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NONSTANDARD REPRESENTATIONS OF GENERALIZED SECTIONS OF VECTOR BUNDLES

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Introduction

In terms of nonstandard analysis, Todorov ([8], [9]) showed that every Schwartz distribution on $\boldsymbol{R}^n$ can be represented by a $*$-integral with ${}^*C^\infty$ internal kernel function without the necessity of saying, "up to an infinitesimal" (for the case $n=1$, see also [5]). From the differential-geometric viewpoint, it would be desirable to obtain nonstandard representations of generalized sections of vector bundles in an intrinsic manner.

The main purpose of this note is to prove in a simple way that every generalized section (see §1 or [2]) T of a C^∞ vector bundle E over a σ-compact manifold M can be represented by a $*$-integral in the sense that there exists a ${}^*C^\infty$ internal section β_T of the nonstandard extension ${}^*E$ of E such that $T(u)=\int_{{}^*M}\beta_T\cdot{}^*u$ for every compactly supported C^∞ section u of $E^\dagger\otimes|\wedge_M|$, where $E^\dagger$ is the dual bundle of E and $|\wedge_M|$ stands for the density bundle over M.

After devoting §1 to some notational preliminaries, we obtain in §2 nonstandard representations of linear maps from the space of C^∞ sections of E with compact support and then the desired representations of generalized sections. In § 3 we get a result on nonstandard representations of linear maps either from the space of C^k sections of E or from its subspace consisting of compactly supported sections.

As for nonstandard analysis, see, e.g., [1], [3], or [4]; we work with a sufficiently saturated nonstandard model.

1. Notational preliminaries

Throughout the paper we let $\boldsymbol{K}$ be either $\boldsymbol{R}$ (the real numbers) or $\boldsymbol{C}$ (the complex numbers). Furthermore, let $\boldsymbol{N}$ be the (strictly) positive integers and ${}^*\boldsymbol{N}_\infty$ the infinite elements in ${}^*\boldsymbol{N}$.

By a vector bundle $\pi_E: E\to M$ we mean a C^∞ vector bundle with typical fiber $\boldsymbol{K}^p$ (for some $p\in\boldsymbol{N}$) over a σ-compact C^∞ manifold M with $\dim M\in\boldsymbol{N}$. For each

$x \in M$, we write $E_x := \pi_E^{-1}(x)$, the fiber of E over x. The dual bundle of E is denoted by $E^\dagger$. For $k \in \{0\} \cup \boldsymbol{N} \cup \{\infty\}$, let $\Gamma^k(E)$ be the space of all C^k sections of E and $\Gamma_0^k(E)$ the space of C^k sections of E with compact support.

The $\boldsymbol{K}$-line bundle of densities over M is denoted by $|\wedge_M|$. Given a C^∞ Riemannian metric g on M, we let $dv_g \in \Gamma^\infty(|\wedge_M|)$ be the Riemannian volume density associated with g (see [7]). A *generalized section* of a vector bundle $E \to M$ is defined as a continuous linear functional on the space $\Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$ (endowed with the canonical LF-topology); cf. [2].

For two vector bundles $E \to M$ and $F \to N$, we denote by $E \boxtimes F$ the vector bundle over $M \times N$ such that $(E \boxtimes F)_{(x,y)} = E_x \otimes F_y$ for every $(x, y) \in M \times N$. We will write $\otimes$ and $\boxtimes$ for $^*\otimes$ and $^*\boxtimes$, respectively.

2. Nonstandard representations of generalized sections of vector bundles

Let $E \to M$ be a vector bundle. Choose a C^∞ Riemannian metric g on M. By the saturation principle, there exists a hyperfinite-dimensional vector subspace V of the internal vector space $^*(\Gamma_0^\infty(E))$ such that $^\sigma(\Gamma_0^\infty(E)) := \{^*s : s \in \Gamma_0^\infty(E))\}$ is an external subset of V. Take a C^∞ fiber metric h in E and pick $\psi_i \in V$ ($i = 1, 2, \ldots, \eta$ with $\eta = {}^*\dim V \in {}^*\boldsymbol{N}_\infty$) such that

$$\int_{^*M} {}^*h(\psi_i, \psi_j)\, {}^*dv_g = \delta_{ij} \text{ (Kronecker delta)}; \; i, j = 1, 2, \ldots, \eta. \tag{2.1}$$

Regard $\psi_j^h := {}^*h(\cdot, \psi_j)$ as an element of $^*(\Gamma_0^\infty(E^\dagger))$ in a natural manner. Define an internal section $\Psi \in {}^*(\Gamma_0^\infty(E^\dagger \boxtimes E))$ by

$$\Psi(x, y) := \sum_{i=1}^{\eta} \psi_i^h(x) \otimes \psi_i(y) \quad (x, y \in {}^*M). \tag{2.2}$$

Moreover, define $\widetilde{\Psi} \in {}^*(\Gamma_0^\infty((E^\dagger \otimes |\wedge_M|) \boxtimes E))$ by

$$\widetilde{\Psi}(x, y) := \sum_{i=1}^{\eta} (\psi_i^h(x) \otimes {}^*dv_g(x)) \otimes \psi_i(y) \quad (x, y \in {}^*M). \tag{2.3}$$

Proposition 2.1. *Given a vector bundle $E \to M$ and a C^∞ Riemannian metric g on M, let Ψ and $\widetilde{\Psi}$ be as in (2.2) and (2.3), respectively.*

(1) *For every $s \in \Gamma_0^\infty(E)$ and every $y \in {}^*M$,*

$$^*s(y) = \int_{x \in {}^*M} {}^*s(x) \cdot \widetilde{\Psi}(x, y). \tag{2.4}$$

[The map $^*M \times {}^*M \ni (x, y) \mapsto {}^*s(x) \cdot \widetilde{\Psi}(x, y) \in {}^*(|\wedge_M| \boxtimes E)_{(x,y)}$ gives an element of $^*(\Gamma_0^\infty(|\wedge_M| \boxtimes E))$ obtained from $^*s(x) \otimes \widetilde{\Psi}(x, y)$ by the canonical pairing between $^*E_x := {}^*(\pi_E)^{-1}(x)$ and $^*E_x^\dagger$.]

(2) *For every $x, z \in {}^*M$,*

$$\int_{y \in {}^*M} \Psi(x, y) \cdot \Psi(y, z) \otimes {}^*dv_g(y) = \Psi(x, z). \tag{2.5}$$

Proof. Since every ${}^*s \in {}^\sigma(\Gamma_0^\infty(E))$ is expressed as

$$ {}^*s = \sum_{j=1}^{\eta} c_j({}^*s)\psi_j \text{ with } c_j({}^*s) = \int_{{}^*M} {}^*s \cdot \psi_j^h \, {}^*dv_g, \tag{2.6}$$

we have (2.4). Formula (2.5) follows immediately from (2.1). □

Proposition 2.2. *Let $E \to M$ and $F \to N$ be vector bundles. Let $\tilde{\Psi} \in {}^*(\Gamma_0^\infty((E^\dagger \otimes |\wedge_M|) \boxtimes E))$ be as in* (2.3). *For a $\boldsymbol{K}$-linear map $L: \Gamma_0^\infty(E) \to \Gamma^\infty(F)$, define $\Psi_L \in {}^*(\Gamma^\infty((E^\dagger \otimes |\wedge_M|) \boxtimes F))$ by*

$$\Psi_L(y, z) := (I_{{}^*(E^\dagger \otimes |\wedge_M|)_y} \otimes {}^*L)(\tilde{\Psi}(y, \cdot))(z) = \sum_{i=1}^{\eta} (\psi_i^h(y) \otimes {}^*dv_g(y)) \otimes {}^*L(\psi_i)(z)$$

for $y \in {}^*M$ and $z \in {}^*N$, where $I_{{}^*(E^\dagger \otimes |\wedge_M|)_y}$ is the identity transformation of ${}^*(E^\dagger \otimes |\wedge_M|)_y$. Then L is represented as

$$ {}^*(L(s))(z) = \int_{y \in {}^*M} {}^*s(y) \cdot \Psi_L(y, z) \ (s \in \Gamma_0^\infty(E),\ z \in {}^*N).$$

Proof. Let $s \in \Gamma_0^\infty(E)$ and $z \in {}^*N$. By the expression (2.6),

$$\begin{aligned} {}^*(L(s))(z) &= {}^*L({}^*s)(z) = {}^*L\Big(\sum_{i=1}^{\eta} c_i({}^*s)\psi_i\Big)(z) \\ &= \sum_{i=1}^{\eta} c_i({}^*s){}^*L(\psi_i)(z) = \int_{y \in {}^*M} {}^*s(y) \cdot \Psi_L(y,z). \end{aligned}$$

□

We can now represent every generalized section of E by a $*$-integral.

Theorem 2.3. *Let $E \to M$ be a vector bundle, and let $\tilde{\Psi}$ be as in* (2.3). *For each generalized section T of E, define $\beta_T \in {}^*(\Gamma_0^\infty(E))$ by*

$$\beta_T(y) := ({}^*T \otimes I_{{}^*E_y})(\tilde{\Psi}(\cdot, y)) = \sum_{i=1}^{\eta} {}^*T(\psi_i^h \otimes {}^*dv_g)\psi_i(y) \ (y \in {}^*M).$$

Then

$$T(u) = \int_{{}^*M} \beta_T \cdot {}^*u, \ u \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|).$$

Proof. Apply Proposition 2.2 to the $\boldsymbol{K}$-linear map $L_T: \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|) \ni u \mapsto L_T(u) \in \Gamma^\infty(M \times \boldsymbol{K})$ defined by $L_T(u)(z) = (z, T(u))$ for $z \in M$. □

REMARK. If there exists a section $s \in \Gamma_0^\infty(E)$ such that $T(u) = \int_M s \cdot u$ for all $u \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$, then ${}^*T(\psi_i^h \otimes {}^*dv_g) = c_i({}^*s)$ (see (2.6)) and thus $\beta_T = {}^*s$.

EXAMPLE. Given a σ-compact C^∞ Riemannian manifold (M, g), we obtain a "nonstandard delta function" with respect to ${}^*dv_g$ using the above results. In fact, let $C^\infty(M\,;\,\boldsymbol{K})$ denote the space of $\boldsymbol{K}$-valued C^∞ functions on M and let $C_0^\infty(M\,;\,\boldsymbol{K}):=\{f\in C^\infty(M\,;\,\boldsymbol{K}):\ \mathrm{supp}(f)\ \text{compact}\}$. Let V_0 be a hyperfinite-dimensional vector space over ${}^*\boldsymbol{R}$ such that

$$^\sigma(C_0^\infty(M\,;\,\boldsymbol{R})):=\{{}^*f:\ f\in C_0^\infty(M\,;\,\boldsymbol{R})\}\subset V_0\subset{}^*(C_0^\infty(M\,;\,\boldsymbol{R})).$$

Pick $\varphi_i\in V_0$ ($i=1, 2, \dots, \nu$ with $\nu={}^*\dim V_0$) such that $\int_{^*M}\varphi_i\varphi_j\,{}^*dv_g=\delta_{ij}$ ($i, j=1, 2, \dots, \nu$) and define an internal function $\delta\in{}^*(C_0^\infty(M\times M\,;\,\boldsymbol{R}))$ by $\delta(x, y):=\sum_{i=1}^{\nu}\varphi_i(x)\varphi_i(y)$ $(x,\ y\in{}^*M)$. Noting that $C_0^\infty(M\,;\,\boldsymbol{C})=C_0^\infty(M\,;\,\boldsymbol{R})+\sqrt{-1}C_0^\infty(M\,;\,\boldsymbol{R})$, for $x, y, z\in{}^*M$ we have:

(1) $\delta(x, x)\geq 0$, $\delta(x, y)=\delta(y, x)$, and $(\delta(x, y))^2\leq\delta(x, x)\delta(y, y)$.

(2) ${}^*f(x)=\int_{y\in{}^*M}\delta(x, y){}^*f(y)\,{}^*dv_g(y)$ for $f\in C_0^\infty(M\,;\,\boldsymbol{C})$.

(3) $\int_{y\in{}^*M}\delta(x, y)\delta(y, z)\,{}^*dv_g(y)=\delta(x, z)$.

(4) If $T:\ \mathscr{D}=C_0^\infty(M\,;\,\boldsymbol{C})\to\boldsymbol{C}$ is a Schwartz distribution on M, then $T(f)=\int_{y\in{}^*M}{}^*f(y)\gamma_T(y)\,{}^*dv_g(y)$ $(f\in\mathscr{D})$, where $\gamma_T\in{}^*(C_0^\infty(M\,;\,\boldsymbol{C}))$ is defined by $\gamma_T(y):={}^*T(\delta(\cdot.\ y))=\sum_{i=1}^{\nu}{}^*T(\varphi_i)\varphi_i(y)$ $(y\in{}^*M)$.

3. Nonstandard representations of linear maps from $\Gamma^k(E)$ or $\Gamma_0^k(E)$ to $\Gamma^r(F)$

Let $E\to M$ and $F\to N$ be vector bundles. For $s\in\Gamma^0(E)$, define $T_s{:}\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)\to\boldsymbol{K}$ by

$$T_s(u)=\int_M s\cdot u,\quad u\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|).$$

Furthermore, for $k\in\{0\}\cup\boldsymbol{N}\cup\{\infty\}$, let U_E^k be either $\Gamma^k(E)$ or $\Gamma_0^k(E)$.

We first note that if $s_1, s_2, \dots, s_m$ $(m\in\boldsymbol{N})$ are linearly independent in U_E^k, then there exist m elements $\sigma_i\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$ such that $T_{s_i}(\sigma_j)=\delta_{ij}$ ($i, j=1, 2, \dots, m$). Indeed, $T_{s_1}, \dots, T_{s_m}$ are linearly independent in the vector space $\{T_s:\ s\in U_E^k\}$ over $\boldsymbol{K}$. Therefore there exist m elements $\tau_i\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$ ($i=1, 2, \dots, m$) such that the $m\times m$ matrix $A=(T_{s_i}(\tau_j))_{1\leq i,j\leq m}$ is nonsingular; for a simple nonstandard proof (in a more general setting), see [6, Lemma 1.1]. Then we have only to put $\sigma_j=\sum_{i=1}^{m}b_{ij}\tau_i$ where (b_{ij}) is the inverse of the matrix A.

Now, for $r\in\{0\}\cup\boldsymbol{N}\cup\{\infty\}$, let $\mathscr{G}^r$ [resp. $\mathscr{G}_0^r$] be the internal set of all $\zeta\in{}^*(\Gamma^r((E^\dagger\otimes|\wedge_M|)\boxtimes F))$ of the form

$$\zeta(x, y)=\sum_{i=1}^{\nu}u_i(x)\otimes v_i(y)\quad(x\in{}^*M,\ y\in{}^*N)\tag{3.1}$$

for some $\nu\in{}^*\boldsymbol{N}$, $u_i\in{}^*(\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|))$, $v_i\in{}^*(\Gamma^r(F))$ [resp. $v_i\in{}^*(\Gamma_0^r(F))$] ($i=1, 2, \dots, \nu$).

Theorem 3.1. *Let k, $r\in\{0\}\cup\boldsymbol{N}\cup\{\infty\}$ be fixed. Let U_E^k be as above. Suppose that $L: U_E^k\to\Gamma^r(F)$ is a $\boldsymbol{K}$-linear map. Then there exists an element $\Phi_L\in\mathcal{G}^r$ such that*

$$*(L(s))(y)=\int_{x\in{}^*M} {}^*s(x)\cdot\Phi_L(x,\ y)\ (s\in U_E^k,\ y\in{}^*N).$$

Moreover, if $L(s)\in\Gamma_0^r(F)$ for all $s\in U_E^k$, then Φ_L can be chosen from $\mathcal{G}_0^r$.

Proof. For $s\in U_E^k$, define an internal map $G_s: \mathcal{G}^r\to{}^*(\Gamma^r(F))$ by

$$G_s(\zeta)(y):=\int_{x\in{}^*M} {}^*s(x)\cdot\zeta(x,\ y)=\sum_{i=1}^{\nu}({}^*(T_s)(u_i))v_i(y)\ (y\in{}^*N),$$

where $\zeta\in\mathcal{G}^r$ is as in (3.1). Let $\mathcal{B}_s$ be the internal set

$$\mathcal{B}_s:=\{\zeta\in\mathcal{G}^r:\ G_s(\zeta)={}^*(L(s))\}.$$

We shall show that the family $\{\mathcal{B}_s: s\in U_E^k\}$ has the finite intersection property. To do this, let $P(m)$ $(m\in\boldsymbol{N})$ be the following proposition:

For $s_i\in U_E^k$, $i=1, 2, \ldots, m$, the system of equations

$$G_{s_i}(\zeta)={}^*(L(s_i))\ (i=1,\ 2,\ \ldots,\ m) \tag{3.2}$$

has a solution ζ in $\mathcal{G}^r$.

Consider first the case $m=1$. If $s_1=0$, any $\zeta\in\mathcal{G}_0^r$ satisfies (3.2) for $m=1$. If $s_i\neq 0$, then there exists an element $\tau\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$ with $T_{s_1}(\tau)\neq 0$ and thus we can choose $\sigma\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$ such that $T_{s_1}(\sigma)=1$; therefore, if we let

$$\zeta(x,\ y)={}^*\sigma(x)\otimes{}^*(L(s_1))(y)\ (x\in{}^*M,\ y\in{}^*N),$$

then $\zeta\in\mathcal{G}^r$ and moreover this ζ satisfies (3.2) for $m=1$, since ${}^*(T_{s_1})({}^*\sigma)=T_{s_1}(\sigma)=1$. Hence $P(1)$ is true.

Next, assume that $m>1$ and that $P(m-1)$ is true. If $s_1, \ldots, s_m$ are linearly dependent in U_E^k, then we may assume that $s_m=\sum_{i=1}^{m-1}a_is_i$ $(a_i\in\boldsymbol{K})$ without loss of generality, so that the system (3.2) is equivalent to the system of equations for $i=1, 2, \ldots, m-1$. If $s_1, \ldots, s_m$ are linearly independent, then, as noticed earlier, we can choose $\sigma_i\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$ such that $T_{s_i}(\sigma_j)=\delta_{ij}$ $(i,\ j=1, \ldots, m)$; so, if we let

$$\zeta(x,\ y)=\sum_{j=1}^{m}{}^*\sigma_j(x)\otimes{}^*(L(s_j))(y)\ (x\in{}^*M,\ y\in{}^*N),$$

then ζ belongs to $\mathcal{G}^r$ and satisfies (3.2). Thus $P(m-1)$ implies $P(m)$.

Hence $P(m)$ is true for all $m\in\boldsymbol{N}$. Then, by the saturation principle, the intersection $\mathcal{B}=\bigcap_{s\in U_E^k}\mathcal{B}_s$ is nonempty; accordingly, there exists an element $\Phi_L\in\mathcal{B}$.

If $L(s)\in\Gamma_0^r(F)$ for all $s\in U_E^k$, then by replacing $\mathcal{G}^r$ with $\mathcal{G}_0^r$ in the above discussion, we see that Φ_L can be chosen from $\mathscr{G}_0^{\tau}$. □

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