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Osaka J. Math.
32 (1995), 783-797

DISCRETE SPECTRUM OF SCHRÖDINGER OPERATORS WITH PERTURBED UNIFORM MAGNETIC FIELDS

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(Received February 16, 1994)

1. Introduction

In this paper we study a Schrödinger operator with a magnetic field:

$$H=(-i\nabla-b(x))^2+V(x) \tag{1.1}$$

defined on $C_0^\infty(\boldsymbol{R}^3)$, where $V\in L^2_{loc}(\boldsymbol{R}^3)$ is a scalar potential and $b\in C^1(\boldsymbol{R}^3)^3$ is a vector potential, both of which are real-valued, and $\vec{B}(x)=\nabla\times b$ is called the magnetic field. Let $x=(x_1, x_2, z)\in\boldsymbol{R}^3$, $\vec{\rho}=(x_1, x_2)$, $r=|x|$, $\rho=|\vec{\rho}|$, and $\nabla_2=(\partial/\partial x_1, \partial/\partial x_2)$. Letting $T=-i\nabla-b(x)$, we define the quadratic form q_H by

$$q_H[\phi, \psi]=\int_{\boldsymbol{R}^3}(T\phi\cdot\overline{T\psi}+V\phi\bar{\psi})dx,$$
$$q_H[\phi]=q_H[\phi, \phi]$$

for ϕ, $\psi\in C_0^\infty(\boldsymbol{R}^3)$. We assume that

$$V(x)\to 0 \text{ as } |x|\to\infty. \tag{V1}$$

Then H admits a unique self-adjoint realization in $L^2(\boldsymbol{R}^3)$ (denoted by the same notation H) with the domain

$$D(H)=\{u\in L^2(\boldsymbol{R}^3)\,;\ |V|^{1/2}u,\ Tu,\ Hu\in L^2(\boldsymbol{R}^3)\},$$

which is associated with the closure of q_H (denoted by the same notation q_H) with the form domain

$$Q(H)=\{u\in L^2(\boldsymbol{R}^3)\,;\ |V|^{1/2}u,\ Tu,\in L^2(\boldsymbol{R}^3)\},$$

This fact can be proved in the same way as in the cases of the constant magnetic fields ([1] and [7]).

It is well known that, if $\vec{B}(x)\equiv 0$, then the finiteness or the infiniteness of the discrete spectrum of H depends on the decay order of the scalar potential V, of which the border is $|x|^{-2}$([6]). On the other hand, if $\vec{B}(x)\equiv(0, 0, B)$, B being a positive constant, then the number of the discrete spectrum of H is infinite under a suitable negativity assumption of the scalar potential, which is independent of the decay order of V. More precisely, the following result was proved by Avron-Herbst-Simon [2].

Theorem 0. ([2]) *Let $\vec{B}(x)=\nabla\times b=(0, 0, B)$, B being a positive constant. Suppose that $V\in L^2+L^\infty_\varepsilon$ and that V is non-positive, not identically zero and azimuthally symmetric. Then the number of the discrete spectrum of H is infinite.*

Here a function $f(x)$ on $\boldsymbol{R}^3$ is called *azimuthally symmetric* (in z-axis) if $f(x)$ depends only on ρ and z. Now a question arises: What occurs for the discrete spectrum when we perturb slightly the constant magnetic field? One may well imagine that the infiniteness or the finiteness of the discrete spectrum depends on both of the magnetic vector potential $b(x)$ and the scalar potential $V(x)$. This is certainly true. In fact, Mohamed [5] gave a sufficient condition for the existence of infinite discrete spectrum with long-range scalar potential $V(x)$ and suitable magnetic fields. The case of short-range scalar potential is also important since in this case the number of discrete spectrum turns to be infinite because of the presence of constant magnetic fields. The aim of this paper is to clearify the relation between $b(x)$ and $V(x)$ for H to have an infinite or a finite discrete spectrum.

To state the main theorem we make some preparations. We assume that

$$\text{(V2)} \qquad \begin{cases} V \textit{ is azimuthally symmetric, bounded above and there exists} \\ R_0>0 \textit{ such that } V\in C^0(|x|\geq R_0),\ V<0 \textit{ for } |x|\geq R_0. \end{cases}$$

Let B be a positive constant and

$$b_c(x)=B/2(-x_2, x_1, 0)$$

which satisfies $\nabla\times b_c=(0, 0, B)$. For given $b\in C^1(\boldsymbol{R}^3)^3$, we put

$$b_p(x)=b(x)-b_c(x)=(a_1(x), a_2(x), a_3(x)).$$

By introducing the polar coordinate (ρ, θ) in $\boldsymbol{R}^2$, we define the set X by

$$X=\{a\in C^1(\boldsymbol{R}^3)\,;\ \text{there exists } N(a)\in\boldsymbol{N} \text{ such that} \\ \int_0^{2\pi} a(\rho, \theta: z)e^{ik\theta}d\theta=0 \text{ for } |k|\geq N(a),\ k\in\boldsymbol{Z}\}.$$

We denote by $\sigma(H)$ the spectrum of H, by $\sigma_d(H)$ the discrete spectrum of H, by $\sigma_e(H)$ the essential spectrum of H and by $\# Y$ the cardinal number of a set Y. For two vector potentials $b_1, b_2\in C^1(\boldsymbol{R}^3)^3$, we denote $b_1\sim b_2$ when b_1 is equivalent to b_2 under a gauge transformation, namely, $b_1-b_2=\nabla\lambda$ for some $\lambda\in C^2(\boldsymbol{R}^3)$. Then our main result is the following theorem.

Theorem 1. *Assume* (V1), (V2) *and that $a_j(x)\in X$ $(j=1, 2, 3)$. Suppose* *that there exist* $R_1>0$ *and positive constants* c_j $(j=1, 2, 3)$ *such that*

$$\begin{cases} |a_j(x)|\leq c_1\min\{|V(x)|^{1/2}, |V(x)|\rho\} \ (j=1, 2), \\ |\nabla_2 a_j(x)|\leq c_2|V(x)| \ (j=1, 2), \\ |a_3(x)|\leq c_3|V(x)|^{1/2} \end{cases} \tag{1.2}$$

for $|x|\geq R_1$,

$$2(c_1^2+c_2)+c_3^2+\sqrt{2}c_1<1, \tag{1.3}$$

and also suppose that

$$\partial a_3/\partial z\to 0 \ \text{as} \ |x|\to\infty. \tag{1.4}$$

Then $\sigma_e(H)=[B, \infty)$ *and*

$$\#\sigma_d(H)=+\infty. \tag{1.5}$$

REMARK 1.1. Let V be as in Theorem 1. If $W\in L^2_{loc}(\boldsymbol{R}^3)$ satisfies (V1) and $W\leq V$, then $\#\sigma_d(T^2+W)=+\infty$ by the min-max principle. Thus we can apply the above theorem to potentials which are not azimuthally symmetric or not continuous on $|x|\geq R_0$.

REMARK 1.2. The above theorem of course holds if we replace the vector potential by an equivalent one.

As an example we consider the perturbation of the constant magnetic field on a compact set.

Proposition 1.3. *If there exists* $R_2>0$ *such that*

$$\vec{B}(x)=(0, 0, B) \ \text{for} \ |x|\geq R_2,$$

then one can replace the magnetic vector potential $b(x)$ *by an equvalent one satisfying* (1.2), (1.3) *and* (1.4).

Proof of Proposition 1.3. It is easy to see that

$$\nabla\times(b-b_c)=0 \qquad (|x|\geq R_2).$$

Hence, there exist $\lambda\in C^2(\boldsymbol{R}^3)$ such that

$$b-b_c=\nabla\lambda \qquad (|x|\geq R_2).$$

We put

$$\tilde{b}=b-\nabla\lambda \ \text{on} \ \boldsymbol{R}^3.$$

Then $\tilde{b}\sim b$ and $\tilde{b}-b_c=0$ for $|x|\geq R_2$. For this $\tilde{b}$, (1.2), (1.3) and (1.4) are always satisfied. $\square$

Let us compare our result with that of Mohamed [5]. Roughly speaking, supposing that $V(x)=O(|x|^{-\alpha})$, he studied the case $0<\alpha<2$. In this case our result is weaker than his, however, our method can also treat the case $\alpha\geq 2$. We shall also construct examples which show that our condition (consequently the condition of Mohamed) is almost optimal to guarantee the infiniteness of the discrete spectrum when lies in an interval $(2-\varepsilon,\ 2]$. These examples also show that some non-constant magnetic fields decrease the number of bound states in spite of the fact that if $\vec{B}(x)\equiv 0$ and $0<\alpha<2$ the number of the discrete spectrum is infinite ([6]).

2. Proof of Theorem 1

We first recall the following facts.

$$\inf \sigma_e(H)=\sup_{E:\ compact} \inf\{(H\phi,\ \phi)_{L^2};\ \phi\in C_0^\infty(\boldsymbol{R}^3\backslash E),\ \|\phi\|_{L^2}=1\} \tag{2.1}$$

$$=\lim_{R\to\infty} \inf\{(H\phi,\ \phi)_{L^2};\ \phi\in C_0^\infty(|x|\geq R),\ \|\phi\|_{L^2}=1\}. \tag{2.2}$$

They can be proved in the same way as in [1]. We devide the proof of Theorem 1 into three steps.

<u>*Step 1.*</u> We prove that, if $|b_p(x)|\to 0$ as $|x|\to\infty$, then

$$\Sigma(H)\equiv\inf \sigma_e(H)=B.$$

In fact, letting

$$T_c=-i\nabla-b_c,$$

we have, for any $\phi\in C_0^\infty(\boldsymbol{R}^3)$ and any $\varepsilon>0$,

$$\begin{aligned}|T\phi|^2&=|T_c\phi-b_p\phi|^2\\&=|T_c\phi|^2+|b_p|^2|\phi|^2-2ReT_c\phi\cdot b_p\overline{\phi}\\&\geq(1-\varepsilon)|T_c\phi|^2+(1-\varepsilon^{-1})|b_p|^2|\phi|^2.\end{aligned}$$

Hence, letting M be the operator of multiplication by the function $|b_p(x)|^2$, we have

$$T^2\geq(1-\varepsilon)T_c^2+(1-\varepsilon^{-1})M$$

in the form sense. By using (2.2) and the fact that $|b_p(x)|\to 0$ as $|x|\to\infty$, we have

$$\Sigma(T^2)\geq(1-\varepsilon)\Sigma(T_c^2)=(1-\varepsilon)B.$$

Similarly one can show

$$\Sigma(T^2)\leq(1+\varepsilon)B.$$

Hence we have

$$\Sigma(T^2)=B,$$

so, by using (2.2) again, we have

$$\Sigma(H)=B.$$

Step 2.

Proposition 2.1. *If* $|b_p(x)|\to 0$, $|\mathrm{div}\ b_p(x)|\to 0$ *as* $|x|\to 0$, *then* $\sigma_e(H)=[B, \infty)$.

Proof of Proposition 2.1. We have only to prove $[B, \infty)\subset\sigma_e(H)$. For $\lambda\geq 0$, we define $\psi_m(x)$ by

$$\psi_m(x)\equiv\psi_{m,\lambda}(x)=e^{i\lambda z}\eta_m(z)\phi_0(\vec{\rho})\ (m\in\boldsymbol{N},\ m\gg 1),$$

where

$$\phi_0(\vec{\rho})=B^{1/2}(2\pi)^{-1/2}e^{-B\rho^2/4},$$
$$\eta_m(z)=2^{-(m-1)/2}\eta(2^{-(m-1)}z)$$

for some fixed $\eta\in C_0^\infty(1\leq|z|\leq 2)$. We remark that

$$\|\phi_0\|_{L^2(\boldsymbol{R}^2)}=\|\eta_m\|_{L^2(\boldsymbol{R}^1)}=\|\psi_m\|_{L^2(\boldsymbol{R}^3)}=1,$$
$$(\psi_j,\ \psi_k)_{L^2}=0\ (j\neq k),$$

$$\{T_c^2-(-\partial^2/\partial z^2)\}\phi_0\equiv\{(-i\partial/\partial x_1+Bx_2/2)^2+(-i\partial/\partial x_2-Bx_1/2)^2\}\phi_0=B\phi_0. \tag{2.3}$$

To prove $[B,\infty)\subset\sigma_e(H)$ it is sufficient to show that

$$(H-(B+\lambda^2))\psi_m\to 0 \text{ strongly in } L^2(\boldsymbol{R}^3) \text{ as } m\to\infty. \tag{2.4}$$

By using (2.3) and $T^2=T_c^2+(i\mathrm{div}\ b_p+|b_p|^2)-2b_p\cdot T_c$, we have

$$T^2\psi_m=B\psi_m-\partial^2\psi_m/\partial z^2+(i\mathrm{div}\ b_p+|b_p|^2)\psi_m-2b_p\cdot T_c\psi_m. \tag{2.5}$$

We compute

$$-\partial^2\psi_m/\partial z^2=\lambda^2\psi_m+(\mathrm{I})+(\mathrm{II}),$$

where

$$(\mathrm{I})=-2i\lambda e^{i\lambda z}\eta'_m(z)\phi_0(\vec{\rho}),$$
$$(\mathrm{II})=-e^{i\lambda z}\eta''_m(z)\phi_0(\vec{\rho}).$$

By the change of variable: $\xi=2^{-(m-1)}z$, we have

$$\|(\mathrm{I})\|_{L^2}^2\leq\lambda^2 4^{2-m}\|\eta'\|_{L^2(\boldsymbol{R}^1)}^2\to 0 \text{ as } m\to\infty,$$
$$\|(\mathrm{II})\|_{L^2}^2\leq 16^{1-m}\|\eta''\|_{L^2(\boldsymbol{R}^1)}^2\to 0 \text{ as } m\to\infty.$$

Hence we have

$$-\partial^2\psi_m/\partial z^2-\lambda^2\psi_m\to 0 \text{ strongly in } L^2(\boldsymbol{R}^3) \text{ as } m\to\infty. \tag{2.6}$$

Since $\|T_c\psi_m\|_{L^2}^2=(T_c^2\psi_m,\ \psi_m)$ and $T_c^2=(B+\lambda^2)\psi_m+(\mathrm{I})+(\mathrm{II})$, there exists a constant $c_0>0$ independent of m such that

$$\|T_c\psi_m\|_{L^2}<c_0<+\infty \tag{2.7}$$

Using the assumption on b_p and the fact that

$$\operatorname{supp}\psi_m\subset\{x\in\boldsymbol{R}^3\,;\ 2^{m-1}\le|z|\le 2^m\},$$

gives

$$(i\operatorname{div} b_p+|b_p|^2)\psi_m\to 0 \text{ strongly in } L^2(\boldsymbol{R}^3) \text{ as } m\to\infty. \tag{2.8}$$

By (2.7) we also have

$$2b_p\cdot T_c\psi_m\to 0,\ V\psi_m\to 0 \text{ strongly in } L^2(\boldsymbol{R}^3) \text{ as } m\to\infty. \tag{2.9}$$

By (2.5), (2.6), (2.8) and (2.9), we obtain (2.4). □

By the assumption of Theorem 1, the condition in Proposition 2.1 is fulfilled. Hence we have $\sigma_e(H)=[B,\ \infty)$.

<u>*Step 3.*</u> We can assume that $R_1\ge R_0$. To prove that $\#\sigma_d(H)=+\infty$, by using the Rayleigh-Ritz method ([6]), it is sufficient to construct $\{\Phi_m\}_{m=1}^{\infty}\subset Q(H)$ such that

$$\begin{cases}\|\Phi_m\|_{L^2}=1,(\Phi_j,\ \Phi_k)_{L^2}=0\ (j\ne k),\\ q_H[\Phi_j,\ \Phi_k]=0\ (j\ne k),\\ q_H[\Phi_m]<B.\end{cases} \tag{2.10}$$

We define ψ_m^s by

$$\psi_m^s(x)=h_s(z)\phi_m(\vec{\rho})\ (0<s\ll 1,\ m\in\boldsymbol{N},\ m\gg 1),$$

where in terms of $(\rho,\ \theta)$-coordinates

$$\phi_m(\vec{\rho})=a_m e^{im\theta}\rho^m e^{-B\rho^2/4}=a_m(x_1+ix_2)^m e^{-B\rho^2/4}\quad([3]), \tag{2.11}$$

$$a_m=(\pi m!)^{-1/2}(B/2)^{(m+1)/2}, \tag{2.12}$$

$$h_s(z)=\sqrt{s}^{-s|z|}.$$

They satisfy the following relations.

$$\begin{aligned}&\|\phi_m\|_{L^2(\boldsymbol{R}^2)}=\|h_s\|_{L^2(\boldsymbol{R}^1)}=\|\psi_m^s\|_{L^2(\boldsymbol{R}^3)}=1,\ \psi_m^s\in Q(H),\\ &(\psi_j^s,\ \psi_k^t)_{L^2}=0\ (j\ne k),\\ &\{T_c-(-\partial^2/\partial z^2)\}\phi_m=\{(-i\partial/\partial x_1+Bx_2/2)^2+(-i\partial/\partial x_2-Bx_1/2)^2\}\phi_m\\ &\qquad=B\phi_m.\end{aligned} \tag{2.13}$$

We first show that

$$\|T\psi_m^s\|_{L^2}^2 = B + s^2 + \int(-\sin\theta\nabla_2 a_1 + \cos\theta\nabla_2 a_2)\cdot\vec{\rho}\rho^{-1}|\psi_m^s|^2 dx \\ + \int\{\rho^{-2}(-x_2a_1 + x_1a_2) + a_1^2 + a_2^2 + a_3^2\}|\psi_m^s|^2 dx. \tag{2.14}$$

On one hand, by (2.13) and a straightforward calculation,

$$\begin{aligned}\|T\psi_m^s\|_{L^2}^2 &= \|T_c\psi_m^s\|_{L^2}^2 + \int\{-2\mathrm{Im}(\nabla\psi_m^s\cdot b_q\overline{\psi_m^s}) + (|b_p|^2 + 2b_c\cdot b_p)|\psi_m^s|^2\}dx \\ &= B + s^2 - 2m\int\rho^{-2}(-x_2a_1 + x_1a_2)|\psi_m^s|^2 dx \\ &\quad + \int\{a_1^2 + a_2^2 + a_3^2 + B(-x_2a_1 + x_1a_2)\}|\psi_m^s|^2 dx.\end{aligned}$$

On the other hand, passing to the cylindorical coordinates and integrating by parts in ρ, we have

$$\begin{aligned}&(2m+1)\int\rho^{-2}(-x_2a_1 + x_1a_2)|\psi_m^s|^2 dx \\ &= \int(\sin\theta\nabla_2 a_1 - \cos\theta\nabla_2 a_2)\cdot\vec{\rho}\rho^{-1}|\psi_m^s|^2 dx + \int B(-x_2a_1 + x_1a_2)|\psi_m^s|^2 dx.\end{aligned} \tag{2.15}$$

By using (2.15) and a simple manipulation, we have (2.14) which implies

$$\|T\psi_m^s\|_{L^2}^2 \le B + s^2 + \int(|\nabla_2 a_1| + |\nabla_2 a_2| + \rho^{-1}\sqrt{a_1^2 + a_2^2} + a_1^2 + a_2^2 + a_3^2)|\psi_m^s|^2 dx.$$

Here we use the assumption (1.2) to see that

$$\begin{aligned}\|T\psi_m^s\|_{L^2}^2 &\le B + s^2 + \int_{|x|\ge R_1}\{2(c_1^2 + c_2) + c_3^2 + \sqrt{2}c_1\}|V(x)||\psi_m^s|^2 dx \\ &\quad + \int_{|x|<R_1}(c_4 + c_5\rho^{-1})|\psi_m^s|^2 dx\end{aligned}$$

for some constants c_4, $c_5 > 0$. Since $V<0$ for $|x|\ge R_1$, by letting $\delta = 1 - \{2(c_1^2 + c_2) + c_3^2 + \sqrt{2}c_1\} > 0$, we have

$$\|T\psi_m^s\|_{L^2}^2 \le B + s^2 + (1-\delta) + \int_{|x|\ge R_1}(-V(x))|\psi_m^s|^2 dx + \int_{|x|<R_1}(c_4 + c_5\rho^{-1})|\psi_m^s|^2 dx.$$

We add $(V\psi_m^s, \psi_m^s)_{L^2}$ to the both side, noting that V is bounded from above by assumption (V2), we have

$$q_H[\psi_m^s] \le B + s^2 + \delta\int_{|x|\ge R_1}V(x)|\psi_m^s|^2 dx + \int_{|x|<R_1}(c_6 + c_5\rho^{-1})|\psi_m^s|^2 dx \tag{2.16}$$

for some constant $c_6 > 0$. Let

$$\begin{aligned}\Omega_1 &= \{(\vec{\rho}, z)\in\boldsymbol{R}^3\,;\ 2R_1\le\rho\le 3R_1,\ 0\le|z|\le 1\}, \\ \Omega_2 &= \{(\vec{\rho}, z)\in\boldsymbol{R}^3\,;\ 0\le\rho\le R_1,\ 0\le|z|\le R_1\}.\end{aligned}$$

We estimate the integral of the right-hand side as follows.

$$\int_{\Omega_1}|\psi_m^s|^2dx=2\int_0^1 se^{-2sz}dz\int_0 2\pi d\theta\int_{2R_1}^{3R_1}\alpha_m^2\rho^{2m+1}e^{-B\rho^2/2}d\rho$$
$$\geq 2se^{-2s}\cdot 1\cdot 2\pi\alpha_m^2(2R_1)^{2m+1}e^{-B(3R_1)^2/2}\cdot R_1.$$

Therefore there exists a constant $c(R_1)>0$ independent of m and s such that

$$(2.17)\qquad \delta\int_{|x|\geq R_1}V(x)|\psi_m^s|^2dx\leq\delta\int_{\Omega_1}\sup_{x\in\Omega_1}V(x)|\psi_m^s|^2dx$$
$$\leq -c(R_1)s\alpha_m^2(2R_1)^{2m}.$$

We also have by a similar calculation

$$(2.18)\qquad \int_{|x|\leq R_1}(c_6+c_5\rho^{-1})|\psi_m^s|^2dx\leq\int_{\Omega_2}(c_6+c_5\rho^{-1})|\psi_m^s|^2dx$$
$$\leq c'(R_1)s\alpha_m^2R_1^{2m}$$

for some constant $c'(R_1)>0$ which is independent of m and s. Hence, by (2.16), (2.17) and (2.18),

$$q_H[\psi_m^s]\leq B+s^2+s\alpha_m^2R_1^{2m}(c'(R_1)-c(R_1)4^m).$$

There exists $m_1>0$ such that

$$c'(R_1)-c(R_1)4^m\leq -1 \text{ for } m\geq m_1,$$

so we have

$$q_H[\psi_m^s]\leq B+s(s-\alpha_m^2R_1^{2m}) \text{ for } m\geq m_1.$$

Let

$$s=s(m)=1/2\ \alpha_m^2R_1^{2m},\ \Phi_m=\psi_m^{s(m)}.$$

Then the above inequality implies

$$(2.19)\qquad q_H[\Phi_m]\leq B-(1/2\ \alpha_m^2R_1^{2m})^2<B \text{ for } m\geq m_1.$$

Next, by the assumption of Theorem 1, there exists $N_1\in\boldsymbol{N}$ such that each $a_j(x)(j=1, 2, 3,)$ is a linear combination of $\{e^{il\theta}\}_{|l|\leq N_1,\, l\in Z}$ as a function of θ with coefficients depending on ρ and z. We show that

$$(2.20)\qquad q_H[\Phi_j,\ \Phi_k]=\iiint G(\rho,\ \theta,\ z)e^{i(j-k)\theta}\,d\rho d\theta dz$$

where

$$G(\rho,\ \theta,\ z)=\sum_{|l|\leq 2N_1+2}e^{il\theta}G_l(\rho,\ z),\ G_l(\rho,\ z)\in L^1((0,\ \infty)\times\boldsymbol{R}).$$

In fact, we examine each term of the expression

$$q_H[\Phi_j,\ \Phi_k]=\int\{\nabla\Phi_j\cdot\nabla\overline{\Phi_k}+i(\nabla\Phi_j\cdot b\overline{\Phi_k}-\nabla\overline{\Phi_k}\cdot b\Phi_j)+(|b|^2+V)\Phi_j\overline{\Phi_k}\}dx.$$

Since V is azimuthally symmetric, it is easy to see (2.20). Then we have

$$q_H[\Phi_j,\ \Phi_k]=0\ (|j-k|\geq 2N_1+3).$$

Therefore by choosing a subsequence of $\{\Phi_m\}$ which we denote again by $\{\Phi_m\}$, one can assume that

$$q_H[\Phi_j,\ \Phi_k]=0\ (j\neq k). \tag{2.21}$$

Summing up, we have obtained $\{\Phi_m\}$ satisfying (2.10). Hence

$$\#\sigma_d(H)=+\infty.$$

This completes the proof of Theorem 1. □

3. Examples

In this section we illustrate some examples showing that the conditions in Theorem 1 are almost optimal. For the sake of convenience, we strengthen slightly the conditions in Theorem 1 as follows.

Theorem 1*. *Assume* (V1), (V2) *and that* $a_j(x)\in X(j=1,\ 2,\ 3)$. *Suppose that*

$$\begin{cases} a_j(x)=o(\min\{|V(x)|^{1/2},\ |V(x)|\rho\})\ (j=1,\ 2) \\ \nabla_2 a_j(x)=o(|V(x)|)\ (j=1,\ 2), \\ a_3(x)=o(|V(x)|^{1/2}) \\ \partial a_3/\partial z=o(1) \end{cases} \tag{3.1}$$

as $|x|\to\infty$. *Then* $\sigma_e(H)=[B,\ \infty)$ *and*

$$\#\sigma_d(H)=+\infty.$$

We give the above mentioned examples in the following form.

$$b=f(r)(-x_2,\ x_1,\ 0) \tag{3.2}$$

where $f\in C^1([0,\ \infty))$, $f'(0)=0$ and f is real-valued. In this case $a_1(x)=-(f(r)-B/2)x_2$, $a_2(x)=(f(r)-B/2)x_1$, $a_3(x)=0$, so the assumption that $a_j\in X(j=1,\ 2,\ 3,)$ is satisfied. We assume that $V(x)$ is a function of $r=|x|$. Then (3.1) is equivalent to the following

$$\begin{cases} |f(r)-B/2|=o(\min\{|V(x)|^{1/2}r^{-1},\ |V(x)|\}), \\ |f'(r)|=o(|V(x)|r^{-1}). \end{cases} \tag{3.3}$$

Now we put $V=-r^{-\alpha}(\alpha>0)$ for $|x|\geq 2$, then (3.3) is equivalent to

$$\begin{cases} |f(r)-B/2|=o(r^{\min\{-1-\alpha/2,-\alpha\}}), \\ |f'(r)|=o(r^{-1-\alpha}). \end{cases} \tag{3.4}$$

Before showing the examples, we prepare the following proposition.

Proposition 3.1. *For $\phi\in C_0^\infty(\boldsymbol{R}^3)$, we have the following inequality.*

$$\int|T\phi|^2dx\geq\int(\partial b_2/\partial x_1-\partial b_1/\partial x_2)|\phi|^2dx, \tag{3.5}$$

where $b=(b_1(x),\ b_2(x),\ b_3(x))$.

Cororally. *In the case of* (3.2) *we have*

$$\int|T\phi|^2dx\geq\int(f'(r)\rho^2r^{-1}+2f(r))|\phi|^2dx \text{ for } \phi\in C_0^\infty(\boldsymbol{R}^3).$$

In particular, if $f'(r)\leq 0$, then

$$\int|T\phi|^2dx\geq\int F_f(r)|\phi|^2dx \text{ for } \phi\in C_0^\infty(\boldsymbol{R}^3), \tag{3.6}$$

where $F_f(r)=rf'(r)+2f(r)$.

Proof of Proposition 3.1. We put

$$A_1=\partial/\partial x_1+b_2,\ A_2=\partial/\partial x_2-b_1,\ A=A_1+iA_2 \text{ and } P=\partial/\partial z-ib_3.$$

Then by a straightforward calculation,

$$\begin{aligned}A^*A=&-\partial^2/\partial x_1^2-\partial^2/\partial x_2^2+2i(b_1\partial/\partial x_1+b_2\partial/\partial x_2)+i(\partial b_1/\partial x_1+\partial b_2/\partial x_2)\\&+|b_1|^2+|b_2|^2-\partial b_2/\partial x_1+\partial b_1/\partial x_2,\\P^*P=&-\partial^2/\partial z^2+2ib_3\partial/\partial z+i\partial b_3/\partial z+|b_3|^2.\end{aligned}$$

Therefore we have

$$P^*P+A^*A=T^2-(\partial b_2/\partial x_1-\partial b_1/\partial x_2).$$

Hence, for $\phi\in C_0^\infty(\boldsymbol{R}^3)$,

$$\begin{aligned}\int|T\phi|^2dx&=((P^*P+A^*A)\phi,\phi)_{L_2}+\int(\partial b_2/\partial x_1-\partial b_1/\partial x_2)|\phi|^2dx\\&\geq\int(\partial b_2/\partial x_1-\partial b_1/\partial x_2)|\phi|^2dx. \quad\square\end{aligned}$$

EXAMPLE 1. We first take $\alpha=2$, namely, let

$$V(x)=\begin{cases}-r^{-2} & (r\geq e^{1/2}),\\ 0 & (r<e^{1/2}).\end{cases}$$

If $f(r)-B/2=r^{-\beta}$ for $r\geq e^{1/2}(\beta>2)$, the condition (3.4) is fulfilled, hence $\#\sigma_d(H)=+\infty$. We next see what occurs when this condition is violated. We define $f(r)$ by

$$f(r)=\begin{cases}B/2+r^{-2}\log r & (r\geq e^{1/2}),\\ B/2+1/(2e) & (r<e^{1/2}).\end{cases}$$

Then $f\in C^1([0, \infty))$, $f'(0)=0$, $f'(r)\leq 0$, and

$$F_f(r)=\begin{cases} B+r^{-2} & (r\geq e^{1/2}),\\ B+e^{-1} & (r<e^{1/2}).\end{cases}$$

Hence, by using (3.6),

$$(3.7)\qquad (H\phi, \phi)_{L^2}\geq \int (F_f(r)+V)|\phi|^2dx\geq B\|\phi\|_{L^2} \text{ for } \phi\in C_0^\infty(\boldsymbol{R}^3).$$

By Proposition 2.1, it is easy to see that $\sigma_e(H)=[B, \infty)$. Hence, by (3.7), we have

$$\#\sigma_d(H)=\emptyset.$$

EXAMPLE 2. To consider the case of $0<\alpha<2$ we use the almost same but slightly complicated method.

Let

$$V(x)=\begin{cases} -r^{-\alpha} & (r\geq 2), \quad 0<\alpha<2,\\ 0 & (r<2).\end{cases}$$

If $f(r)-B/2=(\text{constan})\cdot r^{-\beta}$ for $r\geq 2(\beta>1+\alpha/2)$, the condition (3.4) is fulfilled, hence $\#\sigma_d(H)=+\infty$. When $\beta=\alpha(<1+\alpha/2)$, H does not always have infinitely many bound states, although the difference $(1+\alpha/2)-\alpha\rightarrow 0$ as $\alpha\rightarrow 2$. In fact, We define $f(r)$ by

$$f(r)=\begin{cases} B/2+r^{-\alpha}/(2-\alpha) & (r\geq 2),\\ B/2+\{2^{-\alpha}+2^{-\alpha-2}\alpha r(2-r)\}/(2-\alpha) & (1<r<2),\\ B/2+2^{-\alpha-2}(4+\alpha)/(2-\alpha) & (r\leq 1).\end{cases}$$

Then $f\in C^1([0, \infty))$, $f'(0)=0$, $f'(r)\leq 0$, and

$$F_f(r)=\begin{cases} B+r^{-\alpha} & (r\geq 2),\\ B+2^{-\alpha-1}\{-2\alpha r^2+3\alpha r+4\}/(2-\alpha) & (1<r<2),\\ B+2^{-\alpha-2}(4+\alpha)/(2-\alpha) & (r\leq 1),\end{cases}$$

so

$$F_f(r)+V(x)\geq B \ (0<r<\infty).$$

Hence, by using (3.6), we have

$$(H\phi, \phi)_{L^2}\geq B\|\phi\|_{L^2}^2 \text{ for } \phi\in C_0^\infty(\boldsymbol{R}^3).$$

So, in the case of $1<\alpha<2$, by the same reasoning as before, we have $\sigma(H)=\sigma_e(H)=[B, \infty)$, hence

$$\sigma_d(H)=\emptyset.$$

In the case of $0<\alpha\leq 1$, we need another proof that $\sigma_e(H)=[B, \infty)$, which is due to [4] (p117).

Proof. We have only to prove $[B, \infty)\subset\sigma_e(H)$. Since $f(r)-B/2\to 0$ as $r\to +\infty$, there exist $\{x_n\}_{n\in N}\subset \boldsymbol{R}^3$ such that

$$\text{(3.8)}\qquad \begin{cases} x_n=(0,\ 0,\ z_n),\ z_n>0,\\ z_n/n^2\to+\infty \text{ as } n\to+\infty \text{ and}\\ \sup\{|f(r)-B/2|\rho^2; |z-z_n|\le n,\ \rho\le n\}\le n^{-1}.\end{cases}$$

For $\lambda\ge 0$, we define $\Psi_n(x)$ by

$$\Psi_n(x)\equiv\Psi_{n,\lambda}(x)=e^{i\lambda z}\xi_n(z)\phi_0(\vec{\rho})\ (n\in\boldsymbol{N}),$$

where $\phi_0(\vec{\rho})$ is in the proof of Proposition 2.1 and

$$\xi_m(z)=n^{-1/2}\xi((z-z_n)/n)$$

for some fixed $\xi\in C_0^\infty(|z|\le 1)$. We remark that

$$\text{(3.9)}\qquad \|\phi_0\|_{L^2(\boldsymbol{R}^2)}=\|\xi_n\|_{L^2(\boldsymbol{R}^1)}=\|\Psi_n\|_{L^2(\boldsymbol{R}^3)}=1.$$

To prove $[B, \infty)\subset\sigma_e(H)$ it is sufficient to show that

$$\text{(3.10)}\qquad \begin{cases}(H-(B+\lambda^2))\Psi_n\to 0 \text{ strongly in } L^2(\boldsymbol{R}^3) \text{ as } m\to\infty \text{ and}\\ (\Psi_j,\ \Psi_k)_{L^2}=0\ (j\neq k).\end{cases}$$

Since div $b=\mathrm{div}(f(r)(-x_2,\ x_1,\ 0))=0$, we have

$$T^2=T_c^2+2ib_p\cdot\nabla+(2b_c\cdot b_p+|b_p|^2).$$

Moreover, since Ψ_n is independent of θ,

$$b_p\cdot\nabla\Psi_n=(f(r)-B/2)(-x_2,\ x_1,\ 0)\cdot((\partial\Psi_n/\partial\rho)\rho^{-1}x_1,\ (\partial\Psi_n/\partial\rho)\rho^{-1}x_2,\ \partial\Psi_n/\partial z)=0.$$

Hence we have

$$\text{(3.11)}\qquad T^2\Psi_n=T_c^2\Psi_n+(2b_c\cdot b_p+|b_p|^2)\Psi_n\ (n\in\boldsymbol{N}).$$

By a simple calculation,

$$\begin{aligned}|(2b_c\cdot b_p+|b_p|^2)\Psi_n|&=|(f(r)-B/2)(f(r)+B/2)\rho^2\Psi_n|\\ &\le d_1|f(r)-B/2|\rho^2|\Psi_n|\end{aligned}$$

for some constant $d_1>0$. By using the above inequality,

$$\begin{aligned}\int_{\boldsymbol{R}^3}|(2b_c\cdot b_p+|b_p|^2)\Psi_n|^2dx&\le d_1^2\int_{\boldsymbol{R}^3}(|f(r)-B/2|\rho^2)^2|\Psi_n|^2dx\\ &\le d_1^2\left\{\int_{\rho\le n}+\int_{\rho>n}\right\}(|f(r)-B/2|\rho^2)^2|\Psi_n|^2dx.\end{aligned}$$

Using (3.8) and the fact that supp $\xi_n\subset\{|z-z_n|\le n\}$ gives

$$\int_{\rho\le n}(|f(r)-B/2|\rho^2)^2|\Psi_n|^2dx\le n^{-2}\int_{\rho\le n}|\Psi_n|^2dx\le n^{-2}\to 0 \text{ as } n\to\infty.$$

On the other hand, by using (3.9),

$$\int_{\rho>n}(|f(r)-B/2|\rho^2)^2\rho^4|\Psi_n|^2dx\leq d_2\int_n^\infty \rho^5 e^{-B\rho^2/2}d\rho\to 0 \text{ as } n\to\infty.$$

Therefore we obtain

$$(2b_c\cdot b_p+|b_p|^2)\Psi_n\to 0 \text{ strongly in } L^2(\boldsymbol{R}^3) \text{ as } n\to\infty. \tag{3.12}$$

By a similar argument as in the proof of Proposition 2.1, we also have

$$(T_c^2-(B+\lambda^2))\Psi_n\to 0 \text{ strongly in } L^2(\boldsymbol{R}^3) \text{ as } n\to\infty, \tag{3.13}$$

and

$$V\Psi_n\to 0 \text{ strongly in } L^2(\boldsymbol{R}^3) \text{ as } n\to\infty. \tag{3.14}$$

By (3.11), (3.12), (3.13) and (3.14), we obtain

$$(H-(B+\lambda^2))\Psi_n\to 0 \text{ strongly in } L^2(\boldsymbol{R}^3) \text{ as } n\to\infty.$$

Using (3.8) and choosing a subsequence of $\{\Psi_n\}$ (denoted by the same notation $\{\Psi_n\}$), one can assume that

$$(\Psi_j,\ \Psi_k)_{L^2}=0\ (j\neq k).$$

Thus we obtain (3.10). □

We next show that the negativity assumption (V2) is necessary for the infiniteness of the discrete spectrum under the situation that V is bounded above.

EXAMPLE 3. Let

$$f(r)=\begin{cases} B/2\ (r\geq 2),\\ B/2+\exp(1/(r-2))\ (3/2\leq r<2),\\ B/2+2e^{-2}-\exp(-1/(r-1))\ (1\leq r<3/2),\\ B/2+2e^{-2}\ (0\leq r<1).\end{cases}$$

Then we have $f\in C^1([0,\infty))$, $f'(0)=0$, $f'(r)\leq 0$, and

$$F_f(r)=\begin{cases} B\ (r\geq 2)\\ B+4e^{-2}\ (0\leq r\leq 1).\end{cases}$$

Now we define $V(x)$ by

$$V(x)=\begin{cases} 0\ (r\geq 2),\\ \max(0,\ B-F_f(r))\ (1<r<2),\\ v(r)\ (0\leq r\leq 1),\end{cases}$$

where $|v(r)|\leq 4e^{-2}$. We remark that, in this case, (3.3) is satisfied but $V(x)$ does not satisfy (V2). We also have

$$(H\phi,\ \phi)_{L^2} \geq \int (F_f(r)+V)|\phi|^2 dx \geq B\|\phi\|_{L^2}^2,$$

so

$$\sigma_e(H)=[B,\ \infty),\ \sigma_d(H)=\emptyset.$$

Finally we show an example of the magnetic bottle (see [2]) which means a magnetic Schrödinger operator without the static potential term having a non-empty discrete spectrum.

EXAMPLE 4. Let

$$\beta=\inf\{(-\Delta\phi,\ \phi)_{L^2};\ \phi\in C_0^\infty(|x|\leq 1),\ \|\phi\|_L^2=1\} \tag{3.15}$$

We pick up $f\in C^1([0,\ \infty))$ such that

$$f(r)=\begin{cases}0\ (0\leq r\leq 1),\\(\beta+1)/2\ (r\geq 2).\end{cases}$$

Then it follows from Proposition 2.1 that $\sigma_e(T^2)=[\beta+1,\ \infty)$, so by (3.15) it is easy to see that

$$\inf\sigma(T^2)\leq\beta<\inf\sigma_e(T^2),$$

which implies $\sigma_d(T^2)\neq\emptyset$.

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