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WELLPOSEDNESS AND REGULARITY OF SECOND ORDER ABSTRACT EQUATIONS ARISING IN HYPERBOLIC-LIKE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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1. Introduction

This paper is concerned with well-posedness of abstract nonlinear differential equations of the form

$$(1.1)\quad \begin{cases} Mu_{tt}(t)+Au(t)+AGG^*Au_t(t)+AGf(u)(t)=\mathcal{F}(u)(t)\,; & t>0 \\ u(0)=u_0\,;\ u_t(0)\equiv u_1 \end{cases}$$

under the following assumptions:

(1.2) If $\tilde{A}: \mathcal{D}(\tilde{A})\subset H\rightarrow H$ is a closed, linear, positive self-adjoint operator acting on the Hilbert space H, then A denotes its realization as an operator: $\mathcal{D}(\tilde{A}^{1/2})\rightarrow[\mathcal{D}(\tilde{A}^{1/2})]'$.

(1.3) Let V be another Hilbert space such that

$$\mathcal{D}(\tilde{A}^{1/2})\subset V\subset H\subset V'\subset[\mathcal{D}(\tilde{A}^{1/2})]',$$

all injections being continuous and dense. We assume that $M\in\mathcal{L}(V\,; V')$ and $(Mu,\ u)\geq a|u|_V^2$ where (,) is understood here as a duality pairing between V and V'. Hence $M^{-1}\in\mathcal{L}(V',V)$. As is well known, setting $\tilde{M}=M|_H$, the restriction of M on H with $\mathcal{D}(\tilde{M})=\{u\in V\,;\ Mu\in H\}$, we have $V=\mathcal{D}(\tilde{M}^{1/2})$.

(1.4) Let U be another Hilbert space with scalar product denoted by $\langle\cdot,\ \cdot\rangle$. We assume that the bounded linear operator $G: U\rightarrow H$ satisfies $\tilde{A}^{1/2}G\in\mathcal{L}(U\,; H)$. Hence, $G^*A\in\mathcal{L}(\mathcal{D}(\tilde{A}^{1/2})\,; U)$.

(1.5) The nonlinear bounded operator $\mathcal{F}: \mathcal{D}(\tilde{A}^{1/2})\rightarrow V'$ is assumed to be Fréchet differentiable and its Fréchet derivative, denoted by $D\mathcal{F}$, satisfies

$$|D\mathcal{F}(u)h|_{V'}\leq C(\|u\|)\|h\|, \text{ where } \|h\|=\|h\|\mathcal{D}(\tilde{A}^{1/2})$$

(1.6) The nonlinear bounded operator $f: \mathcal{D}(\tilde{A}^{1/2})\rightarrow U$ is Fréchet differentiable and its Fréchet derivative Df satisfies

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$$|Df(u)h|_U \le C(\|u\|)\|h\|.$$

Here and throughout this paper, $C(\|u\|)$ denotes a generic function which is bounded for bounded values of the argument $\|u\|$. Equations of the type (1.1) can be considered as abstract models of second order (in time) nonlinear problems with *nonlinear* boundary conditions (see [10] and [28] for the treatment of linear equations). In fact, the composition operator $AG: U \to [\mathscr{D}(\tilde{A}^{1/2})]'$ (whose domain as $U \to H$ typically contains only the "zero" element) represents various boundary operators (see section 4). A distinctive feature of our problem is that the nonlinear "boundary" operator $M^{-1}AGf$ is *not* Lipschitz on a basic space on which the evalution is defined (i.e.: V). Examples motivating the above framework are equations of nonlinear elasticity with nonlinear boundary conditions. They include: nonlinear wave equations, von Kārmān plate equations, nonlinear Euler-Bernoulli and Kirchoff plate equations, etc. To fix our attention, we shall present three nonlinear plate equations exemplifying the abstract model (1.1).[1]

I. Nonlinear Euler-Bernoulli plate model with nonlinear boundary conditions

(1.7) $$u_{tt} + \Delta^2 u = g\left(\int_\Omega |\nabla u|^2 d\Omega\right)\Delta u \text{ in } \Omega \times (0, T),$$

with the boundary conditions

(1.8) $$\begin{cases} u|_\Gamma = 0, & \text{on } \Gamma \times (0, T) \\ \Delta u = -\frac{\partial}{\partial \nu} u_t + f(u, \nabla u) & \text{on } \Gamma \times (0, T) \end{cases}$$

and the initial conditions

(1.9) $$u(0) = u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \ u_t(0) = u_1 \in L_2(\Omega)$$

Here Ω is an open, bounded domain in R^2 with "smooth" (say C^4) boundary Γ. The operator f is a substitution operator (Nemytskii operator) represented by a C^1 function with a polynomial growth. The real valued function $g \in C^1(R)$ satisfies $g(s)s \ge 0$, $s \in R$. Equation (1.7) describes nonlinear vibrations of the plate. Its special case where g is linear is often referred to as "Berger's approximation" (see [29]).

II. Von Kārmān plate model with nonlinear boundary conditions

(1.10) $$u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u = [F(u), u] \text{ in } \Omega \times (0, T), \ \Omega \subset R^2$$

with the boundary conditions

[1] Other examples can be provided as well.

$$(1.11)\quad \begin{cases} u=0 & \text{on } \Gamma\times(0,\ T) \\ \Delta u=-\dfrac{\partial}{\partial\nu}u_t+\tilde{f}(u,\ u_t,\ \nabla u) & \text{on } \Gamma\times(0,\ T), \end{cases}$$

and the initial conditions

$$(1.12)\quad u(0)=u_0\ ;\ u_t(0)=u_1$$

Here, the nonlinear operator $F(u)$ (Airy stress function) is defined by

$$(1.13)\quad \begin{cases} \Delta^2F(u)=-[u,\ u] & \text{in } \Omega \\ F=\dfrac{dF}{\partial\nu}=0 & \text{on } \Gamma \end{cases}$$

where $[\psi,\phi]\equiv\psi_{xx}\phi_{yy}+\phi_{xx}\psi_{yy}-2\psi_{xy}\phi_{xy}$.

We shall consider two cases in the model (1.10) : (i) $\gamma>0$, i.e. when rotational forces are accounted for, and (ii) $\gamma=0$.

III. Parallely connected plates

We consider a system of two plates which are connected (via springs) at the boundary. This leads to the following system of plate equations, with nonlinear coupled boundary conditions.

$$(1.14)\quad \begin{cases} y_{tt}+\Delta^2y=[F(y),\ y] & \text{in } \Omega\times(0,\ T), \\ w_{tt}+\Delta^2w=[F(w),\ w] & \text{in } \Omega\times(0,\ T), \end{cases}$$

with the boundary conditions on $\Gamma\times(0,\ T)$

$$(1.15)\quad \begin{cases} y=w=0 \\ \Delta y=-\dfrac{\partial}{\partial\nu}y_t+f_1(\nabla(y-w),\ y-w) \\ \Delta w=-\dfrac{\partial}{\partial\nu}w_t+f_2(\nabla(w-y),\ y-w) \end{cases}$$

and the initial conditions in Ω

$$(1.16)\quad \begin{cases} y(0)=y_0,\ y_t(0)=y_1, \\ w(0)=w_0,\ w_t(0)=w_1. \end{cases}$$

Here $F(y)$(resp. $F(w)$) are Airy's stress functions defined as in (1.13). One could also consider the same models with other types of boundary conditions (moments and shears, etc.).

The nonlinear Euler-Bernoulli equation (1.7) and von Kārmān equation (1.10) are well known elastic models describing nonlinear vibrations of plates. These equation, when accompanied by *homogeneous* boundary conditions (i.e. the terms on the right hand side of (1.10) (resp. (1.11)) are equal to zero) have been studied extensively in the literature with several results related to the existence and uniqueness of solutions available in [18], [33], [7], [11], [29], [36], etc. Recent developments in *boundary* stabilization theory for elastic systems (see [18] and references therein) have brought to focus models with nonhomogeneous feedback boundary conditions (physically they represent forces, shears, moments applied on the edge or portion thereof of a plate). This, of course, raises the questions of *well-posedness* and *regularity* of the solutions to such models. While there are results dealing with well-posedness and regularity issues for *linear* equations (linear waves, plates) with either (i) linear boundary feedback (see [18], [21], [28], and references therein), or else (ii) nonlinear but monotone boundary feedbacks (see [22]), very few results are available in the *nonlinear* and non monotone cases, as considered in this paper. Indeed, the only results known to the authors are in the case of one dimensional von Kārmān systems (see [23]).

We note that the main technical difficulties of the problem at the abstract level stem from two reasons:
(i) the presence of the unbounded operator AG in model (1.1) which does not admit a nontrivial realization from U to the basic space H,
(ii) lack of smoothing effects of the original dynamics such as it occurs in "parabolic problems" (see for instance [5], [13]), where the smoothing character of the underlying evolution "makes up" for the unboundedness of the nonlinear terms.

With reference to the abstract equation (1.1), the main contribution of this paper is twofold:
(i) to provide a theory of well-posedness (existence and uniqueness) for nonlinear equations, with nonlinearities which are neither monotone nor locally Lipschitz (Theorems 2.1, 2.4), where known results and methods for studying abstract nonlinear equations (see for instance [3], [4], [19], [9], [32], [34], [24], etc.) are not applicable;
(ii) to provide a regularity theory which includes, in particular, existence of *classical* solutions (Theorems 2.2, 2.3). We note that our results are new even in the context of linear problems with linear, but nonhomogeneous, boundary conditions (i.e. when f in (1.1) is affine)

The abstract results are then applied (in Section 4), to several specific problems arising in nonlinear elasticity (Problems I-III above). Here again, the results obtained in the context of these particular equations are new in the literature. We illustrate this point, more specifically, in the case of the von Karman system (1.10)-(1.13). In this instance, the results available in the literature (see e.g. [19], [7], [17], [33]) deal mostly with well-posedness and regularity for problems with *zero* boundary conditions. While some of the well-posedness results for problems with nonhomogeneous, but *linear* boundary conditions can be obtained by extending the techniques available for zero boundary conditions (see [18]), the presence of nonlinearities in the boundary conditions raises much more delicate questions (it is here where the presence of the boundary damping—the term $AGG^* Au_t$— may be critical). The well-posedness results of Theorems 4.1, 4.2, 4.5, 4.7 provide an answer to this problem. Moreover, in the case of the van Kārmān plate with $\gamma=0$, the result of Theorem 4.5 is new *even* in the case of *zero* boundary conditions. Indeed, the question of uniqueness of weak solutions for this model has been an open problem in the literature (see [19], [18]).

Regarding the issue of *regularity* of solutions, Theorems 4.3, 4.4, 4.6 extend to the case of *nonlinear* boundary conditions those available for the case of zero boundary conditions (see [17], [8], [35]) with proofs which are considerable simpler (see Remark 4.5). We conclude by pointing out the relevance of these regularity results to other problems in the literature. Available stabilization estimates such as those of the main Theorem in [18] refer to *postulated* classical solutions to von Kārmān systems with *linear* but nonhomogeneous (i.e.: $\tilde{f}$ in (1.11) is linear) boundary conditions. On the other hand, the existence of such classical solutions has been an open problem in the literature. The result of Theorem 4.6 provides precisely the existence of *classical* solutions, and hence fully justifies the contribution in [18].

The outline of the paper is as follows. In section 2 we formulate the results pertinent to the existence and regularity of local and global solutions to the abstract model (1.1). The proofs of these results are given in section 3. Section 4 deals with applications of the abstract theory to the specific model of the von Kārmān plate equation (1.10)-(1.13) and to the nonlinear Euler-Bernoulli equation (1.7)-(1.9). In Section 4.1 (under additional structural hypothesis on the function $\tilde{f}$), we prove the existence and uniqueness of local and global weak solutions to (1.10)-(1.13), with $\gamma>0$, in the space $C([0, T]; [H^2(\Omega)\cap H_0^1(\Omega)]\times H_0^1(\Omega))$. Moreover, under additional assumptions on regularity and compatibility of initial conditions, we prove that these weak solutions are, in fact, classical solutions i.e. $u\in C([0, T]; H^4(\Omega))$ and $u_{tt}\in C([0, T]; H^2(\Omega))$. To our knowledge, this result is original and new, as all other results available in the literature deal either with homogeneous linear boundary conditions (see for instance [19], [7], [33] and references therein), or if the boundary conditions are nonlinear, the problem is treated in the one dimensional case only (see [23])(i.e. dim $\Omega=1$). In subsection 4.2 we treat the von Kārmān model with $\gamma=0$. Here, the existence and uniqueness of solutions is established in the space $C([0, T]; H^2(\Omega)\times L_2(\Omega))$. It should be noted that the uniqueness result for this model is new even in the case of *homogeneous* boundary conditions (see [19], [20]). Finally, section 4.3 deals with applications of the abstract theory to the nonlinear Euler-Bernoulli model (1.7)-(1.9) where the existence and uniqueness of weak solutions in the space $C([0, T]; H^2(\Omega)\times L_2(\Omega))$ is proved. Here, again, to our best knowledge, the results are new.

Other works available in the literature on this topic consider either one-dimensional models or problems with homogeneous boundary conditions (see for instance [11], [36], [31]).

2. Statement of the main results

We treat the equation

$$(2.1)\qquad \begin{cases} Mu_{tt}(t)+Au(t)+\beta AGG^*Au_t(t)+AGf(u(t))=\mathcal{F}(u(t)), t>0 \\ u(0)=u_0\in\mathcal{D}(\tilde{A}^{1/2}),\ u_t(0)=u_1\in V \end{cases}$$

under the assumptions (1.2)-(1.6) where β is a positive constant.

DEFINITION 2.1 We say that the function $\tilde{u}(t)=(u(t),\ u_t(t))$ is a *strong* solution to (2.1) on $[0,\ T]$ iff $\tilde{u}\in C([0,\ T];\ \mathcal{D}(\tilde{A}^{1/2})\times\mathcal{D}(\tilde{A}^{1/2}))$, $u_{tt}\in C([0,\ T];\ V)$, $\tilde{u}(0)=(u_0,\ u_1)$ and relation (2.1) holds for all $t\in[0,\ T]$ in the sense of the $[\mathcal{D}(\tilde{A}^{1/2})]'$-topology.

In order to define weak solutions to problem (2.1), we first define weak solutions to the following nonhomogeneous linear problem

$$(2.2)\qquad \begin{cases} Mu_{tt}(t)+Au(t)+\beta AGG^*Au_t(t)=-AGf+\mathcal{F}\ ; \\ u(0)=u_0\ ;\ u_t(0)=u_1. \end{cases}$$

where f (resp $\mathcal{F}$) are given elements in $L_1(0,\ T\ ;\ U)$(resp. $L_1(0,\ T\ ;\ V')$).

DEFINITION 2.2. We say that the function $\tilde{u}\in C([0,\ T];\ \mathcal{D}(\tilde{A}^{1/2})\times V)$ is a *weak* solution to (2.2) iff there exists a sequence of functions $f_n\in L_1(0,\ T\ ;\ U)$, $\mathcal{F}_n\in L_1(0,\ T\ ;\ V')$ and corresponding strong solutions $\tilde{u}_n(t)$ of (2.2) such that $f_n\to f$ in $L_1(0,\ T\ ;\ U)$, $\mathcal{F}_n\to\mathcal{F}$ in $L_1(0,\ T\ ;\ V')$ and $\tilde{u}_n\to\tilde{u}$ in $C([0,\ T];\ \mathcal{D}(\tilde{A}^{1/2})\times V)$. □

DEFINITION 2.3. We say that the function $\tilde{u}\in C([0,\ T];\ \mathcal{D}(\tilde{A}^{1/2})\times V)$ is a *weak* solution to (2.1) iff $\tilde{u}$ is a weak solution to the nonhomogeneous problem (2.2) with $f=f(u)$ and $\mathcal{F}=\mathcal{F}(u)$. □

Theorem 2.1. (local existence). *For each initial data* $(u_0,\ u_1)\in\mathcal{D}(\tilde{A}^{1/2})\times V$, *there exists* $T_0>0$ *such that problem* (2.1) *has a unique weak solution* $(u(t),\ u_t(t))$ *on* $(0,\ T_0)$. *Moreover,*

$$(2.3)\qquad \int_0^{T_0}|G^*Au_t(t)|_U^2dt\le C_{T_0,\beta}(\|u_0\|,\ |u_1|_V)$$

and the weak solution $\tilde{u}(t)$ *satisfies*

(2.4) $\frac{d}{dt}(Mu_t(t), \phi)+(Au(t), \phi)+\beta\langle G^*Au_t(t), G^*A\phi\rangle+\langle f(u(t)), G^*A\phi\rangle$
$=(\mathcal{F}(u(t), \phi)$

for all $\phi\in\mathcal{D}(\tilde{A}^{1/2})$, *where the above equality holds in* $H^{-1}(0, T_0)$. □

Theorem 2.2 (regularity) *Assume that the initial data* (u_0, u_1) *satisfy*

(2.5) $u_1\in\mathcal{D}(\tilde{A}^{1/2})$;
(2.6) $A(u_0+\beta GG^*Au_1+Gf(u_0))\in V'$.

Moreover, assume that

(2.7) (i) $|\tilde{A}^{-1/2}D\mathcal{F}(u)h|_H\leq C(\|u\|)|h|_V$;
(ii) $|\tilde{A}^{1/2}GDf(u)h|_H\leq C(\|u\|)[|h|_V+|G^*Ah|_U]$.

Then the solution to (2.1) *is strong on* $[0, T_0]$. *Moreover,*

(2.8) $A(u+\beta GG^*Au_t+Gf(u))\in C([0, T_0]; V')$,

and (2.4) *holds for all* $t\in[0, T_0]$ *and* $\phi\in\mathcal{D}(\tilde{A}^{1/2})$. □

REMARK 2.1. In the linear case (when $f\equiv 0$ and $\mathcal{F}\equiv 0$), the result of Theorem 2.1 can be obtained by using variational techniques as, for example, in [32]. Also, if $\mathcal{F}\neq 0$ but *still* $f=0$, a combination of the variational approach with a contraction argument would lead to the result. What makes this problem more interesting is the presence of the nonlinear term represented by the function f. In fact, in this case, the result depends critically on the strict positivity of the constant. The reason for this is that, in general, the regularity of the "undamped" linear model is not sufficient to control the "boundary" terms $AG\ f(u)$. □

In order to obtain *more* regular solutions, additional hypotheses on the nonlinear term need to be imposed.

Theorem 2.3 (regularity revisited). *In addition to the assumptions of* Theorem 2.2, *we assume that* $f=0$ *and* $\mathcal{F}$ *is twice Fréchet differentiable* $\mathcal{D}(\tilde{A}^{1/2})\rightarrow V'$. *Moreover, we assume that*

(2.9) $\tilde{M}^{-1}\in\mathcal{L}(H; \mathcal{D}(\tilde{A}^{1/2}))$;
(2.10) $\mathcal{F}(u_0)\in H$;
(2.11a) $u_0+\beta GG^*Au_1\in\mathcal{D}(\tilde{A})$;
(2.11b) $A(-u_1+\beta GG^*AM^{-1}[\tilde{A}(u_0+\beta GG^*Au_1)-\mathcal{F}(u_0)])\in V'$.

Then,

(2.12) $u_{tt}\in C([0, T_0]; \mathcal{D}(\tilde{A}^{1/2}))$;
(2.13) $u_{ttt}\in C([0, T_0]; V)$;

$$(2.14)\quad \begin{cases} \tilde{A}(u+\beta GG^*Au_t)-\mathcal{F}(u)\in C([0,\ T_0];\ H), \\ \tilde{A}(u_t+\beta GG^*Au_{tt})-D\mathcal{F}(u)u_t\in C([0,\ T_0]; V'); \end{cases}$$

Finally, $Mu_{tt}(t)+A(u(t)+\beta GG^*Au_t(t))-\mathcal{F}(u(t))=0$ *for all* $t\geq 0$*, where the above equation holds in* H. □

To obtain global solutions, we need to impose some structural conditions on the functions f and $\mathcal{F}$.

Theorem 2.4. (global existence). *In addition to the assumptions of* Theorem 2.1 *we assume that for all* $\tilde{u}\equiv(u,\ u_t)\in C([0,\ T_0];\ \mathcal{D}(\tilde{A}^{1/2})\times V)$ *and such that* $G^*Au_t\in L_2(0,\ T_0;\ U)$*, the following inequalities hold for all* $t\in[0,\ T_0]$

$$(2.15)\quad \int_0^t(\mathcal{F}(u(\tau)),\ u_t(\tau))d\tau\leq C_1\int_0^t[\|u(\tau)\|^2+|u_t(\tau)|_V^2]d\tau+C_2(\|u_0\|,\ |u_1|_V)\equiv C_0;$$

$$(2.16)\quad -\int_0^t<f(u(\tau)),\ G^*Au_t(\tau)>d\tau\leq C_0.$$

Then, the weak solution $(u(t),\ u_t(t))$ *of Theorem 2.1 is global on* $[0,\ T]$ *for any* $T>0$. □

3. Proofs of Theorems 2.1-2.4

3.1. Preliminary Lemmas

We define a linear operator

$$(3.1)\quad \begin{aligned} &\mathcal{A}:\ \mathcal{H}\rightarrow\mathcal{H},\ \mathcal{H}\equiv\mathcal{D}(\tilde{A}^{1/2})\times V; \\ &\mathcal{A}\begin{bmatrix}u\\ v\end{bmatrix}\equiv\begin{bmatrix}-v\\ M^{-1}A(u+\beta GG^*Av)\end{bmatrix}; \\ &\mathcal{D}(\mathcal{A})=\{(u,\ v)\in\mathcal{D}(\tilde{A}^{1/2})\times\mathcal{D}(\tilde{A}^{1/2});\ A(u+\beta GG^*Av)\in V'\}. \end{aligned}$$

Proposition 3.1. *The operator* $\mathcal{A}$ *generates a* C_0*-semigroup of contractions on* $\mathcal{H}$ *which we denote by* $e^{-\mathcal{A}t}$.

Proof. It is rather standard and based on the application of the Lumer-Phillips Theorem (see [30]). It suffices to show that $\mathcal{A}$ is maximal monotone.

Step 1. $\mathcal{A}$ is monotone. Indeed, with $\tilde{u}=(u,\ v)\in\mathcal{D}(\mathcal{A})$ we have

$$\begin{aligned} (\mathcal{A}(u,\ v)\ ,(u,\ v))_{\mathcal{H}} &=-(\tilde{A}^{1/2}u,\ \tilde{A}^{1/2}v)+(M^{-1}(A(u+\beta GG^*Au)),\ v)_V \\ &=-(Au,\ v)+(A(u+\beta GG^*Av),\ v)=\beta|G^*Av|_V^2\geq 0. \end{aligned}$$

Step 2. $\mathcal{A}$ is maximal monotone. By Minty's Theorem (see [3]) it suffices to prove that there exists a solution $(u,\ v)\in\mathcal{D}(\mathcal{A})$ to the following equations

$$(3.2)\quad \begin{cases} \lambda u-v=g \\ \lambda v+M^{-1}A[u+\beta GG^*Av]=h, \end{cases}\quad \text{with } \lambda>0 \text{ and } g\in\mathcal{D}(\tilde{\mathcal{A}}^{1/2}),\qquad h\in V$$

System (3.2) reduces to

$$(3.3) \qquad Au+\lambda^2 Mu+\beta\lambda AGG^*Au=\lambda Mg+Mh+\beta AGG^*Ag\in[\mathcal{D}(\tilde{A}^{1/2})]'.$$

The operator A is maximal monotene and coercive $\mathcal{D}(\tilde{A}^{1/2})\to[\mathcal{D}(\tilde{A}^{1/2})]'$. The sum of two operators $\lambda^2 M+\lambda\beta AGG^*A$ is continuous and monotone $\mathcal{D}(\tilde{A}^{1/2})\to[\mathcal{D}(\tilde{A}^{1/2})]'$. Hence (see [3]) $A+\lambda^2 M+\beta AGG^*A$: $\mathcal{D}(\tilde{A}^{1/2})\to[\mathcal{D}(\tilde{A}^{1/2})]'$ is maximal monotone and coercive, hence boundedly invertible. This implies that there exists $u\in\mathcal{D}(\tilde{A}^{1/2})$, solution to (3.3), and from (3.2) we obtain that $v=\lambda u-g\in\mathcal{D}(\tilde{A}^{1/2})$. Going back to the second equation in (3.2) we infer that $M^{-1}[A(u+\beta GG^*Av)]=h-\lambda v\in V$, hence $A(u+\beta GG^*Av)\in V'$ as desired. ■

We now consider linear part of equation (2.1)

$$(3.4) \qquad \begin{cases} Mu_{tt}+Au+\beta AGG^*Au_t=0 \\ u(0)=u_0\in\mathcal{D}(\tilde{A}^{1/2})\,;\ u_t(0)=u_1\in V. \end{cases}$$

Corollary 3.1.

(i) *For each* $(u_0, u_1)\in\mathcal{D}(\mathcal{A})$ *there exists a unique strong solution to* (3.4).

(ii) *For each* $(u_0, u_1)\in\mathcal{D}(\tilde{A}^{1/2})\times V$ *there exists a unique weak solution to* (3.4). *Moreover, the weak solution* $\tilde{u}=(u, u_t)$ *satisfies the estimate*

$$(3.5) \qquad \int_0^t |G^*Au_t(t)|_U^2 dt\leq\frac{1}{2\beta}[\|u_0\|^2+|u_1|_V^2].$$

Proof. All the statements except (3.5) follow from Proposition 3.1 combined with standard results in linear semigroup theory (see [3]). To prove (3.5) we consider first strong solutions $\tilde{u}_n(t)$ corresponding to the initial data $(u_{0n}, u_{1n})\in\mathcal{D}(\mathcal{A})$, such that $u_{0n}\to u_0$ in $\mathcal{D}(\tilde{A}^{1/2})$ and $u_{1n}\to u_1$ in V. Since $\tilde{u}_n(t)$ is a strong solution, each term in equation (3.4) is a continuous function on $[0, T]$ with the values in $[\mathcal{D}(\tilde{A}^{1/2})]'$. Hence for all $t\geq 0$ and $\phi\in\mathcal{D}(\tilde{A}^{1/2})$,

$$(Mu_{ntt}(t), \phi)+(Au_n(t), \phi)+\beta(G^*Au_{nt}(t), G^*A\phi)=0.$$

Setting $\phi\equiv u_{nt}(t)\in\mathcal{D}(\tilde{A}^{1/2})$ yields

$$(3.6) \qquad |u_{nt}(t)|_V^2+\|u_n(t)\|^2+2\beta\int_0^t |G^*Au_{nt}(\tau)|_U^2 d\tau=\|u_{0n}\|^2+|u_{1n}|_V^2.$$

Similarly, we obtain

$$(3.7) \qquad \lim_{n,m\to\infty} |(u_{nt}-u_{mt})(t)|_V^2+\|(u_n-u_m)(t)\|^2+2\beta\int_0^2 |G^*A(u_{nt}-u_{mt})|_U^2 d\tau=0.$$

Hence

$$(3.8) \qquad \tilde{u}_n\to\tilde{u} \text{ in } C([0, T]\,;\ \mathcal{D}(\tilde{A}^{1/2})\times V,$$

$$(3.9) \qquad G^*Au_{nt}\to g \text{ in } L_2(0, T\,;\ U).$$

From (3.8) and the regularity $G^*\tilde{A}^{1/2}\in\mathcal{L}(H\,;\ U)$ we infer that

(3.10) $G^*Au_n\to G^*Au$ in $C([0,\ T]\,;\ U)$ and $\frac{d}{dt}G^*Au_n\to\frac{d}{dt}G^*Au$ in $H^{-1}(0,\ T\,;\ U)$.

By the uniqueness of the strong limit we must have that $g=\frac{d}{dt}G^*Au\in L_2(0,\ T\,;\ U)$ and

$$G^*Au_{nt}\to\frac{d}{dt}G^*Au \text{ in } L_2(0,\ T\,;\ U) \tag{3.11}$$

On the other hand we also have from (3.8) $u_{nt}\to u_t$ in $H^{-1}(0,\ T\,;\ \mathcal{D}(\tilde{A}^{1/2}))$, and since $G^*\tilde{A}^{1/2}\in\mathcal{L}(H\,;\ U)$

$$G^*Au_{nt}\to G^*Au_t \text{ in } H^{-1}(0,\ T\,;\ U). \tag{3.12}$$

Comparing (3.11) with (3.12) yields $g=G^*Au_t$ and

$$G^*Au_{nt}\to G^*Au_t \text{ on } L_2(0,\ T\,;\ U). \tag{3.13}$$

Passage to the limit on (3.6) after taking into account (3.8) and (3.13) yields (3.5). ■

We introduce the following operators:

$$\mathcal{B}:U\to[\mathcal{D}(\mathcal{A}^*)]',\ \text{where } \mathcal{D}(\mathcal{A}^*)\subset\mathcal{H}\subset[\mathcal{D}(\mathcal{A}^*)]',$$

$$\mathcal{B}g\equiv\begin{bmatrix}0\\ M^{-1}AGg\end{bmatrix}. \tag{3.14}$$

Notice that

$$\mathcal{A}^{-1}\mathcal{B}g=\begin{bmatrix}A^{-1}M^{-1}AGg\\ 0\end{bmatrix}\in\mathcal{H}. \tag{3.15}$$

$\mathcal{L}:\ L_2(0,\ T\,;\ U)\to C([0,\ T]\,;\ [\mathcal{D}(\mathcal{A}^*)]')$ defined by

$$(\mathcal{L}g)(t)=\int_0^t e^{-\mathcal{A}(t-s)}\mathcal{B}g(s)ds. \tag{3.16}$$

The following regularity result plays a crucial role in the proof of Theorem 2.1.

Lemma 3.1. *The operator* $\mathcal{L}$ *defined by* (3.16) *admits a bounded extension from* $L_2(0,\ T\,;\ U)\to C([0,\ T]\,;\ \mathcal{H})$.

Proof. From (3.15) and (3.16) it follows that $\mathcal{A}^{-1}\mathcal{L}\in\mathcal{L}(L_2(0,\ T\,;\ U)\to C([0,\ T]\,;\ \mathcal{H})$. Hence (see [15]) $\mathcal{L}$ is closeable. It is straightforward to verify (see [26]) that $H_0^1(0,\ T\,;\ U)\subset\mathcal{D}(\mathcal{L})$, which implies that $\mathcal{L}$ is densely defined. Thus, by using the duality argument of [26], it suffices to prove that

$$\text{(3.17)} \quad \int_0^T |\mathcal{B}^* e^{-\mathcal{A}^* T} \tilde{u}|_U^2 dt \leq C_T |\tilde{u}|_{\mathcal{H}}^2 \text{ for } \tilde{u}=(u, v) \in \mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\tilde{A}^{1/2}) \times \mathcal{D}(\tilde{A}^{1/2}).$$

Here $\langle \mathcal{B}^* v, g \rangle \equiv (v, Bg)_{\mathcal{H}}$ for $g \in U$, $v \in \mathcal{D}(\mathcal{A}^*)$ and $(\cdot, \cdot)_{\mathcal{H}}$ denotes the duality pairing in $\mathcal{D}(\mathcal{A}^*) \times [\mathcal{D}(\mathcal{A}^*)]'$. Straghtforward computations show that with $(u, v) \in \mathcal{D}(\mathcal{A}^*)$, then $\tilde{z}(t) \equiv (z(t), -z_t(t)) \equiv e^{-\mathcal{A}^* t}(u, v)$ is characterized as a strong solution to

$$\text{(3.18)} \quad \begin{cases} Mz_{tt} + Az + \beta AGG^* Az_t = 0 \\ z(0) = u, \; z_t(0) = -v. \end{cases}$$

Notice that $(u, v) \in \mathcal{D}(\mathcal{A}^*)$ is equivalent to $(u, -v) \in \mathcal{D}(\mathcal{A})$. Thus

$$z \in C([0, T]; \mathcal{D}(\tilde{A}^{1/2})), \; z_t \in C([0, T]; \mathcal{D}(\tilde{A}^{1/2})) \text{ and } z_{tt} \in C([0, T]; V).$$

Applying inequality (3.5) to (3.18) yields

$$\text{(3.19)} \quad \int_0^T |G^* A z_t|_U^2 dt \leq \frac{1}{2\beta} [\|u\|^2 + |v|_V^2].$$

On the other hand with $(u, v) \in \mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\tilde{A}^{1/2}) \times \mathcal{D}(\tilde{A}^{1/2})$, we have

$$\langle g, \mathcal{B}^* \begin{bmatrix} u \\ v \end{bmatrix} \rangle = (\mathcal{B} g, \begin{bmatrix} u \\ v \end{bmatrix})_{\mathcal{H}} = (M^{-1} AGg, v)_V = (AGg, v) = (Gg, Av) = \langle g, G^* Av \rangle.$$

Hence with $(u, v) \in \mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\tilde{A}^{1/2}) \times \mathcal{D}(\tilde{A}^{1/2})$

$$\text{(3.20)} \quad \mathcal{B}^* \begin{bmatrix} u \\ v \end{bmatrix} = G^* Av.$$

Combining (3.18)-(3.20) yields the desired inequality in (3.17). ■

REMARK 3.1. Notice that inequality (3.17) or—equivalently— the result of Lemma 3.1 *does not* follow from the regularity properties of the solutions provided by the semigroup theory. (3.17) is an independent regularity result which critically relies on the assumption that $\beta > 0$. In fact, it can be shown, in a number of pde examples, that the "trace regularity" property (3.17) is not valid if $\beta = 0$ (see [27]).

Our next step is to obtain regularity properties of the solution to the *non-homogeneous* problem (2.2)

Lemma 3.2

(i) *For every* $(f, \mathcal{F}) \in H_0^1(0, T; U \times V')$ *and* $\tilde{u}(0) \in \mathcal{D}(\mathcal{A})$ *there exists a unique strong solution to problem* (2.2).

(ii) *For each* $(f, \mathcal{F}) \in L_1(0, T; U \times V')$, $\tilde{u}(0) \in \mathcal{H}$, *there exists a unique weak solution to problem* (2.2). *Moreover, this weak solution* $\tilde{u}(t)$ *is represented by the following formula*

(3.21) $\tilde{u}(t)=(\mathcal{L}f)(t)+(\hat{\mathcal{L}}\mathcal{F})(t)+e^{-\mathcal{A}t}\tilde{u}(0)$, *where* $\mathcal{L}$ *is defined by* (3.16) *and*

$$(\hat{\mathcal{L}}\mathcal{F})(t)\equiv\int_0^t e^{-\mathcal{A}(t-s)}\begin{bmatrix}0\\ M^{-1}\mathcal{F}(s)\end{bmatrix}ds.$$

(iii) *Weak solutions to the problem* (2.2) *satisfy the following inequalities*:

(3.22) $\int_0^T|G^*Au_t(t)|_U^2dt\le C_{T,\beta}[\|u_0\|^2+|u_1|_V^2+|f|^2_{L_2(0,T;U)}+|\mathcal{F}|^2_{L_1(0,T;V')}]$,

$$|u_t(t)|_V^2+\|u(t)\|^2+2\beta\int_0^t|G^*Au_t(s)|_U^2ds+2\int_0^t\langle f(s),\ G^*Au_t(s)\rangle ds$$

(3.23) $$-2\int_0^t((\mathcal{F}(s),\ u_t(s))ds\le\|u_0\|^2+|u_1|_V^2.$$

Proof. Notice that with $f\in H_0^1(0,T;U)$, $\frac{d}{dt}(\mathcal{L}f)(t)=\mathcal{L}\left[\frac{d}{dt}f\right](t)$. Hence, by the result of Lemma 3.1 and (3.15)

(3.24) $$\frac{d}{dt}\mathcal{L}\in\mathcal{L}(H_0^1(0,T;U);\ C([0,T];\mathcal{H}).$$

Assuming also that $\mathcal{F}\in H_0^1(0,T;V')$, we obtain $\frac{d}{dt}(\hat{\mathcal{L}}\mathcal{F})(t)=\hat{\mathcal{L}}\left[\frac{d}{dt}\mathcal{F}\right](t)$, and

(3.25) $$\frac{d}{dt}\hat{\mathcal{L}}\in\mathcal{L}(H_0^1(0,T;V');\ C([0,T];\mathcal{H})).$$

By using (3.24), (3.25) and Proposition 3.1 along with standard semigroup arguments, one easily shows that strong solutions to problerm (2.2) are given by the formula (3.21). This proves part (i) of Lemma 3.2. To obtain weak solutions of part (ii), it is just enough to recall the boundedness of the opetator $\mathcal{L}$: $L_2(0,T;U)\to C([0,T];\mathcal{H})$ (Lemma 3.1) and of the operator $\hat{\mathcal{L}}$: $L_1(0,T;V')\to C([0,T];\mathcal{H})$. As for part (iii) of the Lemma, it suffices to establish inequalities (3.22) and (3.23) for strong solutions and then, by the same arguments as those in Corollary 3.1, to pass to the limit. Let $\tilde{u}(t)=(u(t),u_t(t))$ be the strong solution to (2.2). Then $u_{tt}\in C([0,T];V)$, $u_t\in C([0,T];\mathcal{D}(\tilde{A}^{1/2}))$, and $G^*Au_t\in C([0,T];U)$. Thus, $\tilde{u}(t)$ satisfies

(3.26) $(Mu_{tt}(t),\ \phi)+(Au(t),\ \phi)+\beta\langle G^*Au_t(t),\ G^*A\phi\rangle=-\langle f(t),\ G^*A\phi\rangle$ $+(\mathcal{F}(t),\ \phi)$

for *all* $\phi\in\mathcal{D}(\tilde{A}^{1/2})$ and $t\ge0$. Thus, setting $\phi\equiv u_t(t)$ and integrating (3.26) from 0 to t (as in Corollary 3.1) yields inequality (3.23). From (3.23) we obtain

$$|u_t(t)|_V^2+\|u(t)\|^2+2\beta\int_0^t|G^*Au_t(\tau)|_U^2d\tau\le2\int_0^t|f(\tau)|_U|G^*Au_t(\tau)|_Ud\tau$$

$$+2\int_0^t|\mathcal{F}(\tau)|_{V'}|u_t(\tau)|_V d\tau+\|u_0\|^2+|u_1|_V^2.$$

Hence

$$(3.27)\quad |u_t(t)|_V^2+\|u(t)\|^2+\int_0^t|G^*Au_t(\tau)|_U^2 d\tau\le C_\beta\int_0^t|f(\tau)|_U^2 d\tau$$
$$+\int_0^t|\mathcal{F}(\tau)|_{V'}|u_t(\tau)|_V d\tau+\|u_0\|^2+|u_1|_V^2.$$

By using Lemma A-5 in [4] we obtain

$$|u_t(t)|_V\le C_\beta|f|_{L_2(0,\,T;\,U)}\int_0^t|\mathcal{F}(\tau)|_{V'}d\tau+\|u_0\|+|u_1|_V,$$

which inequality together with (3.27) leads to the desired result in (3.22) for strong solutions. Passage to the limit along the same arguments as in Corollary 3.1 proves these estimates for weak solutions (here, careful attention must be paid—as in Corollary 3.1—in passing to the limit on the term G^*Au_t, since this term is not bounded for $\tilde{u}\in C([0,\,T];\,\mathcal{H})$ and G^*A is typically unclosable).

3.2. Proof of Theorem 2.1

To prove this Theorem, we shall construct a fixed point for the map $\tilde{u}\to\Lambda(\tilde{u})$ where

$$(3.28)\quad (\Lambda\tilde{u})(t)\equiv e^{-\mathcal{A}t}\tilde{u}_0+\mathcal{L}f(u)(t)+\hat{\mathcal{L}}\mathcal{F}(u)(t).$$

Let B_R denote a closed ball in $\mathcal{H}$ with a radius R. We shall show that Λ admits the unique fixed point in the closed subspace $C([0,\,T_0];\,B_R)$ for sufficiently large R and sufficiently small T_0. To accomplish this, we need to prove that Λ is a contraction and that

$$(3.29)\quad \Lambda(C([0,\,T_0];\,B_R)\subset C([0,\,T_0];\,B_R).$$

The contraction property of Λ follows now from Lemma 3.1 and the following computations.

$$|(\mathcal{L}f(u_1)-\mathcal{L}f(u_2))(t)|_{\mathcal{H}}^2\le C_\beta\int_0^t|f(u_1(s))-f(u_2(s))|_U^2 ds,\ \text{by assumption (1.6)}$$

$$(3.30)\quad \le C_\beta(R)\int_0^t\|u_1(s)-u_2(s)\|^2 ds\le C_\beta(R)t|\tilde{u}_1-\tilde{u}_2|_{C([0,\,T];\,\mathcal{H})}^2.$$

Similarly, using hypothesis (1.5) and the boundedness $\hat{\mathcal{L}}:L_1(0,\,T;\,V')\to C([0,\,T];\,\mathcal{H})$ we obtain

$$(3.31)\quad |(\hat{\mathcal{L}}\mathcal{F}(u_1)(t)-\hat{\mathcal{L}}\mathcal{F}f(u_2))(t)|_{\mathcal{H}}\le C\int_0^t|\mathcal{F}(u_1(s))-\mathcal{F}(u_2(s))|_{V'}ds$$
$$\le C(R)\int_0^t\|u_1(s)-u_2(s)\|ds\le C(R)t|\tilde{u}_1-\tilde{u}_2|_{C([0,\,T];\,\mathcal{H})}.$$

Thus, for a given R, we select a sufficiently small $T_0(R)$ so that Λ is a contraction. To prove (3.29) it is enough to take R large enough (depending on the initial data $\tilde{u}_0$) and to perform computations similar to these in (3.30), (3.31). Application of the Fixed Point Theorem yields the existence of weak solutions. To complete the proof of Theorem 2.1 we need to justify the validity of inequality (2.3). To do this we shall use the result of Lemma 3.2. Indeed, since for all weak solutions by assumption (1.5), (1.6) we have

$$|f(u)|_{L_2(0,\,T;\,U)} \le C_{T_0,\beta}(\|u_0\|,\ |u_1|_V)\,;$$
$$|\mathcal{F}(u)|_{L_1(0,\,T;\,V')} \le C_{T_0}(\|u_0\|,\ |u_1|_V),$$

we are in a position to apply inequality (3.22) of Lemma 3.2. This yields the result in (2.3). Derivation of (2.4) is now straightforward, via the usual semigroup argument (see [30] or [3]). ■

3.3. Proof of Theorem 2.2

By using the regularity properties (2.7) and (2.3) one easily shows that $\tilde{z} \equiv \tilde{u}_t$ (derivative in the sense of distributions)) with $\tilde{u}$- weak solution guaranteed by Theorem 2.1 satisfies the equation

$$(3.32)\quad \tilde{z}(t) = e^{-\mathcal{A}t}\tilde{z}(0) + \int_0^t e^{-\mathcal{A}(t-s)}\begin{bmatrix} 0 \\ M^{-1}D\mathcal{F}(u)u_t(s)\end{bmatrix} ds + \mathcal{L}(Df(u)u_t)(t)$$
in $[\mathcal{D}(\mathcal{A}^*)]'$,
$$\tilde{z}(0) = -\mathcal{A}\,\tilde{u}_0 + \begin{bmatrix} 0 \\ M^{-1}\mathcal{F}(u_0)\end{bmatrix} + \begin{bmatrix} 0 \\ -M^{-1}AGf(u_0)\end{bmatrix}.$$

Properties (2.7) and (2.3) are used to assert that the weak solutions $\tilde{u}$ satisfy

$$(3.33)\quad \mathcal{A}^{-1}\begin{bmatrix} 0 \\ M^{-1}D\mathcal{F}(u)u_t\end{bmatrix} \in C([0,\ T_0]\,;\ \mathcal{H})\,;$$
$$(3.34)\quad \mathcal{A}^{-1}Df(u)u_t \in L_2(0,\ T_0\,; U).$$

These regularity properties allow us to compute $\tilde{z} = \tilde{u}_t$ in $[\mathcal{D}(\mathcal{A}^*)]'$ as in (3.32). In view of (3.32), to prove Theorem 2.2 it suffices to show that the following integral equation in the variable $\tilde{z} = (z,\ z_t)$

$$(3.35)\quad \tilde{z}(t) = e^{-\mathcal{A}t}\tilde{z}(0) + \int_0^t e^{-\mathcal{A}(t-s)}\begin{bmatrix} 0 \\ M^{-1}D\mathcal{F}(u(s))z(s)\end{bmatrix} ds + \mathcal{L}(Df(u(\cdot))z(\cdot))(t)$$

admits a *unique* solution in $C([0,\ T]\,;\ \mathcal{H})$ for any $\tilde{z}(0) \in \mathcal{H}$ and fixed $\tilde{u}$-weak solution to (2.1). Indeed, assuming for the present the solvability of (3.35), we easily check that a unique solution $\tilde{z}(t)$ of (3.35) with

$$(3.36)\quad \tilde{z}(0) \equiv -\mathcal{A}\,\tilde{u}_0 + \begin{bmatrix} 0 \\ M^{-1}\mathcal{F}(u_0)\end{bmatrix} - \begin{bmatrix} 0 \\ M^{-1}AGf(u_0)\end{bmatrix}.$$

$$=\begin{bmatrix} u_1 \\ -M^{-1}[A(u_0+\beta GG^*Au_1+Gf(u_0))-\mathcal{F}(u_0)] \end{bmatrix}$$

is precisely the solution of (3.32), hence it coincides with $\tilde{u}=(u_t, u_{tt})\in C([0, T_0]; \mathcal{H})$. To claim this, we use hypotheses (2.5) and (2.6) which give

$$M^{-1}[A(u_0+\beta GG^*Au_1+Gf(u_0))]\in V,\ M^{-1}\mathcal{F}(u_0)\in V,\ u_1\in\mathcal{D}(\tilde{A}^{1/2}).$$

Hence, by (3.36), $\tilde{z}(0)\in\mathcal{H}$. Thus, to complete the proof of the Theorem, we need to prove the solvability of (3.35) in $C([0, T]; \mathcal{H})$ with $z(0)\in\mathcal{H}$. To this end, notice first that for a fixed $u\in C([0, T_0]; \mathcal{H})$, equation (3.35) is *linear* in $\tilde{z}$. Thus, provided that the appropriate Lipschitz continuity of the terms in (3.35) (3.3) (in the variable $\tilde{z}$) holds, we are in a position to use the Contraction Mapping Principle. This is first done locally on $C([0, T_1]; \mathcal{H})$ where $T_1\ll T_0$ and then, by linearity, extended globally for all $t\in[0, T_1]$. The afore-mentioned Lipschitz continuity follows from the following estimates

$$(3.37)\quad |M^{-1}D\mathcal{F}(u)[z_1-z_2]|_V\le C(\|u\|)\|z_1-z_2\|,$$
$$(3.38)\quad |Df(u)(z_1-z_2)|_U\le C(\|u\|)\|z_1-z_2\|.$$

By Lemma 3.1 and (3.38)

$$(3.39)\quad |\mathcal{L}(Df(u))(z_1-z_2)|^2_{C([0,T];\mathcal{H})}\le C_T\int_0^T|Df(u(t))(z_1-z_2)(t)|^2_U dt$$
$$\le C_T(\|u\|_{C([0,T];\mathcal{D}(\tilde{A}^{1/2}))}T\|z_1-z_2\|^2_{C([0,T];\mathcal{D}(\tilde{A}^{1/2}))}.$$

Similarly the operator

$$(\hat{\mathcal{L}}_1 z)(t)\equiv\int_0^t e^{-\mathcal{A}(t-s)}\begin{bmatrix} 0 \\ M^{-1}D\mathcal{F}(u(s))z(s) \end{bmatrix}ds$$

satisfies the Lipschitz condition

$$(3.40)\quad |\hat{\mathcal{L}}_1(z_1-z_2)|_{C([0,T];\mathcal{H})}\le C_T[\|u\|_{C([0,T];\mathcal{D}(\tilde{A}^{1/2}))}]T\|z_1-z_2\|_{C([0,T];\mathcal{D}(\tilde{A}^{1/2}))}.$$

The bounds in (3.39), (3.40) allow application of the Contraction Mapping Principle on $C([0, T_1]; \mathcal{D}(\tilde{A}^{1/2}))$, where T_1 is sufficiently small and depends on the norms of the initial data and $C([0, T]; \mathcal{H})$ norm of the weak solution $\tilde{u}(t)$. This completes the proof of the existence of strong solutions. Relation (2.8) in Theorem 2.2 can be directly read off from the equation. ■

3.4. Proof of Theorem 2.3

Having already solutions $u(t)$ with regularity as in Theorem 2.2, we differentiate once more formula (3.32) (with respect to time). This leads us to the following equation in the variable $\tilde{z}\equiv(u_{tt}, u_{ttt})$.

$$(3.41)\quad \tilde{z}(t)=e^{-\mathcal{A}t}\tilde{z}(0)+\int_0^t e^{-\mathcal{A}(t-s)}\begin{bmatrix} 0 \\ M^{-1}D\mathcal{F}(u)u_{tt}(s) \end{bmatrix}ds$$

$$+\int_0^t e^{-\mathscr{A}(t-s)}\begin{bmatrix} 0 \\ M^{-1}D^2\mathscr{F}(u)(u_t,\ u_t)\end{bmatrix}ds.$$

where

$$\tilde{z}(0)\equiv\mathscr{A}\left[\mathscr{A}\tilde{\mathrm{u}}_0-\begin{bmatrix} 0 \\ M^{-1}\mathscr{F}(u_0)\end{bmatrix}\right]+\begin{bmatrix} 0 \\ M^{-1}D\mathscr{F}(u_0)u_1\end{bmatrix}.$$

Notice that since $u_{tt}\in C([0,\ T_0];\ V)$, $u_t\in C([0,\ T_0];\ \mathscr{D}(\tilde{A}^{1/2}))$ and for a fixed $u\in\mathscr{D}(\tilde{A}^{1/2})$, $D^2\mathscr{F}(u)$ is a bilinear continuous transformation

$$\mathscr{D}(\tilde{A}^{1/2})\times\mathscr{D}(\tilde{A}^{1/2})\rightarrow V' \tag{3.42}$$

(see [1] p.21, Theorem 4.3), all the terms on the RHS of (3.41) are elements in $C([0,\ T_0];\ [\mathscr{D}(\mathscr{A}^*)]')$. What we need to prove is that

$$\tilde{z}\in C([0,\ T_0];\ \mathscr{H}). \tag{3.43}$$

Rewriting (3.41) as an integral equation yields

$$\tilde{z}(t)=e^{-\mathscr{A}t}\tilde{z}(0)+\int_0^t e^{-\mathscr{A}(t-s)}\begin{bmatrix} 0 \\ M^{-1}D\mathscr{F}(u)z(s)\end{bmatrix}ds + \int_0^t e^{-\mathscr{A}(t-s)}\begin{bmatrix} 0 \\ M^{-1}D^2\mathscr{F}(u)(u_t,\ u_t)(s)\end{bmatrix}ds. \tag{3.44}$$

Define

$$a(t)\equiv\int_0^t e^{-\mathscr{A}(t-s)}\begin{bmatrix} 0 \\ M^{-1}D^2\mathscr{F}(u(s))(u_t(s),\ u_t(s))\end{bmatrix}ds.$$

From (3.42) and the regularity of $u_t\in C([0,\ T_0];\ \mathscr{D}(\tilde{A}^{1/2}))$, we obtain

$D^2\mathscr{F}(u)(u_t,u_t)\in C([0,\ T_0];\ V')$, hence

$$a\in C([0,\ T_0];\ \mathscr{H}). \tag{3.45}$$

The regularity of the initial data u_0, u_1 postulated by (2.9)-(2.11) implies that

$$\tilde{z}(0)\in\mathscr{H}. \tag{3.46}$$

Returning to (3.44) we obtain

$$\tilde{z}(t)=e^{-\mathscr{A}t}\tilde{z}(0)+\hat{\mathscr{L}}_1(z(\cdot))(t)+a(t) \tag{3.47}$$

where we recall that $\hat{\mathscr{L}}_1$ is defined in the formula below (3.39)). By using estimate (3.40), the regularity in (3.45), and (3.46), we easily show (as in Theorem 2.2) that the linear equation (3.47) has a unique global solution on $C([0,\ T_0];\ \mathscr{H})$. This completes the proof of regularity in (2.12), (2.13). The remaining statement of the Theorem follows directly from the equation and the regularity of u_{tt}. ■

3.5. Proof of Theorem 2.4

To prove the Theorem it suffices to establish the following a-priori bound.

Lemma 3.3. *Let* $\tilde{u}=(u, u_t)$ *be a weak solution to* (2.1). *Assume the hypotheses of* Theorem 2.4. *Then*

$$\|u(t)\|+|u_t(t)|_V\leq C(\|u_0\|, |u_1|_V), \text{ for } t\leq T_0.$$

Proof. We shall use the result of Lemma 3.2 with

$$f(t)\equiv f(u(t)), \ \mathcal{F}(t)\equiv\mathcal{F}(u(t)).$$

Since $u\in C([0, T_0]; \mathcal{D}(\tilde{A}^{1/2}))$, by the assumptions imposed on f and $\mathcal{F}$ (see (1.5) and (1.6)), we obtain

$$|f|_{C([0, T_0]; U)}\leq C_{T0}(\|u_0\|; |u_1|_V),$$
$$|\mathcal{F}|_{C([0, T_0]; V')}\leq C_{T0}(\|u_0\|; |u_1|_V).$$

Thus, we are in a position to apply inequality (3.23) of Lemma 3.2. This yields

$$\begin{aligned}(3.48)\quad & |u_t(t)|_V^2+\|u(t)\|^2+2\beta\int_0^t|G^*Au_t(s)|_U^2ds+2\int_0^t(f(u(s)), G^*Au_t(s)ds\\ & -2\int_0^2(\mathcal{F}(u(s)), u_t(s))ds\leq\|u_0\|^2+|u_1|_V^2.\end{aligned}$$

From inequality (2.3) in Theorem 2.1 we conclude that hypotheses (2.15), (2.16) of Theorem 2.4 are applicable to weak solutions (u, u_t). Hence, from (3.48) and (2.15), (2.16) we infer that

$$(3.49)\quad |u_t(t)|_V^2+\|u(t)\|^2\leq\|u_0\|^2+|u_1|_V^2+C\int_0^t(|u_t(\tau)|_V^2+\|u(\tau)\|^2)d\tau, \ t\leq T_0.$$

Application of Gronwall's inequality to (3.49) completes the proof of Lemma 3.3. ■

4. Examples motivating the abstract theory

4.1 von Kārmān plate model accounting for rotational forces (i.e. $\gamma>0$)

Let Ω be an open bounded domain in R^2 with sufficiently smooth boundary Γ and let the parameter $\gamma>0$. We consider the following model of a dynamic von Kārmān plate in the variable $u(t, x)$

$$(4.1)\quad u_{tt}-\gamma\Delta u_{tt}+\Delta^2u=[F(u), u] \text{ in } \Omega\times(0, T)$$

with initial conditions

$$(4.2)\quad u(0)=u_0\in H^2(\Omega)\cap H_0^1(\Omega), \ u_t(0)=u_1\in H_0^1(\Omega) \text{ in } \Omega,$$

and boundary conditions

$$(4.3)\qquad \begin{cases} u|_{\Gamma}=0 \\ \Delta u|_{\Gamma}=-\beta\frac{\partial}{\partial\nu}u_t+\tilde{f}\left(\frac{\partial}{\partial\nu}u(t,x)\right) \text{ on } \Gamma\times(0,T). \end{cases}$$

The nonlinear operator $F: H^2(\Omega)\to H^2(\Omega)$ is defined by

$$(4.4)\qquad \begin{cases} \Delta^2 F(u)=-[u,u] \text{ in } \Omega, \\ F=\frac{\partial F}{\partial\nu}=0 \text{ on } \Gamma; \end{cases}$$

with $[\Psi, \phi]\equiv\Psi_{xx}\phi_{yy}+\Psi_{yy}\phi_{xx}-2\Psi_{xy}\phi_{xy}$. Here $\tilde{f}\in C^1(R)$ is assumed to be polynomially bounded i.e.

$$|\tilde{f}'(s)|\le C[1+|s|^p] \text{ for } 0\le p<\infty: s\in R.$$

The constants β and γ are strictly positive.

REMARK 4.1. One could also consider the von Kārmān plate equation with boundary conditions different than in (4.3), (for instance clamped or hinged boundary conditions). Since the technicalities are similar to those in (4.3), we shall concentrate only on the latter. Also, one may consider a more general structure of the operator $\tilde{f}$, for instance, $\tilde{f}(u, \nabla u, u_t)$ subject to an analogous growth condition as above. Since this level of generality does not introduce new (conceptual) difficulties, for simplicity of exposition we take $\tilde{f}\left(\frac{\partial}{\partial\nu}u\right)$.

Von Kārmān plate equations have attracted considerable attention in the past. However, to the authors' best knowledge the results on well-posedness available in the literature for two dimensional problems deal with the case when the boundary conditions are *homogeneous*, i.e. the right hand side of (4.3) is equal to zero (see [7], [9],[32]). In fact, in [19], an existence and uniqueness result for the *homogeneous* (on the boundary) problem was established, by using Faedo-Gelerkin method. The problem becomes more difficult when the boundary conditions are nonlinear and nonmonotone (as they often arise in boundary stabilization problems, see [18]). In this case, the existing techniques (see [19]) are not applicable. The reason for this is that in order to "handle" the nonlinear term on the boundary, $\tilde{f}\left(\frac{\partial}{\partial\nu}u\right)$, the regularity of the solutions of the homogeneous von Kārmān plate is not sufficient (this precludes the use of standard perturbation or approximation techniques). On the other hand, as we shall see below, the results on well-posedness (and regularity) will follow from the abstract theory presented in section 2. To accomplish this, we need to put problem (4.1) into the abstract framework. We introduce the following spaces and operators

$$H=L_2(\Omega);\quad V=H_0^1(\Omega);\quad U=L_2(\Gamma).$$

$$A_D: L_2(\Omega)\supset\mathcal{D}(A_D)\to L_2(\Omega) \text{ defined by}$$
$$A_D u=-\Delta u\ ;\ u\in\mathcal{D}(A_D)\equiv H^2(\Omega)\cap H_0^1(\Omega),$$
$$D: L_2(\Gamma)\to L_2(\Omega) \text{ given by}: Dg=v \text{ iff } \Delta v=0 \text{ in } \Omega \text{ and } v|_\Gamma=g.$$

We set

$$\tilde{A}\equiv A_D^2 \text{ hence } \tilde{A}^{1/2}=A_D\ ;\quad \mathcal{D}(\tilde{A}^{1/2})=H^2(\Omega)\cap H_0^1(\Omega) \text{ and}$$
$$\|u\|=|A_D u|=|\Delta u|_{L_2(\Omega)},\ u\in\mathcal{D}(A_D),\ \tilde{M}\equiv(I+\gamma A_D), \text{ hence}$$
$$M\in\mathcal{L}(H_0^1(\Omega)\ ;\ H^{-1}(\Omega)) \text{ and } |u|_V^2=|\tilde{M}^{1/2}u|^2=((I-\gamma\Delta)u,u)=|u|_{L_2(\Omega)}^2+\gamma|\nabla u|_{L_2(\Omega)}^2.$$
$$G\equiv A_D^{-1}D, \text{ hence } \tilde{A}^{1/2}G=A_D A_D^{-1}D\in\mathcal{L}(U\ ;\ H).$$

From [28] we also have

$$G^*Au=D^*A_D u=-\frac{\partial}{\partial\nu}u|_\Gamma \text{ for } u\in\mathcal{D}(A_D). \tag{4.5}$$

$$\mathcal{F}(u)\equiv[F(u),\ u] \text{ where } \Delta^2F(u)=[-u,\ u] \text{ in } \Omega \tag{4.6}$$
$$F=0 \text{ in } \Gamma,\ \frac{\partial F}{\partial\nu}=0 \text{ on } \Gamma.$$

$$f(u)(t.x)\equiv\tilde{f}\left(\frac{\partial u}{\partial\nu}(t,x)\right). \tag{4.7}$$

Notice that by (4.5), (4.7)

$$\frac{\partial}{\partial\nu}u_t=-G^*Au_t \text{ and} \tag{4.8}$$

$$\tilde{f}\left(\frac{\partial}{\partial\nu}u\right)=\tilde{f}(-G^*Au)\equiv f(u) \tag{4.9}$$

With the above notation, it is known (see [6]) that the abstract form of equation (4.1) becomes precisely equation (2.1). Thus, in order to apply the results of Section 2, we need to verify hypotheses (1.2)-(1.6). Notice that hypotheses (1.2)-(1.4) follow directly from the definitions of the operators. As for hypothesis (1.5), we must show that

$$\mathcal{F}: H^2(\Omega)\cap H_0^1(\Omega)\to H^{-1}(\Omega) \tag{4.10}$$

is Fréchet differentiable. This will be done by using arguments similar to those in [19] or [18]. We first prove that the operator $\mathcal{F}$ is bounded. Let $u\in H^2(\Omega)\cap H_0^1(\Omega)$. Then

$$|[u,u]|_{L_1(\Omega)}\le C|u|_{H^2(\Omega)}^2. \tag{4.11}$$

Since $L_1(\Omega)\subset H^{-1-\epsilon}(\Omega)$, (see [1]) from elliptic regularity combined with explicit representations of fractional powers of elliptic operator (see [14]), we obtain that

$$|F(u)|_{H^{3-\epsilon}(\Omega)}\le C|u|_{H^2(\Omega)}^2, \text{ hence } |[F(u),\ u]|_{H^{-\epsilon}(\Omega)}\le C|u|_{H^2(\Omega)}^3 \text{ and, in particular} \tag{4.12}$$
$$|\mathcal{F}(u)|_{H^{-1}(\Omega)}\le C|u|_{H^2(\Omega)}^3 \tag{4.13}$$

which proves the boundedness of $\mathcal{F}$. To compute the Fréchet derivative of $\mathcal{F}$, we introduce the operator:

(4.14) $A_0 u \equiv \Delta^2 u$ in Ω; $u=\dfrac{\partial u}{\partial \nu}=0$ on Γ.

Then, $F(u)$ can be written explicitly in terms of the solution operator of (4.14) as

$$F(u)=-A_0^{-1}[u, u] \text{ and } \mathcal{F}(u)=-[A_0^{-1}[u, u], u].$$

It is now straightforward to verify that

(4.15) $D\mathcal{F}(u)h=-[A_0^{-1}[u, u], h]-2[A_0^{-1}[u, h], u]$.

By using the same arguments as above (i.e. (4.11)-(4.13)), one easily shows that

$$|D\mathcal{F}(u)h|_{H^{-1}(\Omega)} \leq C|u|^2_{H^2(\Omega)}|h|_{H^2(\Omega)}$$

as desired for (4.10). It remains to verify (1.6). From (4.8) and (4.9), $f(u)=\tilde{f}\left(\dfrac{\partial}{\partial \nu}u\right)$. Since $\dfrac{\partial}{\partial \nu} \in \mathcal{L}(H^2(\Omega); H^{1/2}(\Gamma))$, and (see [1])

(4.16) $H^{1/2}(\Gamma) \subset L_{2p+1}(\Gamma)$ for any $0<p<\infty$,

by using the well known result (see [2]) according which the substitution operator generated by functions with polynomally bounded derivatives is differentiable from $L_{2p+1}(\Gamma) \rightarrow L_2(\Gamma)$, we arrive at (1.6). Thus, we are in a position to apply Theorem 2.1, which specialized to our situation gives:

Theorem 4.1. (local existence). *For any* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, *there exists a unique solution* (u, u_t) *to* (4.1)-(4.4) *such that*

(4.17) $u \in C([0, T_0]; H^2(\Omega) \cap H_0^1(\Omega))$,

(4.18) $u_t \in C([0, T_0]; H_0^1(\Omega))$,

(4.19) $\dfrac{\partial}{\partial \nu} u_t \in L_2(0, T_0; L_2(\Gamma))$, for some $T_0>0$.

REMARK 4.2. Notice that the boundary regularity in (4.19) does not follow from the interior regularity in (4.17)-(4.18). It is an additional regularity result. □

We shall now turn to the question of global existence of the solutions to (4.1)-(4.4). At this point we need to assume some structural condition on the function $\tilde{f}$. We shall make the following hypothesis

(4.20) $\tilde{f}(s)s \leq 0$ for $s \in R$.

Theorem 4.2 (global existence). *Under the additional hypothesis* (4.20), *the solutions to* (4.1)-(4.3) *are global.*

Proof. It suffices to verify hypotheses (2.15), (2.16) and to apply the result of Theorem 2.4. To accomplish this, we first note that (see [16])

(4.21) $([\Psi, \phi], f)_{L_2(\Omega)}=([f, \phi], \Psi)_{L_2(\Omega)}=([\Psi, f], \phi)_{L_2(\Omega)}$

for all $\Psi, \phi, f\in H^2(\Omega)\cap H_0^1(\Omega)$. With $y, y_t\in C([0, T]; H^2(\Omega)\cap H_0^1(\Omega))$, using (4.21) and $\Delta^2 Fy=-[y, y]$ and $\frac{\partial F}{\partial\nu}=F=0$ on Γ

$$\int_\Omega \mathcal{F}(y)y_t d\Omega=\int_\Omega [F(y), y]y_t d\Omega=\int_\Omega [y, y_t]F(y)d\Omega=\frac{1}{2}\int_\Omega \frac{\partial}{dt}([y, y])F(y)d\Omega$$
$$=-\frac{1}{2}\int_\Omega \frac{d}{dt}\Delta^2 F(y)F(y)d\Omega=-\frac{1}{4}\frac{d}{dt}|\Delta F(y)|^2_{L_2(\Omega)}.$$

Hence, by (4.12)

(4.22) $$\int_0^t (\mathcal{F}(y), y_t)_{L_2(\Omega)}d\tau=-\frac{1}{4}|\Delta F(y(t))|^2_{L_2(\Omega)}+\frac{1}{4}|\Delta F(y(0))|^2_{L_2(\Omega)}\le C|y(0)|^4_{H^2(\Omega)}.$$

From (4.13) it follows that

(4.23) $|(\mathcal{F}(y), v)_{L_2(\Omega)}|\le C|y|^3_{H^2(\Omega)}|v|_{H_0^1(\Omega)}$.

The inequality in (4.22) can be extended to all $y\in C([0, T]; H^2(\Omega)\cap H_0^1(\Omega))$ and $y_t\in C([0, T]; H_0^1(\Omega))$; this proves (2.15) with $C_1\equiv 0$. As for (2.16), we write with $\tilde{f}_1(s)\equiv\int_0^s \tilde{f}(\tau)d\tau\le 0$:

$$-\int_0^t \left\langle f(y(\tau)), \frac{\partial}{\partial\nu}y_t\right\rangle_{L_2(\Gamma)}d\tau\equiv-\int_0^t \left\langle \tilde{f}\left(-\frac{\partial}{\partial\nu}y\right), \frac{\partial}{\partial\nu}y_t\right\rangle_{L_2(\Gamma)}d\tau$$
$$=\int_0^t \frac{d}{d\tau}\int_\Gamma \tilde{f}_1\left(-\frac{\partial}{\partial\nu}y\right)d\Gamma d\tau=\int_\Gamma \tilde{f}_1\left(-\frac{\partial}{\partial\nu}y(t)\right)d\Gamma-\int_\Gamma \tilde{f}_1\left(-\frac{\partial}{\partial\nu}y(0)\right)d\Gamma$$

by (4.20), (4.16) and the Trace Theorem

$$\le C\left|\frac{\partial}{\partial\nu}y(0)\right|^{\frac{1}{p+1}}_{L_{p+1}(\Gamma)}\le C\left|\frac{\partial}{\partial\nu}y(0)\right|^{\frac{1}{p+1}}_{H^{\frac{1}{2}}(\Gamma)}\le C(|y(0)|_{H^2(\Omega)})$$

which proves the desired inequality in (2.16). ■

We finally turn to the question of the regularity of solutions to (4.1)-(4.3). To simplify the exposition, we shall assume $\tilde{f}=0$ (this restriction is, of course, not essential at the *regularity* level).

Theorem 4.3 (regularity). *Assume that* $\tilde{f}=0$ *and that the initial data* u_0, u_1 *satisfy*

(4.24) $u_1\in H^2(\Omega)\cap H_0^1(\Omega),\ u_0\in H^3(\Omega)\cap H_0^1(\Omega),$

(4.25) $\Delta u_0|_{\Gamma}=-\beta\frac{\partial}{\partial\nu}u_1.$

Then, the global solution (u, u_t) *to* (4.1)-(4.3) *guaranteed by* Theorem 4.2 *enjoys the following regularity properties*:

$u\in C([0, T]; H^3(\Omega)\cap H_0^1(\Omega))$; $u_t\in C([0, T]; H^2(\Omega)\cap H_0^1(\Omega))$; $u_{tt}\in C([0, T]; H_0^1(\Omega))$.

Proof. It suffices to verify hypotheses (2.5)-(2.7) and to apply Theorem 2.2. Hypothesis (2.5) is satisfied by virtue of (4.24). As for (2.6), we note that in our case this is equivalent to

(4.26) $$A\left(u_0-\beta A_D^{-1}D\frac{\partial}{\partial\nu}u_1\right)\in H^{-1}(\Omega),$$

or in PDE form to:

$$\begin{cases}\Delta^2u_0\in H^{-1}(\Omega);\\ u_0|_{\Gamma}=0;\\ \Delta u_0|_{\Gamma}=-\beta\frac{\partial}{\partial\nu}u_1.\end{cases}$$

Thus, if $u_0\in H^3(\Omega)\cap H_0^1(\Omega)$, $u_1\in H^2(\Omega)$ and (4.25) holds, then u_0, u_1 comply with (4.26), and hence with (2.6). Finally, hypothesis (2.7) follows from the following estimates. For any $u\in H^2(\Omega)\cap H_0^1(\Omega)$, $h\in H_0^1(\Omega)$, and $\phi\in\mathcal{D}(\tilde{A})\subset H^4(\Omega)$, since $F(u)\in H^{3-\varepsilon}(\Omega)\cap H_0^2(\Omega)$ we have

$$\begin{aligned}|([A_0^{-1}[u, u], h],\phi)_{L_2(\Omega)}|&=|(F(u), \phi], h)_{L_2(\Omega)}|\le|h|_{H_0^1(\Omega)}|[F(u), \phi]|_{H^{-1}(\Omega)}\\ &\le C|h|_{H_0^1(\Omega)}|F(u)|_{H^{3-\varepsilon}(\Omega)}|\phi|_{H^2(\Omega)}\end{aligned}$$

and by (4.12)

(4.27) $$\le C|h|_{H_0^1(\Omega)}|u|^2_{H^2(\Omega)}|\phi|_{H^2(\Omega)}.$$

Similarly, by [14]

(4.28) $$\begin{aligned}&|([A_0^{-1}[u, h], u],\phi)_{L_2(\Omega)}|=|([u, \phi], A_0^{-1}[u, h])_{L_2(\Omega)}|\\ &\le|[u, \phi]|_{H^{-1-\varepsilon}(\Omega)}|A_0^{-1}[u, h]|_{H_0^{1+\varepsilon}(\Omega)}\le|u|_{H^2(\Omega)}|A_0^{-3/4+\varepsilon/4}[u, h]|_{L_2(\Omega)}|\phi|_{H^2(\Omega)}.\end{aligned}$$

On the other hand, we have

$$|([u, h], \Psi)_{L_2(\Omega)}|=|([u, \Psi], h)_{L_2(\Omega)}|\le|\Psi|_{H^{2+\varepsilon}(\Omega)}|u|_{H^2(\Omega)}|h|_{H_0^1(\Omega)}.$$

Hence, using again [14],

(4.29) $$|A_0^{-1/2-\varepsilon/4}[u,h]|_{L_2(\Omega)}\le C|u|_{H^2(\Omega)}|h|_{H_0^1(\Omega)}.$$

Combining (4.28) with (4.29) yields

(4.30) $$|([A_0^{-1}[u,h],u],\phi)_{L_2(\Omega)}|\le C|u|^2_{H^2(\Omega)}|h|_{H_0^1(\Omega)}|\phi|_{H^2(\Omega)}.$$

The estimate in part (i) of (2.7) follows now from (4.15), (4.27) and (4.30). As for part (ii), this is a consequence of (4.9) and Sobolev's imbedding $H^{1/2}(\Gamma)\subset L_p(\Gamma)$. Thus, we are in a position to apply Theorem 2.2 which yields

(4.31) $u_t \in C([0, T]; H^2(\Omega)\cap H_0^1(\Omega), u_{tt} \in C([0, T]; H_0^1(\Omega)),$

and

(4.32) $A\left(u-\beta A_D^{-1}D\frac{\partial}{\partial\nu}u_t\right)\in C([0, T]; H^{-1}(\Omega)),$

which in PDE version is equivalent to

(4.33) $$\begin{cases}\Delta^2 u\in C([0, T]; H^{-1}(\Omega)),\\ u|_\Gamma=0,\\ \Delta u=-\beta\frac{\partial}{\partial\nu}u_t\in C([0, T]; H^{1/2}(\Gamma)).\end{cases}$$

Using standard elliptic estimates [25], we obtain from (4.33) that

(4.34) $u\in C([0, T]; H^3(\Omega)),$

which together with (4.31) completes the proof of Theorem 4.3. ■

Our final result shows that if we assume more smoothness on the initial data, the solutions to (4.1) are classical. Indeed,

Theorem 4.4. (regularity revisited-classical solutions). *In addition to the assumptions of* Theorem 4.3, *we assume that*

(4.35) $u_0\in H^4(\Omega), u_1\in H^3(\Omega)\cap H_0^1(\Omega),$

(4.36) $\Delta u_1|_\Gamma=\beta\frac{\partial}{\partial\nu}\bar{M}^{-1}[\Delta^2 u_0-\mathcal{F}(u_0)]$ on Γ.

Then,

(4.37) $u_{tt}\in C([0, T]; H^2(\Omega)),$

(4.38) $u_{ttt}\in C([0, T]; H_0^1(\Omega)),$

(4.39) $u\in C([0, T]; H^4(\Omega))$

Proof. It suffices to verify conditions (2.9)-(2.11) and to apply the result of Theorem 2.3. Since $\widetilde{M}=I+\gamma\widetilde{A}^{1/2}$, (2.9) is trivially satisfied. Conditions (4.35), (4.36) imposed on the initial data u_0, u_1 imply (after straightforward verification) that (2.10)-(2.11) hold true. Thus, to apply the result of Theorem 2.3, we need to verify that $\mathcal{F}$ is twice differentiable: $H^2(\Omega)\cap H_0^1(\Omega)\to H^{-1}(\Omega)$. Indeed, straightforward computations yield

(4.40) $D^2\mathcal{F}(u)(h, v)=[DF(u)h, v]+[D^2F(u)(h, v), u]+[h, DF(u)v]$

where

$$(4.41)\quad DF(u)h=-2A_0^{-1}[u,\ h]$$
$$(4.42)\quad D^2F(u)(h,\ v)=-2A_0^{-1}[v,\ h].$$

Since $u,h,v\in H^2(\Omega)\cap H_0^1(\Omega)$, by elliptic regularity

$$(4.43)\quad |A_0^{-1}[u,\ h]|_{H^{3-\epsilon}(\Omega)}\leq C|[u.\ h]|_{H^{-1-\epsilon}(\Omega)}\leq C|u|_{H^2(\Omega)}|h|_{H^2(\Omega)}.$$

Hence (see the estimate below (4.28))

$$(4.44)\quad |[DF(u)h,\ v]|_{H^{-1}(\Omega)}\leq C|v|_{H^2(\Omega)}|DF(u)h|_{H^{2+\epsilon}(\Omega)}\leq C|v|_{H^2(\Omega)}|u|_{H^2(\Omega)}|h|_{H^2(\Omega)}$$

where we have used (4.41), (4.13) and $\varepsilon<\frac{1}{2}$. Similarly, by using again (4.43)

$$(4.45)\quad |D^2F(u)(h,\ v)|_{H^{3-\epsilon}(\Omega)}\leq C|h|_{H^2(\Omega)}|v|_{H^2(\Omega)}$$
$$(4.46)\quad |[D^2F(u)(u,\ v),u]|_{H^{-1}(\Omega)}\leq C|u|_{H^2(\Omega)}|h|_{H^2(\Omega)}|v|_{H^2(\Omega)}.$$

Combining (4.40) with (4.44) and (4.46), we conclude that $\mathcal{F}$ is twice Frechet differentiable. This allows us to use the result of Theorem 2.3, which yields that

$$(4.47)\quad u_{ttt}\in C([0,\ T];\ H_0^1(\Omega)),$$
$$(4.48)\quad u_{tt}\in C([0,\ T];\ H^2(\Omega)),$$
$$(4.49)\quad \tilde{A}\Big(u-\beta A_D^{-1}D\frac{\partial}{\partial\nu}u_t\Big)-\mathcal{F}(u)\in C([0,\ T];\ L_2(\Omega)).$$

Since $\mathcal{F}(u)=[u,\ F(u)]$ is bounded from $H^3(\Omega)\rightarrow L_2(\Omega)$, invoking (4.34) we infer that

$$\tilde{A}\Big(u-\beta A_D^{-1}D\frac{\partial}{\partial\nu}u_t\Big)\in C([0,\ T];\ L_2(\Omega)),$$

which, in turn, is equivalent to

$$(4.50)\quad \begin{cases}\Delta^2u\in C([0,\ T];\ L_2(\Omega)),\\ u|_\Gamma=0,\\ \Delta u=-\beta\frac{\partial}{\partial\nu}u_t.\end{cases}$$

From (2.14) and $|D\mathcal{F}(u)u_t|_{H^{-1}(\Omega)}\leq C(|u|_{H^2(\Omega)})|u_t|_{H^2(\Omega)}$ (see the estimate below (4.15)), we obtain that

$$(4.51)\quad \tilde{A}\Big(u_t-\beta A_D^{-1}G\frac{\partial}{\partial\nu}u_{tt}\Big)\in C([0,\ T];\ H^{-1}(\Omega)).$$

Hence

$$(4.52)\quad \begin{cases} \Delta^2 u_t \in C([0, T]; H^{-1}(\Omega)), \\ u_t|_\Gamma = 0, \\ \Delta u_t = -\beta \frac{\partial}{\partial \nu} u_{tt} \text{ on } \Gamma. \end{cases}$$

From (4.48) and the Trace Theorem, we have that

$$(4.53)\quad \frac{\partial}{\partial \nu} u_{tt} \in C([0, T]; H^{1/2}(\Gamma)).$$

Elliptic theory applied to (4.52) provides now

$$(4.54)\quad u_t \in C([0, T]; H^3(\Omega)).$$

Hence

$$(4.55)\quad \frac{\partial}{\partial \nu} u_t \in C([0, T]; H^{3/2}(\Gamma)).$$

Combining (4.55) with (4.50) and using again elliptic theory, we conclude that $u \in C([0, T]; H^4(\Omega))$ as desired for (4.39) ■

4.2. Von Kármán plate model with $\gamma=0$

Let Ω be a bounded domain with C^∞ boundary. We consider the equation

$$(4.56)\quad u_{tt} + \Delta^2 u = [F(u), u], \text{ in } \Omega \times (0, T),$$

with initial conditions

$$(4.57)\quad u(0) = u_0 \in H^2(\Omega); \; u_t(0) = u_1 \in L_2(\Omega),$$

boundary conditions as in (4.3). The Airy's stress function $F(u)$ satisfies equation (4.4).

To put this problem into the abstract framework, we set

$$H \equiv V = L_2(\Omega); \; U = L_2(\Gamma); \; M = I.$$

The operators A, G, f, and are $\mathcal{F}$ the same as in subsection 4.1. Thus, the arguments of subsection 4.1 apply and the hypotheses (1.2), (1.4) and (1.6) are satisfied. We need to verify (1.3) and (1.5). Since $V = H = V'$ and $M = I$, (1.3) holds trivially. As to (1.5), this requires a more delicate argument (notice that we do not have any longer the smoothing effect of the operator M^{-1}, which was essential in the previous case when $\gamma=0$). Indeed, the following "sharp" regularity result for Airy's stress function established in [12] is critical. Assume that the boundary Γ is C^∞. Then, the operator A_0 introduced in (4.14) satisfies

$$(4.58)\quad |A_0^{-1}[u, v]|_{W_\infty^2(\Omega)} \leq C|u|_{H^2(\Omega)}|v|_{H^2(\Omega)}.$$

REMARK 4.2. Notice that the regularity ih (4.58) improves by "ε" a known result on regularity of Airy's stress function which states that

$$|A_0^{-1}[u, v]|_{W_\infty^2(\Omega)} \le C|u|_{H^{2+\varepsilon}(\Omega)}|v|_{H^2(\Omega)}.$$

This gain of "ε" in (4.58) is critical. Indeed, from (4.58) and formula (4.15), we infer that

$$(4.59) \quad |D\mathcal{F}(u)h|_{L_2(\Omega)} \le C|u|^2_{H^2(\Omega)}|h|_{H^2(\Omega)}$$

which proves (1.5). Thus we are in a position to apply Theorems 2.1, 2.4, which lead to the following result.

Theorem 4.5 (local existence and uniqueness). *For any* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in L_2(\Omega)$, *there exists a unique solution* (u, u_t) *to* (4.56), (4.57) *such that*

$$(4.60) \quad u \in C([0, T_0]; H^2(\Omega) \cap H_0^1(\Omega)),$$
$$(4.61) \quad u_t \in C([0, T]; L_2(\Omega)),$$
$$(4.62) \quad \frac{\partial}{\partial \nu} u_t \in L_2(0, T_0; L_2(\Omega)) \text{ for some } T_0 > 0.$$

REMARK 4.3. The statement of uniqueness in Theorems 4.5 is new even in the case of homogeneous boundary conditions. Indeed, as pointed out in [18], [19] (see also [33]), the question of uniqueness of weak solutions to the von Kármán system (4.56) has been an open problem in the literature. □

We turn next to regularity of the solutions. By using the result in (4.58), we can show that condition (2.7) of Theorem 2.2 holds true. Indeed, this follows from the following estimates.
For every $u \in H^2(\Omega)$, $\Phi \in H^2(\Omega)$, $h \in L_2(\Omega)$ we have:

$$|([A_0^{-1}[u, u], h], \Phi)_{L_2(\Omega)}| = |([A_0^{-1}[u, u], \Phi], h)_{L_2(\Omega)}| \le C|A_0^{-1}[u, u]|_{W_\infty^2(\Omega)}|\Phi|_{H^2(\Omega)}|h|_{L_2(\Omega)}$$
(by (4.58))
$$(4.63) \quad \le C|u|_{H^2(\Omega)}|\Phi|_{H^2(\Omega)}|h|_{L_2(\Omega)},$$

and with $0 < \varepsilon < \frac{1}{2}$

$$(4.64) \quad \begin{aligned} |([A_0^{-1}[u, h], u], \Phi)_{L_2(\Omega)}| &= |([u, \Phi], A_0^{-1}[u, h])_{L_2(\Omega)}| \\ &\le |[u, \Phi]|_{H^{-1-\varepsilon}(\Omega)}|A_0^{-1}[u,h]|_{H_0^{1+\varepsilon}(\Omega)} \le C|u|_{H^2(\Omega)}|\Phi|_{H^2(\Omega)}|[u, h]|_{H^{-2-\varepsilon}(\Omega)} \\ &\le C|u|^2_{H^2(\Omega)}|\Phi|_{H^2(\Omega)}|h|_{L_2(\Omega)} \end{aligned}$$

where we have used
$$|([u,h], \Psi)_{L_2(\Omega)}| = |([u, \Psi], h)_{L_2(\Omega)}| \le C|u|_{H^2(\Omega)}|\Psi|_{H_0^{2+\varepsilon}(\Omega)}|h|_{L_2(\Omega)}, \text{ for all } \Psi \in H_0^{2+\varepsilon}(\Omega).$$

Combining (4.15) with (4.63) and (4.64) yields part (i) in (2.7). Thus, the conclusion of Theorem 2.2 applies and supported with arguments similar to those of

Theorem 4.3 gives

Theorem 4.6 (regularity). *Let the initial data* u_0, u_1 *satisfy*:

$$u_0 \in H^4(\Omega) \cap H_0^1(\Omega)\,;\; u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$$

subject to the compatibility conditions $\Delta u_0|_\Gamma = -\beta \frac{\partial}{\partial \nu} u_1 + \tilde{f}\left(\frac{\partial}{\partial \nu} u_0\right)$. *Then, a solution* u *of* Theorem 4.5 *is a strong solution i.e.*: $u \in C([0, T]\,;\, H^3(\Omega))$, $u_t \in C([0, T_0]\,;\, H^2(\Omega))$, $u_{tt} \in C([0, T]\,;\, L_2(\Omega))$.

REMARK 4.4. Notice that due to the presence of boundary terms in the equation, in general, we do not obtain $u(t) \in H^4(\Omega)$. This is in contrast with regularity results available for von Karman model with *homogeneous* boundary conditions (see[33], [35]). □

Equipped with the regularity result of Theorem 4.6, we are ready to establish, subject to the structural hypothesis (4.20), global existence of solutions to (4.56), (4.57).

Theorem 4.7 (global existence). *Under the additional hypothesis* (4.20), *local solutions of* Theorems 4.5 *and* 4.6 *become global i.e.*: *they are defined for all* $T_0 > 0$.

Proof. it suffices to establish the following a-priori bound

$$|u(t)|_{H^2(\Omega)} + |u_t(t)|_{L_2(\Omega)} \le C(|u_0|_{H^2(\Omega)}, |u_1|_{L_2(\Omega)}). \tag{4.65}$$

We notice first that the computations of the proof of Theorem 4.2, which give an a-priori bound in (4.65), can be justified properly for strong solutions. Thus, in the case of strong solutions, an a-priori bound in (4.65) holds true. We need to justify this inequality for all weak solutions. Let u and v be any two strong solutions corresponding to initial data (u_0, u_1) and (v_0, v_1) respectively. By using the inequalities in (3.30), (3.31) together with (4.59), we easily obtain

$$\begin{aligned}
&|u(t) - v(t)|_{H^2(\Omega)} + |u_t(t) - v_t(t)|_{L_2(\Omega)} \\
&\quad \le \sup_{0 \le \tau \le t} C(|u(\tau)|_{H^2(\Omega)}, |v(\tau)|_{H^2(\Omega)}) \int_0^t |u(\tau) - v(\tau)|_{H^2(\Omega)} d\tau \\
&\quad + C[|u_0 - v_0|_{H^2(\Omega)} + |u_1 - v_1|_{L_2(\Omega)}].
\end{aligned}$$

From (4.65) and the Gronwall's inequality

$$\begin{aligned}
&|u(t) - v(t)|_{H^2(\Omega)} + |u_t(t) - v_t(t)|_{L_2(\Omega)} \\
&\quad \le C_T(|u_0|_{H^2(\Omega)}, |v_0|_{H^2(\Omega)}, |u_1|_{L_2(\Omega)}, |v_1|_{L_2(\Omega)})[|u_0 - v_0|_{H^2(\Omega)} + |u_1 - v_1|_{L_2(\Omega)}].
\end{aligned}$$

Since the above inequality (satisfied for *strong* solutions) is stable for all weak solutions, we conclude, by standard density argument, that the a-priori bound in (4.65) holds true for weak solutions as well. ■

REMARK 4.5. Notice that Theorem 4.7 provides us with global existence and *uniqueness* of weak solutions, as well as with global existence of strong (regular) solutions to the von Kārmān model (4.56), (4.57), subject to the nonlinear boundary conditions (4.3). This is a new result even in the context of *homogeneous* boundary conditions. Indeed, uniqueness of weak solutions to the von Kārmān plate equation (4.56) has been an open problem in the literature (see [19], [18]). Global existence of regular (classical) solutions has been known in the case of *homogeneous boundary condition only* (see [35], [8], [17]). Thus, Theorem 4.7 extends these regularity results to the case of *nonhomogeneous* and *nonlinear* boundary conditions as treated in (4.3). Moreover, our techniques/methods of proofs appear considerably simpler when compared with the ones employed in the above references (where either complicated nonlinear interpolation arguments were used [35], or lengthy computations leading to the a-priori bounds in higher norms were necessary [8], [17]).

4.3. Nonlinear Euler-Bernoulli plate model

Here, we shall consider a more general equation than (1.7)

$$u_{tt}(t)+\Delta^2 u(t)=\mathcal{F}(u(t)) \text{ in } \Omega\times(0,\ T) \tag{4.66}$$

$$\begin{cases} u|_{\Gamma}=0 \\ \Delta u=-\dfrac{\partial}{\partial\nu}u_t+\tilde{f}\left(\dfrac{\partial}{\partial\nu}u(\cdot)\right) \text{ on } \Gamma\times(0,\ T) \end{cases} \tag{4.67}$$

$$\begin{cases} u(0)=u_0\in H_0^1(\Omega)\cap H^2(\Omega) \\ u_t(0)=u_1\in L_2(\Omega) \end{cases} \tag{4.68}$$

under the assumptions

(4.69) $\mathcal{F}: H^2(\Omega)\to L_2(\Omega)$ is Frěchet differentiable

(4.70) $\tilde{f}\in C^1(R)$ and $|\tilde{f}'(s)|\leq C[1+|s|^p]$ for some $0\leq p<\infty$.

Notice that the nonlinear term in equation (1.7) satisfies assumption (4.69). Indeed, the operator

$$\mathcal{F}(u)(x)\equiv g\left(\int_\Omega|\nabla u|^2 d\Omega\right)\Delta u(x) \tag{4.71}$$

is Frechet differentiable: $H^2(\Omega)\to L_2(\Omega)$. To put problem (4.66)-(4.68) into the abstract framework, we set:

$$H=V=V'=L_2(\Omega);\ U=L_2(\Gamma);$$
$$M\equiv I;\ \tilde{A}\equiv A_D^2;\ G=A_D^{-1}D;$$

where both A_D and D are the same as in section 4.1 and

$$f(u) \equiv \tilde{f}\left(\frac{\partial}{\partial \nu} u\right) = \tilde{f}(-G^* A u).$$

We already know, from section 4.1, that $f: H^2(\Omega) \supset \mathcal{D}(\tilde{A}^{1/2}) \to L_2(\Gamma) = U$ is Fréchet differentiable. This combined with (4.53) allows us to apply the result of Theorem 2.1 which yields in our case

Theorem 4.8. *Assume* (4.69), (4.70). *Then, for any* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in L_2(\Omega)$, *there exist* $T_0 > 0$ *and a unique solution* (u, u_t) *to* (4.66)-(4.68) *such that*

(4.72) $u \in C([0, T_0]; H^2(\Omega))$, $u_t \in C([0, T_0]; L_2(\Omega))$;

(4.73) $\frac{\partial}{\partial \nu} u_t \in L_2(0, T_0; L_2(\Gamma))$.

In order to obtain global solutions to (4.66), we assume that the nonlinear operator $\mathcal{F}$ has a structure as in (4.71).

Theorem 4.9. *Assume* (4.20), (4.70) *and* (4.71). *Then, weak solutions to* (4.66)-(4.68) *are global on* $[0, T]$ *where* $T > 0$ *is arbitrary.*

Proof. By Theorem 2.4, it suffices to verify hypotheses (2.15) and (2.16). Validity of (2.16), under the structural assumption (4.20), has been verified in sect. 4.1. To assert (2.15) with $\mathcal{F}$ as in (4.6), we perform the following computations

$$\int_0^t \int_\Omega \mathcal{F}(u(s)) u_t(s) d\Omega ds = \int_0^t g\left(\int_\Omega |\nabla u(s)|^2 d\Omega\right) \int_\Omega \Delta u(s) u_t(s) d\Omega ds$$
$$= -\frac{1}{2} \int_0^t g\left(\int_\Omega |\nabla u(s)|^2 d\Omega\right) \frac{d}{ds} \int_\Omega |\nabla u(s)|^2 d\Omega ds$$

(where we have used the boundary condition $u|_\Gamma = 0$)

$$= -\frac{1}{2} \int_0^t \frac{d}{ds} g_1\left(\int_\Omega |\nabla u(s)|^2 d\Omega\right) ds \qquad \left(\text{with } g_1(t) \equiv \int_0^t g(s) ds\right)$$
$$= -\frac{1}{2} g_1\left(\int_\Omega |\nabla u(t)|^2 d\Omega\right) + \frac{1}{2} g_1\left(\int_\Omega |\nabla u_0|^2 d\Omega\right)$$
$$\leq \frac{1}{2} g_1\left(\int_\Omega |\nabla u_0|^2 d\Omega\right) \leq C(|u_0|_{H^2(\Omega)})$$

as desired for (2.15). ∎

In a similar manner as in section 4.1, one could study questions related to the regularity of local/global solutions. Since the analysis here is very much like before, this topic will not be pursued.

4.4. Parallely connected plates

We consider (1.14)-(1.16) under the following assumptions imposed on the nonlinear functions f_1 and f_2.

(4.74) $f_i(s_1, s_2) \in C^1(R^2), \qquad i=1, 2.$

(4.75) $\left|\frac{\partial f_i}{\partial s_1}(s_1, s_2)\right| \leq C(s_2)[|1+s_1|^p], \qquad i=1, 2.$

where $C(s_2)$ is continuous in $s_2 \in R$.
We set $u \equiv (y, w)$, and

$$H \equiv L_2(\Omega) \times L_2(\Omega) = V = V'; \; U \equiv L_2(\Gamma) \times L_2(\Gamma); \; M = I.$$
$$\tilde{A} = \begin{bmatrix} A_D^2 & 0 \\ 0 & A_D^2 \end{bmatrix}. \quad G = \begin{bmatrix} A_D^{-1}D & 0 \\ 0 & A_D^{-1}D \end{bmatrix}$$

The operator $\mathcal{F}$ is the same as in (4.6). We easily verify (in a manner similar to that in section 4.1, 4.3 that all hypotheses of Theorem 2.1 are verified, hence

Theorem 4.10. *Assume* (4.74), (4.75). *Then for any*

$$y_0, \; w_0 \in H^2(\Omega) \cap H_0^1(\Omega); \; y_1, \; w_1 \in L_2(\Omega),$$

there exists $T_0 > 0$ *such that there exists a unique weak solution* (y, y_t), (w, w_t) *to* (1.14)-(1.16) *such that*

(4.76) $y, \; w \in C([0, T]; H^2(\Omega)); \; y_t, \; w_t \in C([0, T_0]; L_2(\Omega))$

(4.77) $\frac{\partial}{\partial \nu} y_t, \; \frac{\partial}{\partial \nu} w_t \in L_2(0, T_0; L_2(\Gamma)).$

Under suitable structural assumptions imposed on the functions f_i, one obtains global solutions. For instance, it is enough to consider

$$f_1\left(\frac{\partial}{\partial \nu}(y-w)\right) \text{ and } f_2\left(\frac{\partial}{\partial \nu}(y-w)\right)$$

where $f_i(s)s \leq 0$ for $s \in R$.

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