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REIDEMEISTER TORSION OF THE FIGURE-EIGHT KNOT EXTERIOR FOR SL(2; *C*)-REPRESENTATIONS

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0. Introduction

Reidemeister torsion was originally defined by Reidemeister, Franz, and de Rham. A generalized version of this invariant is defined for a pair of a manifold together with a representation of its fundamental group in certain Lie groups(see section 1 below). We consider it as a function defined on the space of representations of the fundamental group. Recently Johnson obtained an explicit formula of the Reidemeister torsion of Brieskorn homology 3-spheres for $SL(2; \boldsymbol{C})$-representations. The sets of values of the Reidemeister torsion of these manifolds turn to be finite sets in $\boldsymbol{R}$, though they are subsets in $\boldsymbol{C}$ by the definition. In our previous paper [2], we obtained an explicit formula of the Reidemeister torsion of Seifert fibered spaces for $SL(2; \boldsymbol{C})$-representations. This is an extension of Johnson's result mentioned above. In particular the sets of values of the Reidemeister torsion are again finite subsets in $\boldsymbol{R}$. It follows that it has no continuous variations, although the dimension of the space of the representations of the fundamental group of these manifolds is generally positive.

In this paper we consider the problem of determining whether there exists a closed 3-manifold with continuous variations of the Reidemeister torsion for $SL(2; \boldsymbol{C})$-representations. In order to attack this problem, we first need to investigate the spaces of $SL(2; \boldsymbol{C})$-representations for given manifolds. Applying the method due to Riley to the Wirtinger presentation of the figure-eight knot, we determine the space of representations of the fundamental group of the figure-eight knot exterior. Then by the method of Johnson, we obtain an explicit formula of the Reidemeister torsion of the exterior of the figure-eight knot. The formula shows that it has continuous variations. The main result of this paper is the following.

Main Theorem. *Let $K \subset S^3$ denote the figure-eight knot, and E its exterior; i.e., the complement of an open tubular neighborhood of K. Let M denote the double $E \bigcup_{id} E$ of E. Then the set of values of the Reidemeister torsion $\tau(M; V_p)$ of M for $SL(2; \boldsymbol{C})$-representations is the set of all nonzero complex numbers. Therefore $\tau(M; V_p)$ has continuous variations.*

Now we describe the contents of this paper briefly. In section 1 we give a definition of the Reidemeister torsion according to Milnor. We restrict the definition to the case of $SL(2; \boldsymbol{C})$-representation. In section 2 we review Johnson's theory. It gives an explicit formula of the Reidemeister torsion of knot exteriors. In section 3 we give a proof of the main theorem. To compute the matrix A_1, which is a generalization of the Alexander matrix, and its determinant, we used a computer(Reduce 2.3.).

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1. Definition of Reidemeister torsion

First let us describe the definition of the Reidemeister torsion for $SL(2; \boldsymbol{C})$-representations. See Johnson [1] and Milnor [4], [5], [6] for details.

Let S be an n-dimensional vector space over $\boldsymbol{C}$ and let $\boldsymbol{b} = (b_1, \cdots, b_n)$ and $\boldsymbol{c} = (c_1, \cdots, c_n)$ be two bases for S. Setting $b_i = \sum_{j=1}^{n} p_{ij} c_j$, we obtain a nonsingular matrix $P = (p_{ij})$ with entries in $\boldsymbol{C}$. Let $[\boldsymbol{b}/\boldsymbol{c}]$ denote the determinant of P.

Suppose

$$C_*:\ 0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$$

is an acyclic chain complex of finite dimensional vector spaces over $\boldsymbol{C}$. We assume that a preferred basis $\boldsymbol{c}_q$ for $C_q(C_*)$ is given for each q. Choose some basis $\boldsymbol{b}_q$ for $B_q(C_*)$ and take a lift of it in $C_{q+1}(C_*)$, which we denote by $\tilde{\boldsymbol{b}}_q$.

Since $B_q(C_*) = Z_q(C_*)$, the bases $\boldsymbol{b}_q$ can serve as a basis for $Z_q(C_*)$. Furthermore the sequence

$$0 \to Z_q(C_*) \to C_q(C_*) \to B_{q-1}(C_*) \to 0$$

is exact and the vectors $(\boldsymbol{b}_q, \tilde{\boldsymbol{b}}_{q-1})$ form a basis for $C_q(C_*)$. It is easily shown that $[\boldsymbol{b}_q, \tilde{\boldsymbol{b}}_{q-1}/\boldsymbol{c}_q]$ does not depend on the choice of the lift $\tilde{\boldsymbol{b}}_{q-1}$.

Hence we simply denote it by $[\boldsymbol{b}_q, \boldsymbol{b}_{q-1}/\boldsymbol{c}_q]$.

DEFINITION 1.1. The torsion of the chain complex C_* is given by the alternating product

$$\prod_{q=0}^{m} [\boldsymbol{b}_q, \boldsymbol{b}_{q-1}/\boldsymbol{c}_q]^{(-1)^q}$$

and we denote it by $\tau(C_*)$.

REMARK. $\tau(C_*)$ depends only on the bases $\{\boldsymbol{c}_0, \cdots, \boldsymbol{c}_m\}$.

Now we apply this torsion invariant of chain complexes to the following geometric situation. Let X be a finite cell complex and $\tilde{X}$ a universal covering of X. The fundamental group $\pi_1 X$ acts on $\tilde{X}$ as deck transformations. Then the chain complex $C_*(\tilde{X}; \boldsymbol{Z})$ has the structure of a chain complex of free $\boldsymbol{Z}[\pi_1 X]$-modules. Let ρ: $\pi_1 X \to SL(2; \boldsymbol{C})$ be a representation. We denote the 2-dimensional vector space $\boldsymbol{C}^2$ by V. Using the representation ρ, V has the structure of a $Z[\pi_1 X]$-module and then we denote it by V_ρ. Define the chain complex $C_*(X; V_\rho)$ by $C_*(\tilde{X}; \boldsymbol{Z}) \otimes_{Z[\pi_1 X]} V_\rho$ and choose a preferred basis

$$\{\sigma_1 \otimes \boldsymbol{e}_1, \sigma_1 \otimes \boldsymbol{e}_2, \cdots, \sigma_{k_q} \otimes \boldsymbol{e}_1, \sigma_{k_q} \otimes \boldsymbol{e}_2\}$$

of $C_q(X; V_\rho)$ where $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a canonical basis of V and $\sigma_1, \cdots, \sigma_{kq}$ are q-cells giving the preferred basis of $C_q(\tilde{X}; \boldsymbol{Z})$.

We consider the situation where $C_*(X; V_\rho)$ is acyclic. Namely all homology groups vanish; $H_*(X; V_\rho) = 0$. In this case we call ρ an acyclic representation.

DEFINITION 1.2. Let ρ: $\pi_1 X \to SL(2; \boldsymbol{C})$ be an acyclic representation. Then the Reidemeister torsion of X with V_ρ-coefficients is defined to be the torsion of the chain complex $C_*(X; V_\rho)$. We denote it by $\tau(X; V_\rho)$.

REMARK. The Reidemeister torsion $\tau(X; V_\rho)$ seems to depend on several choices. However it is well known that the Reidemeister torsion is a piecewise linear invariant. See Johnson [1], Milnor [4], [6].

The next proposition, which gives the Mayer-Vietoris argument in our context, is well known but important.

See Johnson [1], Milnor [6].

Proposition 1.3. *Let M be a closed, oriented 3-manifold with torus* *decomposition* $A\bigcup_{T^2}B$ *and* $\rho\colon \pi_1 M\to SL(2;\ \boldsymbol{C})$ *a representation whose restriction to* $\pi_1 T^2$ *is acyclic. Then* $H_*(M;\ V_\rho)=0$ *if and only if* $H_*(A;\ V_\rho)=H_*(B;\ V_\rho)=0$. *Moreover in this case*

$$\tau(M;\ V_\rho)=\tau(A;\ V_\rho)\tau(B;\ V_\rho).$$

Therefore if there exists a knot whose exterior has continuous variations, then the double of it also has continuous variations. In section 3 we prove that the exterior of the figure-eight knot has continuous variations.

2. Reidemeister torsion of the knot exterior

In this section, we give a review of the Reidemeister torsion of a knot exterior. See Johnson [1] and Milnor [5].

Let $K\subset S^3$ be a knot and E its exterior. We fix a Wirtinger presentation of the knot group $\pi_1 E$ as follows;

$$\pi_1 E=<x_1,x_2,\cdots,x_n\,|\,r_1,r_2,\cdots,r_{n-1}>$$

where r_i is the crossing relation for each i. Let $\rho\colon \pi_1 E\to SL(2;\ \boldsymbol{C})$ be a representation. When a representation ρ is fixed, we denote the matrix $\rho(x)$ for $x\in\pi_1 E$ by the corresponding capital letter X. For example, for $x_1\in\pi_1 E$, we denote the matrix $\rho(x_1)$ by X_1. Consider a matrix

$$A=\begin{pmatrix}\rho\left(\frac{\partial r_1}{\partial x_1}\right) & \cdots & \rho\left(\frac{\partial r_1}{\partial x_n}\right)\\ \vdots & \ddots & \vdots\\ \rho\left(\frac{\partial r_{n-1}}{\partial x_1}\right) & \cdots & \rho\left(\frac{\partial r_{n-1}}{\partial x_n}\right)\end{pmatrix}$$

where each $\rho\left(\frac{\partial r_i}{\partial x_j}\right)$ denotes the image of the free derivative $\frac{\partial r_i}{\partial x_j}$ in 2×2-matrixes. More precisely if $\frac{\partial r_i}{\partial x_j}=\sum_k a_k g_k$ where $a_k\in\boldsymbol{Z}$ and $g_k\in\pi_1 E$, we denote $\sum_k a_k\rho(g_k)$ in 2×2-matrixes by $\rho\left(\frac{\partial r_i}{\partial x_j}\right)$. We denote by A_1 the matrix obtained by removing the first column from A. Then Johnson has shown the next formula.

Theorem 2.1. (Johnson) *Let* $\rho\colon \pi_1 E\to SL(2;\ \boldsymbol{C})$ *be a representation such that* $\det(X_1-I)\neq 0$. *Then all homology groups vanish*: $H_*(E;\ V_\rho)=0$ *if and only if* $\det A_1\neq 0$. *In this case the Reidemeister torsion is given by*

$$\tau(E;\ V_\rho)=\frac{\det(X_1-I)}{\det A_1}\ .$$

REMARK.

(1) The definition of $\det A_1$ above is analogous to the standard method of computing the Alexander polynomials of knots. Milnor has shown a parallel result for the Alexander polynomial. See Milnor [5].

(2) Recently Wada [8] defined the twisted Alexander polynomial for finitely presentable groups. It is a generalization of the Alexander polynomial. In the case of the group of a knot, or a link, we will interpret this polynomial as a Reidemeister torsion of its exterior in [3].

Let W be a 2-dimensional complex constructed from n 1-cells $x_1,\cdots,x_n$ and $(n-1)$ 2-cells $D_1,\cdots,D_{n-1}$ with attaching maps $r_1,\cdots,r_{n-1}$. It is well-known that the knot exterior E collapses to the 2-dimensional complex W. Then it holds that

$$\tau(E;\ V_\rho)=\tau(W;\ V_\rho)$$

by the simple homotopy invariance of the Reidemeister torsion. To prove Theorem 2.1, we show that

$$\tau(W;\ V_\rho)=\frac{\det(X_1-I)}{\det A_1}\ .$$

By an easy computation, this chain complex $C_*(W;\ V_\rho)$ can be described as follows;

$$0\to V^{n-1}\xrightarrow{\partial_2}V^n\xrightarrow{\partial_1}V\to 0$$

where

$$\begin{aligned}\partial_2&=A\\ &=\begin{pmatrix}\rho\left(\frac{\partial r_1}{\partial x_1}\right) & \cdots & \rho\left(\frac{\partial r_1}{\partial x_n}\right)\\ \vdots & \ddots & \vdots\\ \rho\left(\frac{\partial r_{n-1}}{\partial x_1}\right) & \cdots & \rho\left(\frac{\partial r_{n-1}}{\partial x_n}\right)\end{pmatrix},\end{aligned}$$

$$\partial_1=\begin{pmatrix}X_1-I\\ X_2-I\\ \cdots\\ X_n-I\end{pmatrix}.$$

Here we briefly denote by V^k the k-times direct sum of V.

Proposition 2.2. *Let ρ: $\pi_1 W \to SL(2; \boldsymbol{C})$ be a representation such that the determinant* $\det(X_1 - I) \neq 0$. *Then all homology groups vanish*: $H_*(W;\ V_\rho)=0$ *if and only if* $\det A_1 \neq 0$. *In this case, we have*

$$\tau(W;\ V_\rho)=\frac{\det(X_1-I)}{\det A_1}\ .$$

Proof. It is obvious that $H_0(W;\ V_\rho)$ is trivial because $\det(X_1-I)\neq 0$ and hence the boundary map ∂_1 is surjective. For a canonical basis $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ of V, we choose lifts $\tilde{\boldsymbol{e}}_1=((X_1-I)^{-1}\boldsymbol{e}_1,0,\cdots,0)$, $\tilde{\boldsymbol{e}}_2=((X_1-I)^{-1}$ $\boldsymbol{e}_2,0,\cdots,0)$ in V_n. Define the $2n\times 2n$ matrix $\tilde{A}$ whose first $2n-2$ rows are ∂_2 and last two rows are $\tilde{\boldsymbol{e}}_1$ and $\tilde{\boldsymbol{e}}_2$. The matrix $\tilde{A}$ takes the form

$$\tilde{A}=\begin{pmatrix} * & A_1 \\ (X_1-I)^{-1}\boldsymbol{e}_1 & 0\cdots 0 \\ (X_1-I)^{-1}\boldsymbol{e}_2 & 0\cdots 0 \end{pmatrix}.$$

It is obvious that $\det\tilde{A}\neq 0$ if and only if $\det A_1\neq 0$. If all homology groups vanish: $H_*(W;\ V_\rho)=0$, then

$$\begin{aligned}\operatorname{rank} A &= \operatorname{rank} A_1\\ &= 2n\text{-}2,\end{aligned}$$

hence $\det A_1\neq 0$. In this case the Reidemeister torsion is given by

$$\begin{aligned}\tau(W;\ V_\rho) &= \frac{1}{\det\tilde{A}}\\ &= \frac{\det(X_1-I)}{\det A_1}\ .\end{aligned}$$

It is obvious that the contrary is also true. Namely if $\det A_1\neq 0$, then $H_*(W;\ V_\rho)=0$. This completes the proof of Proposition 2.2.

By the above propositions, we have Theorem 2.1.

3. Proof of the main theorem

Let $K\subset S^3$ denote the figure-eight knot and E its exterior. At first we determine the space of the representations of the fundamental group of E by the methods due to Riley [7]. Here we choose a Wirtinger presentation of the fundamental group of E as follows;

$$\pi_1 E = <x,y \mid wx = yw>$$

where $w = x^{-1}yxy^{-1}x^{-1}$.

The following lemma is straightforward. See Riley [7].

Lemma 3.1. *Let X and Y be elements of $SL(2; \boldsymbol{C})$ which are conjugate in $SL(2; \boldsymbol{C})$ and not commutative. Then there exists an element U of $SL(2; \boldsymbol{C})$ such that*

$$UXU^{-1} = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \qquad UYU^{-1} = \begin{pmatrix} s & 0 \\ -t & s^{-1} \end{pmatrix}$$

where $s,t \in \boldsymbol{C} - \{0\}$.

We apply this lemma to irreducible representations of knot groups. Let ρ: $\pi_1 E \to SL(2; \boldsymbol{C})$ be an irreducible representation. By the above lemma, we may assume that

$$X = \rho(x) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix},$$

$$Y = \rho(y) = \begin{pmatrix} s & 0 \\ -t & s^{-1} \end{pmatrix}.$$

Here we have

$$\begin{aligned} W &= X^{-1}YXY^{-1}X^{-1} \\ &= \begin{pmatrix} s^{-1} & -1 \\ 0 & s \end{pmatrix}\begin{pmatrix} s & 0 \\ -t & s^{-1} \end{pmatrix}\begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}\begin{pmatrix} s^{-1} & 0 \\ t & s \end{pmatrix}\begin{pmatrix} s^{-1} & -1 \\ 0 & s \end{pmatrix} \\ &= \frac{1}{s^3}\begin{pmatrix} s^2t^2 + 2s^2t + s^2 - 1 & s(s^4t + s^4 - s^2t^2 - 2s^2t - 2s^2 + t) \\ -st(s^2t + s^2 - 1) & -s^2(s^4t - s^2t^2 - s^2t - s^2 + t) \end{pmatrix}. \end{aligned}$$

By an elementary computation, we can see that

$$WX - YW = \frac{1}{s^3}\begin{pmatrix} 0 & f(s,t)s^2 \\ f(s,t)t & -f(s,t)st \end{pmatrix}$$

where $f(s,t) = s^2t^2 - (s^4 - 3s^2 + 1)t - (s^4 - 3s^2 + 1)$. Therefore $WX = YW$ if and only if $f(s,t) = 0$. Let $\hat{R}$ denote the space of conjugacy classes of $SL(2; \boldsymbol{C})$-irreducible representations of $\pi_1 E$. Then by the above observation, we have

Proposition 3.2. $\hat{R}=\{(s,t)\in \boldsymbol{C}^2\,|\,f(s,t)=0,s\neq 0,t\neq 0\}$.

Solving this equation $f(s,t)=0$ for t, we have

$$t=\frac{s^4-3s^2+1\pm\sqrt{s^8-2s^6-s^4-2s^2+1}}{2s^2}.$$

We denote the right-hand sides of the above expression by t_+ or t_-. If we substitute $t=0$ for $f(s,t)=0$, then we get $s=\pm\alpha_\pm$ where $\alpha_\pm=\sqrt{(3\pm\sqrt{5})/2}$. Hence $t\neq 0$ implies $s\neq\pm\alpha_\pm$. We apply Theorem 2.1 to compute the Reidemeister torsion of E. Apply the Fox's free derivative $\frac{\partial}{\partial y}$ to a relation $wxw^{-1}y^{-1}=1$, then

$$\begin{aligned}\frac{\partial(wxw^{-1}y^{-1})}{\partial y}&=\frac{\partial w}{\partial y}+wx\frac{\partial w^{-1}}{\partial y}+wxw^{-1}\frac{\partial y^{-1}}{\partial y}\\&=\frac{\partial w}{\partial y}-wxw^{-1}\frac{\partial w}{\partial y}-wxw^{-1}y^{-1}\\&=(1-y)\frac{\partial w}{\partial y}-1.\end{aligned}$$

Here

$$\begin{aligned}\frac{\partial w}{\partial y}&=\frac{\partial(x^{-1}yxy^{-1}x^{-1})}{\partial y}\\&=x^{-1}-x^{-1}yxy^{-1}\\&=x^{-1}-wx.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial(wxw^{-1}y^{-1})}{y}&=(1-y)\frac{\partial w}{\partial y}-1\\&=(1-y)(x^{-1}-wx)-1.\end{aligned}$$

Therefore we have

$$A_1=(I-Y)(X^{-1}-WX)-I$$

$$=\begin{pmatrix} \dfrac{s^3t^2+2s^3t+s^3-s^2t^2-2s^2t-3s^2-st+s+t}{s^2} & \dfrac{(s^3t+s^3-s^2t-2s+1)}{s} \\ \dfrac{t(s^3t+s^3-s^2t-3s^2t-2s^2+t+1)}{s^2} & \dfrac{s(s^3t-s^2t^2-2s^2t+s^2-st-3s+t+1)}{s^2} \end{pmatrix}.$$

Here the numerator is

$$\begin{aligned}\det(X-I)&=\det\begin{pmatrix} s-1 & 1 \\ 0 & s^{-1}-1\end{pmatrix}\\ &=(s-1)(s^{-1}-1)\\ &=-\frac{(s-1)^2}{s}.\end{aligned}$$

Then we have $\det(X-I)=0$ if and only if $s=1$. By Theorem 2.1, if $s\neq 1$ and $\det A_1\neq 0$, then $\tau(E;\ V_{\rho(s,t)})$ for $\rho_{(s,t)}$ is given by

$$\begin{aligned}\tau(E;\ V_{\rho(s,t)})&=\frac{\det(X-I)}{\det A_1}\\ &=\frac{s(s-1)^2}{(t-1)s^4+6s^3-(t^2+3t+11)s^2+6s+t-1}.\end{aligned}$$

If we respectively substitute t_+ and t_- for t, we get

$$\begin{aligned}\tau(E;\ V_{\rho(s,t_+)})&=\tau(E;\ V_{\rho(s,t_-)})\\ &=-\frac{s}{2(s^2-s+1)}.\end{aligned}$$

We denote this value of the Reidemeister torsion by $\tau_s(E)$. It is obvious that $\tau_s(E)$ is a continuous function for the parameter $s\in \boldsymbol{C}-\{0,1,\pm\alpha_{\pm}\}$.

Let M denote the double $E\cup_{id}E$ of E. Let ρ: $\pi_1 M\to SL(2;\ \boldsymbol{C})$ be an irreducible representation such that the resutriction on each $\pi_1 E$ is $\rho_{(s,t_+)}$. By Proposition 1.3, the Reidemeister torsion of M is given by

$$\begin{aligned}\tau(M;\ V_\rho)&=(\tau_s E)^2\\ &=\frac{s^2}{4(s^2-s+1)^2}.\end{aligned}$$

Hence by an elementary computation, the set of nonzero torsion is just the set of nonzero complex numbers. This completes the proof of the main theorem.

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