. We can define the p-adic derivatives for any prime integer $p\geqq 2$. They all satisfy the similar properties as those of dyadic derivatives stated above. But what we concern in this note is oly the case $p=2$.

REMARK 2.5. The definition of the dyadic derivatives $D^{(n)}$ can be extended to the finite Laurant series ring $\Lambda((x))$, and Propositions 2.2 and 2.3 still holds in $\Lambda((x))$ (Shibata [10]).

3. The formal group law and its logarithm in $\mathfrak{N}_*$-theory

Let $F^N(u, v)=u+v+\sum_{i,j=1}^{\infty}A_{ij}u^iv^j\in\mathfrak{N}_*[[u, v]]\cong\mathfrak{N}^*(RP_\infty\times RP_\infty)$ be the formal group law of $\mathfrak{N}_*$-theory and

$$G^N(u)=\sum_{i=0}^{\infty}X_iu^{i+1}\in\mathfrak{N}_*[[u]]\cong\mathfrak{N}^*(RP_\infty)$$

be its logarithm. The even degree coefficients X_{2i} are represented by the projective spaces RP_{2i} (Shibata [9]). If we set

$$H^N(u, v)=u+v+\sum_{i,j=1}^{\infty}[H_{i,j}^R]u^iv^j\in\mathfrak{N}_*[[u, v]],$$

the Atiyah-Poincare duality between $\mathfrak{N}_*(RP_i\times RP_j)$ and $\mathfrak{N}^*(RP_i\times RP_j)$ applied to the class of the natural inclusion map $[\iota: H_{i,j}^R\subset RP_i\times RP_j]$ gives rise to the following proposition.

Proposition 3.1. (Buchstaber [2], Theorem 4.8.) *It holds that*

$$H^N(u, v)=G^N(u)'G^N(v)'F^N(u, v),$$

where $G^N(u)'=\sum_{i=0}^{\infty}[RP_{2i}]u^{2i}$ *denotes the first derivative* $D^{(1)}G^N(u)$.

Applying (2.3.1) to the equality

$$G^N(F^N(u, v))=G^N(u)+G^N(v),$$

we obtain

$$(D^{(1)}G^N)(F^N(u, v))D_v^{(1)}F^N(u, v)=D^{(1)}G^N(v), \tag{3.2}$$

and thus

$$D_v^{(1)}F^N(u, v)=D^{(1)}G^N(v)B(F^N(u, v)), \tag{3.3}$$

where $B(v)\in\mathfrak{N}_*[[v]]$ denotes the multiplicative inverse of $D^{(1)}G^N(v)$.

Setting $v=0$ in (3.2) above, we obtain Honda's equality [5]

$$G^N(u)'D_v^{(1)}F^N(u, 0)\equiv 1. \tag{3.4}$$

Now, exactly in the same way as in Ochanine [8], we input the Taylor expansion

$$F^N(u, v) \equiv u + D_v^{(1)}F^N(u, 0)v + D_v^{(2)}F^N(u, 0)v^2 + D_v^{(3)}F^N(u, 0)v^3 \mod v^4$$

into Buchstaber's formula above, and then using Honda's equality (3.4), we deduce that

$$H^N(u, v) = G^N(v)'\{G^N(u)'u + v + G^N(u)'D_v^{(2)}F^N(u, 0)v^2 + G^N(u)'D_v^{(3)}F^N(u, 0)v^3\} \mod v^4.$$

If we set $R(u) = \sum_{i=1}^{\infty}[H^R_{3,2i}]u^{2i}$, we see that

$$R(u) = G^N(u)'D_v^{(3)}F^N(u, 0),$$

since $G^N(v)'$ consists only of terms with even exponents of v.

By Lemma 2.1 and (3.3), we have

$$\begin{aligned} D_v^{(3)}F^N(u, v) &= D_v^{(2)} \circ D_v^{(1)}F^N(u, v) \\ &= D_v^{(2)}(D^{(1)}G^N(v)B(F^N(u, v))) \\ &= D^{(3)}G^N(v)B(F^N(u, v)) + D^{(1)}G^N(v)D_v^{(2)}(B(F^N(u, v))) \end{aligned}$$

by virtue of the Leibniz formula and the fact that $D^{(1)} \circ D^{(1)} \equiv 0$.

Now Proposition 2.3, (2.3.2) implies that

$$\begin{aligned} D_v^{(2)}(B(F^N(u, v))) &= (D^{(2)}B)(F^N(u, v))(D_v^{(1)}F^N(u, v))^2 \\ &\quad + (D^{(1)}B)(F^N(u, v))D_v^{(2)}F^N(u, v) \\ &= (D^{(2)}B)(F^N(u, v))(D^{(1)}G^N(v)B(F^N(u, v)))^2, \end{aligned}$$

in view of (3.3). Notice that $D^{(1)}B(v) \equiv 0$ since $B(v)$ consists only of terms of even exponents of v.

Summarizing, we have obtained

$$\begin{aligned} R(u) &= G^N(u)'\{[RP_2]B(u) + (D^{(2)}B(u))B(u)^2\} \\ &= [RP_2] + (D^{(2)}B(u))B(u), \end{aligned}$$

since $D^{(3)}G^N(0) = [RP_2]$. Therefore, if we set

$B(u) = \sum_{i=0}^{\infty} B_{2i}u^{2i}$, it holds that

$$\sum_{i=1}^{\infty}[H^R_{3,2i}]u^{2i} = \sum_{i=1}^{\infty}\{B_{4i+2} + [RP_2]B_{4i} + \sum_{i=0}^{i-1} B_{4j+2}B_{4(i-j)}\}u^{4i} \tag{3.5}$$

$$+\sum_{i=0}^{\infty}(B_{4i+2})^2u^{8i+2}.$$

The comparison of the coefficients of the both sides of the equality above yields the following results.

Proposition 3.6. *Concerning the unoriented boridism classes of the Milnor hypersurfaces, it holds that*

(3.6.1) $[H^R_{3,8i+6}]=0.$

(3.6.2) $[H^R_{3,8i+2}]=(B^R_{4i+2})^2$; *a square.*

4. Multiplicative genera for $\mathfrak{N}_*$

Let Λ be a division algebra over $\mathbf{Z}_2=\mathbf{Z}/(2)$ as in section 2. *A multiplicative* Λ*-genus* is a ring homomorphism

$$\phi : \mathfrak{N}_* \to \Lambda.$$

A Λ-genus ϕ induces algebra homomorphisms

$$\phi_* : \mathfrak{N}_*[[u, v]] \to \Lambda[[u, v]], \text{ and}$$

$$\phi_* : \mathfrak{N}_*[[u]] \to \Lambda[[u]].$$

The images of $F^N(u, v)$, $G^N(u)$, $R(u)$, $B(u)$ by the homomorphism ϕ_* are henceforth denoted as $f(u, v)$, $g(u)$, $r(u)$, $b(u)$, respectively.

From formula (3.5) and Proposition 3.6, we obtain the following.

Lemma 4.1. *If a multiplicative genus* $\phi : \mathfrak{N}_* \to \Lambda$ annihilates the ideal $\mathscr{I}_3$, *then*

(i) $\phi(B_{4i+2})=0$ *for all* $i\geqq 1$, *and*
(ii) *either*
 (ii-a) $\phi(B_2)=0$, *or*
 (ii-b) $\phi(B_{4i})=0$ *for all* $i\geqq 1$.

In case of (ii-b) above, we have $b(u)=1+\phi([RP_2])u^2$, and thus

$$D^{(1)}g(u)=\frac{1}{1+\phi([RP_2])u^2}=1+\phi([RP_2])u^2+\phi([RP_2])^2u^4+\cdots$$

and consequently,

$$g(u)=u+\phi([RP_2])u^3+\phi([RP_2])^2u^5+\phi([RP_2])^3u^7+\cdots\cdots.$$

In the remaining part of this section,we examine the case (ii-a) of the Lemma above.

Lemma 4.2. *Let $n\geqq 2$ be an integer. If $\phi(\mathcal{J}_{2^n+1})=0$ and $\phi(B_{2k})=0$ for all $2k\neq 0 \bmod 2^n$, then it holds that $\phi(B_{2k})=0$ for all $2k\neq 0 \bmod 2^{n+1}$.*

Proof. First, the coefficients $\{B_0=1,\ B_2=[RP_2],\ \cdots,\ B_{2k},\ \cdots\}$ of $B(u)=(G'(u))^{-1}$ and those $\{X_0=1,\ X_2=[RP_2],\ \cdots,\ X_{2j},\ \cdots\}$ of $G'(u)$ are mutually expressed as polynomials of the others, and the hypothesis $\phi(B_{2k})=0$ for all $2k\neq 0$ mod 2^n implies $\phi(X_{2k})=0$ for all $2k\neq 0$ mod 2^n. In that case, we have

$$g(u)=\sum_{i=0}^{\infty}\phi(X_{2^ni})u^{2^ni+1},$$

and the inverse function $g^{-1}(u)$ of $g(u)$ is also of the form

$$g^{-1}(u)=\sum_{k=0}^{\infty}\bar{g}_{2^nk}u^{2^nk+1}.$$

Then

$$f(u,\ v)=u+v+\sum_{i,j=1}^{\infty}\phi(A_{ij})u^iv^j=g^{-1}(g(u)+g(v))$$

$$=\sum_{k=0}^{\infty}\bar{g}_{2^nk}\{\sum_{i=0}^{\infty}\phi(X_{2^ni})^{2^n}(u^{(2^ni+1)2^n}+v^{(2^ni+1)2^n})\}^k$$

$$\{\sum_{i=0}^{\infty}\phi(X_{2^ni})(u^{2^ni+1}+v^{2^ni+1})\}$$

Consequently, $\phi(A_{ij})=0$ unless either (i) $i\equiv 0$ mod 2^n and $j\equiv 1$ mod 2^n, or (ii) $i\equiv 1$ mod 2^n and $j\equiv 0$ mod 2^n. In view of Buchstaber's formula

$$\sum_{i,j}h_{i,j}u^iv^j=g'(u)g'(v)f(u,\ v),$$

the above fact implies that $h_{i,j}=0$ unless either (i) $i\equiv 0$ mod 2^n and $j\equiv 1$ mod 2^n, or (ii) $i\equiv 1$ mod 2^n and $j\equiv 0$ mod 2^n. In particular, $h_{2^n+1,2i}=0$ if

$2i \neq 0 \bmod 2^n$.

Now, exactly in the same way as in the proof of (3.5) for $R(u)=R_3(u)$, we define

$$R_{2^n+1}(u)=\sum_{i=1}^{\infty} [H^R_{2^n+1,2^n i}]u^{2^n i}, \quad r_{2^n+1}(u)=\phi_*(R_{2^n+1}(u)),$$

and we input the Taylor expansion of $f(u, v)$ into Buchstaber's formula;

$$h(u, v)=g'(v)\{g'(u)u+v+\sum_{k=2}^{2^n+1} g'(u)D_v{}^{(k)}f(u, 0)v^k\} \qquad \bmod v^{2^n+2},$$

and deduce that

$$r_{2^n+1}(u)=g'(u)(D_v{}^{(2^n+1)}f)(u, 0),$$

since $g'(v)$ is of the form $\sum_{i=0}^{\infty} \phi([RP_{2^n i}])v^{2^n i}$.

By Lemma 2.1, formula (3.3), and the Leibniz formula, it holds that

$$\begin{aligned}(D_v{}^{(2^n+1)}f)(u, v) &= (D_v{}^{(2^n)} \circ D_v{}^{(1)}f)(u, v)\\ &= D_v{}^{(2^n)}(g'(v)b(f(u, v)))\\ &= \sum_{0 \leqq i \leqq 2^n} D^{(i)}(g'(v))D_v{}^{(2^n-i)}(b(f(u, v)))\\ &= g'(v)D_v{}^{(2^n)}(b(f(u, v)))+D^{(2^n)}(g'(v))b(f(u, v))\end{aligned}$$

since $D^{(i)}g'(v)=0$ for $1 \leqq i < 2^n$.

In order to compute the 2^n-th derivative $D_v{}^{(2^n)}(b(f(u, v)))$, we need the following lemma.

Lemma 4.3. *Let $e(u)$ and $h(u)$ be elements of $\Lambda[[u]]$ such that $e(h(u))$ is well-defined, $D^{(1)}e(u)=0$, and that $D^{(i)}h(u)=0$ unless $i \equiv 0$ or $1 \bmod 2^n$. Then it holds that*

$$D^{(2^n)}(e(h(u)))=(D^{(2^n)}e)(h(u))D^{(1)}(h(u))^{2^n}.$$

Proof. If $e(u)=u^{2m}$, then by the Leibniz formula,

$$\begin{aligned}D^{(2^n)}(e(h(u))) &= D^{(2^n)}(h(u)^{2m})=\binom{2m}{2^n}(D^{(1)}h(u))^{2^n}h(u)^{2m-2^n}\\ &= (D^{(2^n)}e)(h(u))D^{(1)}(h(u))^{2^n}\end{aligned}$$

as desired. The general case follows from the same convergence arguments as in the proof of Proposition 2.3. Q.E.D.

Proof of Lemma 4.2 (Continued). By Lemma 4.3 above, we have

$$\begin{aligned}(D_v^{(2^n+1)}f)(u,\ v)&=g'(v)(D^{(2^n)}b)(f(u,\ v))(D_v^{(1)}f(u,\ v))^{2^n}\\&\quad+(D^{(2^n)}D^{(1)}g(v))b(f(u,\ v))\\&=g'(v)(D^{(2^n)}b)(f(u,\ v))g'(v)^{2^n}b(f(u,\ v))^{2^n}\\&\quad+(D^{(2^n+1)}g(v))b(f(u,\ v)),\end{aligned}$$

and thus

$$\begin{aligned}g'(u)D_v^{(2^n+1)}f(u,\ 0)&=g'(u)\{(D^{(2^n)}b(u))b(u)^{2^n}+x_{2^n}b(u)\}\\&=(D^{(2^n)}b(u))b(u)^{2^n-1}+x_2{}^n.\end{aligned}$$

Therefore, $\sum_{i=1}^{\infty}\ \phi([H^R_{2^n+1,2^n i}])u^{2^n i}$

$$=x_2{}^n+(\textstyle\sum_{i=1}^{\infty}\binom{i}{1}\ b_{2^n i}u^{2^n(i-1)})(1+\sum_{k=1}^{\infty}\ b_{2^n k}u^{2^n k})^{2^n-1}.$$

Up to now, we have only used the second hypothesis of the lemma;

$\phi(B_{2k})=0$ for all $2k\neq 0$ mod 2^n.

The computational result just obtained above implies that the first hypothesis $\phi(\mathcal{J}_{2^n+1})=0$ of the lemma is equivalent to

$$h_{2^n+1,2^n}u^{2^n}=x_{2^n}+(\textstyle\sum_{i=0}^{\infty}b_{2^n(2i+1)}u^{2^{n+1}i})(\sum_{k=0}^{\infty}b_{2^n k}u^{2^n k})^{2^n-1}. \tag{4.4}$$

Comparing the coefficients of u^0, u^{2^n}, $u^{2^{n+1}}$ and $u^{2^n 3}$ in the equality (4.4) above, we deduce that $b_{2^n}=x_{2^n}=0$, $h_{2^n+1,2^n}=(b_{2^n})^2=0$ and $b_{2^n 3}=0$. Therefore the equality (4.4) reduces to

$$\textstyle\sum_{i=2}^{\infty}b_{2^n(2i+1)}u^{2^{n+1}i}=\sum_{j,k=2}^{\infty}b_{2^n(2j+1)}(b_{2^n k})^{2^n}u^{2^{n+1}(2^{n-1}k+j)},$$

from which it is immediately verified by induction that $b_{2^n(2i+1)}=0$ for all $i\geqq 2$. This completes the proof of Lemma 4.2. QED.

Now in case (ii-a) of Lemma 4.1, if $\phi(\mathcal{J}_5)=0$ then the hypotheses of Lemma 4.2 for $n=2$ are satisfied, and we conclude that $\phi(B_{2k})=0$ for all $2k\neq 0$ mod 8. In this way, the inductive applications of Lemma 4.2 show that the hypothesis $\phi(B_{4k+2})=0$ for all $k\geqq 0$ and $\phi(\mathscr{J}_\infty)=0$ imply that $\phi(B_{2k})=0$ for all $k\geqq 1$. This in turn implies that $\phi(X_{2k})=0$ for all $k\geqq 1$, *i.e.* $g(u)=u$. Summarizing, we have proved the following.

Proposition 4.5. *If a multiplicative genus ϕ annihilates $\mathscr{J}_\infty$, then the induced logarithm $g(u)=\phi_*(G^N(u))$ is of the form*

$$g(u)=u+\phi([RP_2])u^3+\phi([RP_2])^2u^5+\phi([RP_2])^3u^7+$$

DEFINITION 4.6. A multiplicative genus ϕ which gives rise to a logarithm of the form given in Proposition 4.5 is called *an unoriented elliptic genus.*

Proposition 4.7. *The bordism formal group law and the related power series induced by an unoriented elliptic genus ϕ are described as follows;*

$$(4.7.1)\qquad g^{-1}(u)=\sum_{j=0}^{\infty}\phi([RP_2]^{2^j-1})u^{2^{j+1}-1},$$

$$(4.7.2)\qquad f(x,y)=g^{-1}(g(u)+g(v))$$
$$=u+v+\sum_{i=1}^{\infty}\phi([RP_2]^i)\ (u^iv^{i+1}+u^{i+1}v^i),\ \textit{and}$$

$$(4.7.3)\qquad h(u,v)=g'(u)g'(v)f(u,v)$$
$$=\sum_{0\leqq i\leqq j} h_{2j+1,2i}(u^{2j+1}v^{2i}+u^{2i}v^{2j+1})$$
$$=\sum_{0\leqq i\leqq j} \phi([RP_2]^{i+j})(u^{2j+1}v^{2i}+u^{2i}v^{2j+1})$$

Proof. If we out

$$\bar{g}(v)=\sum_{j=0}^{\infty}\phi([RP_2]^{2^j-1})v^{2^{j+1}-1},\ \text{then}$$

$$\bar{g}(v)v=\sum_{j=0}^{\infty}\phi([RP_2]^{2^j-1})v^{2^{j+1}},$$

and replacing v by $g(u)$, we obtain

$$\bar{g}(g(u))g(u)=\sum_{j=0}^{\infty}\phi([RP_2]^{2^j-1})(g(u)^{2^{j+1}})$$

$$=\sum_{j=0}^{\infty}\phi([RP_2]^{2^j-1})(\sum_{k=0}^{\infty}\phi([RP_2]^k)u^{2k+1})^{2^j+1}$$

$$=\sum_{j,k=0}^{\infty}\phi([RP_2]^{2^j(2k+1)-1})u^{2\{2^j(2k+1)\}}$$

$$=\sum_{i=1}^{\infty}\phi([RP_2]^{i-1})u^{2i}=ug(u).$$

Thus $\{\bar{g}(g(u))-u\}g(u)=0$, and so $\bar{g}(u)=g^{-1}(u)$, which proves (4.7.1).

Next, by (4.7.1) just proved above, we have

$$\{g^{-1}(g(u)+g(v))\}\{g(u)+g(v)\}=\sum_{j=0}^{\infty}\phi([RP_2]^{2^j-1})(g(u)+g(v))^{2^j+1}$$

$$=\sum_{j=0}^{\infty}\phi([RP_2]^{2^j-1})\{\sum_{i=0}^{\infty}\phi([RP_2]^{2^j+1 i})(u^{2^j+1(2i+1)}+v^{2^j+1(2i+1)})\}$$

$$=\sum_{k=0}^{\infty}\phi([RP_2]^k)(u^{2k+2}+v^{2k+2})$$

$$=\sum_{j=0}^{\infty}\phi([RP_2]^j)(u^jv^{j+1}+u^{j+1}v^j)\}\{\sum_{i=0}^{\infty}\phi([RP_2]^i)(u^{2i+1}+v^{2i+1})\}$$

$$=\sum_{j=0}^{\infty}\phi([RP_2]^j)(u^jv^{j+1}+u^{j+1}v^j)\}\{g(u)+g(v)\}.$$

This proves (4.7.2)

Finally, inputting the result just obtained above into Buchstaber's formula, we have

$$\sum_{i=0}^{\infty}\phi([RP_2]^i)u^{2i}(\sum_{j=0}^{\infty}\phi([RP_2]^j)v^{2j})\{\sum_{k=0}^{\infty}\phi([RP_2]^k)(u^kv^{k+1}+u^{k+1}v^k)\}$$

$$=\sum_{i,j,k\geqq 0}\phi([RP_2]^{i+j+k})(u^{2i+k}v^{2j+k+1}+u^{2i+k+1}v^{2j+k})$$

$$=\sum_{n=0}^{\infty}\phi([RP_2]^n)\sum_{i=0}^{n}(u^{2i}v^{2(n-i)+1}+u^{2i+(n-i)+1}v^{(n-i)})$$

$$=\sum_{0\leqq i\leqq j}\phi([RP_2]^{i+j})(u^{2i}v^{2j+1}+u^{2j+1}v^{2i})$$

as desired. This completes the proof of the lemma. QED.

Corollary 4.8. *A multiplicative genus* ϕ *annihilates the ideal* $\mathscr{I}_\infty$ *if and only if it is an unoriented elliptic genus.*

Proof. By (4.7.3), if ϕ is an elliptic genus, then $h_{2j+1,2i}=0$ for all $2j+1$ and $2i$ such that $2j+1<2i$. And $\phi(\mathfrak{N}_{\text{odd}})=0$ by definition. Thus $\phi(\mathscr{I}_\infty)=0$ and the converse of Proposition 4.5 holds. QED.

5. Multiplicative sequences and a residue theorem

A multiplicative genus

$$\phi : \mathfrak{N}_* \to \Lambda$$

can be described in terms of a multiplicative sequence in the Stiefel-Whitney classes w_i exactly in the same way as in Hirzebruch [4], §1. Namely, *a multiplicative sequence* $\{K_j;\ j=0,1,2,3,\cdots\}$ is a sequence of polynomials K_j in the variables w_i such that $K_0=1$, satisfying the condition that

$$(5.1)\qquad \textstyle\sum_{j=0}^{\infty} K_j(w_1,w_2,\cdots,w_j)x^j$$

$$=\{\textstyle\sum_{i=0}^{\infty} K_i(w'_1,w'_2,\cdots,w'_i)x^i\}\{\sum_{k=0}^{\infty} K_k(w''_1,w''_2,\cdots,w''_k)x^k\}$$

if $1+w_1x+w_2x^2+\cdots$
$$=(1+w'_1x+w'_2x^2+\cdots)(1+w''_1x+w''_2x^2+\cdots).$$

Such a sequence is uniquely determined by its characteristic power series

$$Q(x)=\textstyle\sum_{i=0}^{\infty} K_i(1,0,\cdots,0)x^i.$$

The main purpose of this section is to prove the following proposition, which is a mod 2 analogue of a theorem of Novikov ([7], IV′).

Proposition 5.2. *Let* $\{K_j\}$ *be a multiplicative sequence which gives rise to a multiplicative genus* ϕ: $\mathfrak{N}_*\to\Lambda$ *such that* $\phi(\mathfrak{N}_{\text{odd}})=0$, *and* $Q(x)$ *be its characteristic power series. Let* $g(x)$ *be the logarithm of the formal group law* $f(x,y)$ *induced from that of* $\mathfrak{N}_*$*-theory via* ϕ. *Then it holds that*

$$Q(x)=\frac{x}{g^{-1}(x)}.$$

For the proof of the above proposition, we define the *residue* of a finite Laurant series as follows.

DEFINITION 5.3. Let $\Lambda((x))$ denote the finite Laurant series ring over Λ. The *residue* $R_x(h(x))$ of a finite Laurant series $h(x)=\sum_{i=-N}^{\infty}a_ix^i \in \Lambda((x))$ is defined to be a_{-1}.

Lemma 5.4. (Invariance of residue with respect to a change of variables)

Let $h(w)$ be a finite Laurant series and $w=f(x)$ be a change of variables of the following type;

(5.5) $$f(x)=x+\sum_{i\geqq 1}a_{2i}x^{2i+1}\in\Lambda[[x]],\ i.e.\ a_{2i+1}=0.$$

Then it holds that

$$R_w(h(w))=R_x(h(f(x))f'(x)).$$

REMARK 5.6. The above lemma corresponds to the invariance of integral with respect to a change of variables;

$$\frac{1}{2\pi i}\oint g(w)dw=\frac{1}{2\pi i}\oint g(f(x))f'(x))dx.$$

Proof of Lemma 5.4. Since R_x: $\Lambda((x))\to\Lambda$ is a Λ-module homomorphism by definition, it suffices to prove the lemma for the following three cases.

case (1). $h(w)=\sum_{i=0}^{\infty}a_iw^i\in\Lambda[[w]]$. In this case, we have obviously

$$R_w(h(w))=0=R_x(h(f(x))f'(x)).$$

case (2). $h(w)=a_{-1}w^{-1}$. In this case, we have $R_w(h(w))=a_{-1}$, and

$$R_x(h(f(x))f'(x))=R_x\left(\frac{a_{-1}}{f(x)}f'(x)\right)=R_x\left(\frac{a_{-1}}{x}\right)=a_{-1},$$

since $f(x)=xf'(x)$ by hypothesis (5.5).

case (c). $h(w)=w^{-i}$ for some $i\geqq 2$. In this case, we have obviously

$R_w(h(w))=0$, and on the other hand, we have

$$R_x(h(f(x))f'(x))=R_x\left(\frac{f'(x)}{f(x)^i}\right).$$

When i is even, both $f'(x)$ and $f(x)^i$ consist of only the terms of even exponents of x. Hence their ratio $f'(x)/f(x)^i$ does not contain a monomial of exponent -1. When i is odd$\geqq 3$, we express i as $i=2^a(2b+1)+1$ with $a\geqq 1$, $b\geqq 0$. Then

$$\frac{f'(x)}{f(x)^i}=\frac{1}{x^i f'(x)^{i-1}}=\frac{(f'(x)^{-1})^{2^a(2b+1)}}{x^i}.$$

Since $f'(x)^{-1}$ consists only of monomials of even exponents of x, the numerator in the above formula consists only of monomials with exponents divisible by 2^{a+1}. Thus it does not contain the term with exponent $i-1$. This implies $R_x\left(\frac{f'(x)}{f(x)^i}\right)=0$. QED.

Proof of Proposition 5.2. In view of the fact that $g(x)=\sum_{i=0}^{\infty}\phi([RP_{2i}])x^{2i+1}$ (Shibata [9]), $\phi([RP_{2i}])$ can be interpreted as the residue $R_x(g'(x)/x^{2i+1})$, where $g'(x)=D^{(1)}g(x)=g(x)/x$. If we put $w=g(x)$, then $x=g^{-1}(w)$ and we have

$$R_x\left(\frac{g'(x)}{x^{2i+1}}\right)=R_x\left(\frac{g'(x)}{\{g^{-1}(g(x))\}^{2i+1}}\right)=R_w\left(\frac{1}{g^{-1}(w)^{2i+1}}\right)$$

by Lemma 5.4. Hence we obtain

$$\phi([RP_{2i}])=R_w\left(\frac{1}{w^{2i+1}}\left(\frac{w}{g^{-1}(w)}\right)^{2i+1}\right). \tag{5.7}$$

On the other hand, let τ be the tagent bundle of RP_{2i}. Then $\tau\oplus 1_R=(2i+1)\eta$, where η is the Hopf line bundle over RP_{2i}, and consequently $w_*(\tau)=(1+w_1)^{2i+1}$, with $w_1=w_1(\eta)$. Therefore we obtain

$$\phi([RP_{2i}])=\langle Q(w_1)^{2i+1},[RP_{2i}]\rangle=R_w\left(\frac{1}{w^{2i+1}}\,Q(w)^{2i+1}\right). \tag{5.8}$$

Now that (5.7) and (5.8) hold for every $i\geqq 0$, we conclude that $Q(w)=w/g^{-1}(w)$. QED.

6. Cohomology of projective space bundles

Let $\zeta^m \to M$ be a real m-dimensional vector bundle over a closed smooth manifold M, and $RP(\zeta)\to M$ be its associated projective space bundle. If we denote the Hopf line bundle over $RP(\zeta)$ by η and its first Stiefel-Whitney class $w_1(\eta)$ by $t\in H^1(RP(\zeta);\ \Lambda)$, the structure of the cohomology ring $H^*(RP(\zeta);\ \Lambda)$ is described as follows (Borrel-Hirzebruch [1], §15).

(6.1) The natural projection π: $RP(\zeta)\to M$ induces an injection

$$\pi^*\colon H^*(M;\Lambda)\to H^*(RP(\zeta);\ \Lambda),$$

and $H^*(RP(\zeta);\ \Lambda)$ is a free $H^*(M;\Lambda)$-module via π^* with basis $\{1,t,t^2,\cdots,t^{m-1}\}$. The multiplicative structure is determined by the following equality;

(6.2) $t^m+\pi^*(w_1(\zeta))t^{m-1}+\cdots+\pi^*(w_{m-1}(\zeta))t+\pi^*(w_m(\zeta))=0.$

Hence, each element y of $H^*(RP(\zeta);\ \Lambda)$ can be expressed as

$$y=b_0+b_1t+\cdots+b_{m-1}t^{m-1};\ b_i\in H^*(M:\Lambda),$$

and it holds that

$$<y,\ [RP(\zeta)]>=<b_{m-1},\ [M]>\in\Lambda.$$

Now that $T(RP(\zeta))\oplus 1_R\simeq\pi^*T(M)\oplus(\eta\otimes\pi^*\zeta)$, the multiplicative sequence $K(T(RP(\zeta)))$ factors as $\pi^*K(T(M))K(\eta\otimes\pi^*\zeta)$.

Following exactly in the same way as in Ochanine [8], we examine the conditions for the vanishing of the coefficient of t^{m-1} in $K(\eta\otimes\pi^*\zeta)$.

If we formally split $w_*(\pi^*\zeta)$ as $\prod_{1\leqq k\leqq m}(1+u_k)$, then $K(\eta\otimes\pi^*\zeta)=\prod_{1\leqq k\leqq m}Q(t+u_k)$, and equality (6.2) becomes

(6.2)' $\prod_{1\leqq k\leqq m}(t+u_k)=0.$

According to Proposition 5.2, we consequently have

(6.3) $K(\eta\otimes\pi^*\zeta)=\prod_{1\leqq k\leqq m}\dfrac{(t+u_k)}{g^{-1}(t+u_k)}.$

Notice that the right side of (6.3) above is a non-homogeneous formal power series in $t,u_1,\cdots,u_m$, but we regard each homogeneous part as a polynomial in t with coefficients in $\Lambda[u_1,u_2,\cdots,u_m]$, and divide it by the

left-hand side of (6.2)', which is a polynomial in t of degree m but is also regarded as a homogeneous polynomial in $t,u_1,\cdots,u_m$ so that the formal series computations below should be considered degree-wise as polynomial computations.
Thus we obtain

$$(6.3)' \qquad K(\eta\otimes\pi^*\zeta)=F(u_1,\cdots,u_m,t)+\{\prod_{1\leqq k\leqq m}(t+u_k)\}A(u_1,\cdots,u_m,t),$$

with $F(u_1,\cdots,u_m,t)$ a polynomial in t of degree less than m with coefficients in $\Lambda[[u_1,u_2,\cdots,u_m]]$.

Since $Q(w)=1+\phi([RP_2])w^2+\cdots$, it holds that

$$(6.4) \qquad F(u_1,\cdots,u_m,u_i)=\prod_{k\neq i}Q(u_i+u_k)$$

for every $i=1,2,\cdots,m$. (We are working in characteristic 2.)

Let us put

$$G(u_1,\cdots,u_m,t)=\{\prod_{1\leqq j<k\leqq m}(u_j+u_k)\}F(u_1,\cdots,u_m,t)$$
$$-\textstyle\sum_{1\leqq i\leqq m}\{\prod_{\substack{1\leqq j<k\leqq m\\ j\neq i\neq k}}(u_j+u_k)\}\{\prod_{h\neq i}Q(u_i+u_h)(u_h+t)\}.$$

Then G is a polynomial in t of degree $m-1$ with coefficients in $\Lambda[[u_1,u_2,\cdots,u_m]]$. By vertue of (6.4), $t+u_i$ divide G for every $i=1,\cdots,m$. This implies that the product $\prod_{1\leqq i\leqq m}(t+u_i)$ divides G since $\Lambda[u_1,u_2,\cdots,u_m][t]$ is a unique factorization domain. Hence we conclude that $G(u_1,\cdots,u_m,t)\equiv 0$ by degree arguments. Therefore we have shown the following.

Lemma 6.5. *Let* $F(u_1,\cdots,u_m,t)$ *be as defined in* (6.3)'. *Then the coefficient of* t^{m-1} *in* $\{\prod_{1\leqq j<k\leqq m}(u_j+u_k)\}$ $F(u_1,\cdots,u_m,t)$ *is given by*

$$\textstyle\sum_{1\leqq i\leqq m}\{\prod_{\substack{1\leqq j<k\leqq m\\ j\neq i\neq k}}(u_j+u_k)\}\{\prod_{h\neq i}Q(u_i+u_h)\}.$$

The rest of this section is devoted to the proof that the coefficient of t^{m-1} given in the Lemma above vanishes for elliptic genera in case m is even.

First we have the following lemma.

Lemma 6.6. *Let $Q(u)$ be the characteristic power series associated to an unoriented elliptic genus ϕ: $\mathfrak{N}_* \to \Lambda$. Then it holds that*

$$Q(u)=1+\sum_{j\geqq 0}\{\phi([RP_2])u^2\}^{2^j}.$$

Proof. If we put

$$q(u)=1+\sum_{j\geqq 0}\{\phi([RP_2])u^2\}^{2^j},$$

then, by virtue of (4.7.1), we have

$$q(u)=1+\phi([RP_2])ug^{-1}(u).$$

Consequently, it holds that

$$g^{-1}(u)q(u)=g^{-1}(u)+\phi([RP_2])u\{g^{-1}(u)\}^2$$

$$=\sum_{j\geqq 0}\{\phi([RP_2])^{2^j-1}u^{2^{j+1}-1}\}$$

$$+\phi([RP_2])u\sum_{j\geqq 0}\{\phi([RP_2])^{2^{j+1}-2}u^{2^{j+2}-2}\}$$

$$=u.$$

Therefore $q(u)=u/g^{-1}(u)=Q(u)$. QED.

Corollary 6.7. *Let $Q(u)$ be the characteristic power series associated to an unoriented elliptic genus ϕ: $\mathfrak{N}_* \to \Lambda$. Then it satisfies the following formulas;*

(6.7.1) $Q(u+v)=1+Q(u)+Q(v)$, *and*

(6.7.2) $Q(u)^2=\phi([RP_2])u^2+Q(u)$.

Proof. Since we are working in characteristic 2, it holds that $(u+v)^{2^i}=u^{2^i}+v^{2^i}$. So the corollary follows directly from Lemma 6.6.

Notations and Definitions 6.8.

(6.8.1) Let $S_j(m)$ denote the j-th elementary symmetric function of $Q(u_1)$,

$Q(u_2),\cdots,Q(u_m)\}$, and $S_j^{<i>}(m)$ denote the j-th elementary symmetric function $\{Q(u_1),\ Q(u_2),\cdots\overset{i}{\vee}\cdots,Q(u_m)\}$.

(6.8.2) Let $T_k(u)$, $U_k(u)$ $(k=0,1,2,\cdots)$ be the polynomials in u defined inductively by

$T_0=0$, $U_0=1$, and

$T_{k+1}=U_k$, $U_{k+1}=\phi([RP_2])u^2T_k+U_k$ for $k\geqq 0$.

Lemma 6.9. *Let $Q(u)$ be as in Corollary 6.7. Then it holds that*

(6.9.1) $$\{Q(u)+1\}^k=T_k(u)Q(u)+U_k(u),\ \textit{and}$$

(6.9.2) $$\prod_{\substack{1\leqq h\leqq m\\ h\neq i}}Q(u_i+u_h)=\textstyle\sum_{0\leqq j\leqq m-1}S_j(m)T_{m-j}(u_i).$$

Proof. Equality (6.9.1) is easily proved by induction on k, using (6.7.2). And (6.7.1) implies

$$\prod_{\substack{1\leqq h\leqq m\\ h\neq i}}Q(u_i+u_h)=\prod_{\substack{1\leqq h\leqq m\\ h\neq i}}\{Q(u_i)+1+Q(u_h)\}$$

$$=\textstyle\sum_{0\leqq j\leqq m-1}S_j^{<i>}(m)\{Q(u_i)+1\}^{m-1-j}$$

$$=S_0^{<i>}(m)U_{m-1}(u_i)$$

$$+\textstyle\sum_{0\leqq j\leqq m-2}\{S_j^{<i>}(m)Q(u_i)T_{m-1-j}(u_i)+S_{j+1}^{<i>}(m)U_{m-1-j-1}(u_i)\}$$

$$+S_{m-1}^{<i>}(m)Q(u_i)T_0(u_i)$$

$$=\textstyle\sum_{0\leqq j\leqq m-1}S_j(m)T_{m-j}(u_i),\ \text{as desired.}$$

Lemma 6.10. *Let n be an integer greater than or equal to 2. We have a congruence*

$$\prod_{1\leqq j<k\leqq n}(u_j+u_k)\equiv\sum_{\sigma\in P_n}u_{\sigma(1)}^{n-1}u_{\sigma(2)}^{n-2}\cdots u_{\sigma(n-1)}^{1}u_{\sigma(n)}^{0}$$

modulo 2, where P_n denotes the permutation group of $\{1,2,\cdots,n\}$.

Proof. The lemma is trivial when $n=2$. Assume it is also true up to $n-1$. Then the left-hand side of the above formula is

$$\prod_{2\leqq k\leqq n}(u_1+u_k)\prod_{2\leqq j<k\leqq n}(u_j+u_k)$$

$$=\{\textstyle\sum_{0\leqq i\leqq n-1}u_1^i S_{n-1-i}(u_2,u_3,\cdots,u_m)\}\{\sum_{\rho\in P'_{n-1}}u_{\rho(2)}^{n-2}\cdots u_{\rho(n)}^0\},$$

where $S_{n-1-i}(u_2,u_3,\cdots,u_n)$ denotes the $(n-1-i)$-th elementary symmetric function in $\{u_2,u_3,\cdots,u_n\}$ and P'_{n-1} denotes the permutation group of $\{2,3,\cdots n\}$. Since this is a symmetric function, it is sufficient to consider the terms $u_1^{i_1}u_2^{i_2}\cdots u_n^{i_n}$ with $i_1\geqq i_2\geqq\cdots\geqq i_n$, namely the cases $i=n-1$ and $n-2$. When $i=n-1$, the term in question is $u_1^{n-1}u_2^{n-2}\cdots u_n^0$ and this term gives the desired one in the right-hand side of the formula of the lemma. When $i=n-2$, the terms in question among

$$u_1^{n-2}(u_2+u_3+\cdots+u_n)\{\sum_{\rho\in P'_{n-1}}u_{\rho(2)}^{n-2}\cdots u_{\rho(n)}^0\}$$

are

$$\textstyle\sum_{3\leqq k\leqq n}\{u_1^{n-2}u_k(u_2^{n-2}\cdots u_{k-1}^{n-(k-1)}u_k^{n-k}\cdots u_n^0)$$
$$+u_1^{n-2}u_{k-1}(u_2^{n-2}\cdots u_k^{n-(k-1)}u_{k-1}^{n-k}\cdots u_n^0)\}$$
$$\equiv 0 \text{ modulo } 2.$$

Corollary 6.11. *For any even integer $m\geqq 2$ and any nonnegative integer $n\leqq m-2$, we have a congruence*

$$\sum_{1\leqq i\leqq m}\{\prod_{\substack{1\leqq j<k\leqq m\\ j\neq i\neq k}}(u_j+u_k)\}u_i^n\equiv 0 \ \mathbf{mod}\ 2.$$

Proof. First, let us consider the terms corresponding to $i=m$ in the above summation. By Lemma 6.10, they are congruent to

$$\{\sum_{\sigma\in P_{m-1}}u_{\sigma(1)}^{m-2}u_{\sigma(2)}^{m-3}\cdots u_{\sigma(m-1-n)}^n\cdots u_{\sigma(m-1)}^0\}u_m^n \quad (0\leqq n\leqq m-2).$$

Each of these terms have a unique counterpart among the terms corresponding to $i=\sigma(m-1-n)$. Hence, summation being taken for all the i, all of these terms are cancelled out modulo 2. The situation is the same for all the other cases of i in the summation, too. This proves the Corollary.

Corollary 6.12. *The coefficient*

$$\sum_{1\leqq i\leqq m}\{\prod_{\substack{1\leqq j<k\leqq m\\ j\neq i\neq k}}(u_j+u_k)\}\{\prod_{h\neq i}Q(u_i+u_h)\}.$$

of t^{m-1} given in Lemma 6.5 vanishes when $Q(u)$ is the characteristic power series of an unoriented elliptic genus with m even.

Proof. In view of Lemma 6.9, 6.10 and Corollary 6.11, it is sufficient to show that, when m is even, $T_{m-j}(u)$ is a polynomial in u of degree less than or equal to $m-2$ for every j $(0\leqq j\leqq m-1)$. But it is easily proved by induction arguments that $T_{2n}(u)$ is of degree$\leqq 2n-2$, and that $T_{2n+1}(u)=U_{2n}(u)$ is of degree$\leqq 2n$. This proves the corollary.

Corollary 6.13. *Let ϕ: $\mathfrak{N}_*\to\Lambda$ be a multiplicative genus. Then the following three conditions are equivalent;*

(i) $\phi(\mathscr{I}_\infty)=0$,

(ii) $\phi(\mathscr{I}^R)=0$, *and*

(iii) ϕ *is an unoriented elliptic genus, i.e.* $\phi([RP_{2i}])=\phi([RP_2])^i$ *and* $\phi(\mathfrak{N}_{\text{odd}})=0$.

Proof. The equivalence of (i) and (iii) are already proved in Corollary 4.8. Now by Lemma 6.5 and Corollary 6.12, (iii) implies (ii). And since $\mathscr{I}_\infty\subset\mathscr{I}^R$, (ii) implies (i). QED.

Corollary 6.14. *Let*

$$\mathscr{P}\mathscr{R}(2)\to\mathfrak{N}_*\to\mathfrak{N}_*/\mathscr{I}^R$$

be the natural composite homomorphism defined in the introduction. No polynomial in $\mathscr{P}\mathscr{R}$ (2) consisting of odd number of monomials is annihilated by the homomorphism above.

Proof. Let ϕ_0: $\mathfrak{N}_*\to\mathbf{Z}/2\mathbf{Z}$ be the unoriented elliptic genus defined by $\phi_0([RP_{2i}])=1$ for every $i\geqq 0$. And Let X be any polynomial in $\mathscr{P}\mathscr{R}$ (2) consisting of odd number of monomials. Then $\phi_0(X)=1$ by definition, and so X does not belong to $\mathscr{I}^R$. QED.

REMARK. 6.15. In the complex bordism ring U_*, the classes $[CP_2]$ and $[H_{3,2}]$ are algebraically independent, and Ochaine [8] shows that the image of U_* by an elliptic genus is generated by those of $[CP_2]$ and $[H_{3,2}]$. In contrast, in the unoriented bordism ring $\mathfrak{N}_*$, it holds that $[H^R_{3,2}]=[RP_2]^2$ and we have shown in this note that the image of $\mathfrak{N}_*$ by an unoriented elliptic genus is generated by that of $[RP_2]$. This difference is the main reason why the unoriented elliptic genera are not rich enough to determine whether $\mathscr{I}_\infty=\mathscr{I}^R$ or not, while the elliptic genera in U_*-theory are so.

Added in proof. After writing up the present note, the author recognized that the left-hand side of the formula in Corollary 6.11 is a Vandermonde determinant, and it consequently vanishes not only modulo 2 but integrally if we replace the plus signs by the minus ones.

References

[1] A. Borrel and F. Hirzebruch: *Characteristic classes and homogeneous spaces*, I. Am. J. Math., **80** (1958), 458–538.

[2] V.M. Buchstaber: *The Chern-Dold character in cobordisms*, I. Math. USSR-Sbornik, **12** (1970), 573–594.

[3] F.L. Capobianco: *Cobordism classes represented by fibering with fiber RP(2k+1)*, Michigan Math. J., **24** (1977), 185–192.

[4] F. Hirzebruch: *Topological methods in algebraic geometry*, Third enlarged edition, Berlin-Heidelberg-New York; Springer 1966.

[5] T. Honda: *Formal groups and zeta-functions*, Osaka J. Math. **5** (1968), 199–213.

[6] E. Lucus: *Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques, suivant un module premier*, Bull. Soc. Math. de France, **6** (1878), 49–54.

[7] S.P. Novikov: *Adams operators and fixed points*, Izv. Acad. Nauk SSSR, Ser. Mat. **32** (1968), 1245–1263=Math. USSR Izv. **2** (1968), 1193–1212.

[8] S. Ochanine: *Sur les genres multiplicatifs définis par des intégrales elliptiques*, Topology, **26** (1987), 143–152.

[9] K. Shibata: *On Boardman's generating set of the unoriented bordism ring*, Osaka J. Math., **8** (1971), 219–232.

[10] K. Shibata: *On the dyadic derivatives in finite Laurant series rings*, J. Coll. Liberal Arts, Nat. Sci, Saitama Univ., **28** (1992), 1–4.

[11] K. Shibata: *Quasi-elliptic genera and local genera for the unoriented bordism theory*, in preparation.

Department of Mathematics
College of Liberal Arts
Saitama University
Urawa, Saitama 338, Japan