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ON A THEOREM OF ZARISKI - VAN KAMPEN TYPE AND ITS APPLICATIONS

TAKANORI MATSUNO

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1. Introduction. Zariski constructed a method to calculate $\pi_1(\boldsymbol{P}^2-C)$, where $\boldsymbol{P}^2$ is the complex projective plane and C is a curve on it. In this paper, following the ideas of Zariski [5] and Van Kampen [4], we give a method to calculate $\pi_1(E-S)$, where E is a holomorphic line bundle over a complex manifold M and S is a hypersurface of E under certain conditions. Applying our method and the Reidemeister-Schreier method (see Rolfsen [3]), we can calculate the fundamental groups of regular loci of certain normal complex spaces. We give a few concrete examples in the final section.

This paper is a revised version of the author's master thesis [1]. The author would like to express his thanks to Professor M. Namba for his useful suggestions and encouragements and to Professor M. Sakuma whose suggestions about Lemma 1 (see section 2) was a great help to prove Main Theorem. He also expresses his thanks to the referee for useful comments.

2. Statement of Main Theorem. Let M be a connected -dimensional complex manifold and $\mu: E\to M$ be a holomorphic line bundle over M and S be a hypersurface of E. We assume that E and S satisfy the following conditions:

(1) $\mu: S\to M$ is a finite proper holomorphic map, where μ' is the ristriction of μ to $S(\mu'=\mu|s)$.

(2) There is a hypersurfase B of M such that $\mu'|_{S-\mu^{-1}(B)}: S-\mu^{-1}(B)\to M-B$ is an unbranched covering of degree .

(3) $(\mathrm{d}\mu')_p: T(S-\mu^{-1}(B))_p\to T(M-B)_{\mu'(p)}$ is isomorphic for every point $p\in S-\mu^{-1}(B)$.

Then we have a following lemma whose proof is given in section 4.

Lemma 1. $\mu|_{E-S-\mu^{-1}(B)}: E-S-\mu^{-1}(B)$ *is a continuous fiber bundle.*

We denote a standard fiber od $\mu: E\to M$ by $\hat{F}$ and that of $\mu|_{E-S-\mu^{-1}(B)}: E-S-\mu^{-1}(B)\to M-B$ by F. We assume that there is a continuous section $\xi: M\to E$ of $\mu: E\to M$ such that $\xi(M)\cap S=\phi$ (see Figure 1).

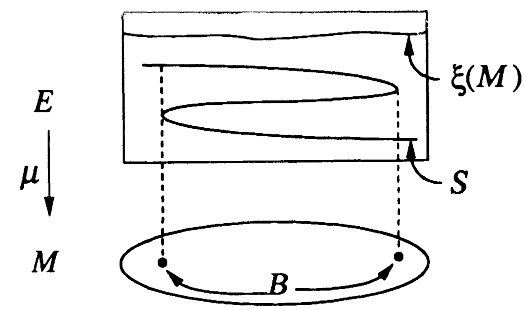

Figure 1

REMARK. Such a continuous sectinuous does not always exist. For example, if E is a negative line bundle and S is the image of the zero section, then there exists no such a continuous section.

In order to describe Main Theorem, we must prepare some more symbols. We choose $F\cap\xi(*)$ as a base point b_0 and we omit the base point hereafter. Since F can be identified with $C-\{n \text{ points}\}$, $\pi_1(F)$ is isomorphic to the n-th free group $F_n=\langle\gamma_1, \cdots, \gamma_n\rangle$ (see Figure 2).

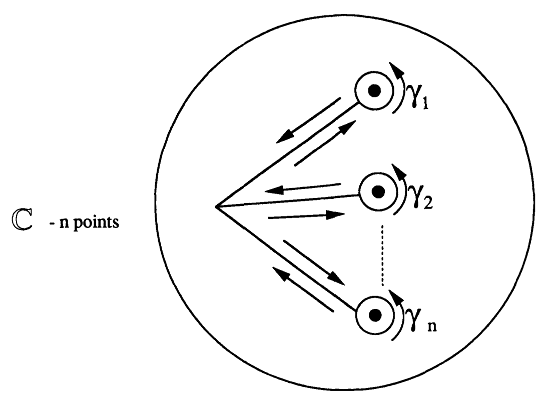

Figure 2

Let $\widehat{Q}$ be the kernel of the surjective homomorphism

$$j_*: \pi_1(M-B)\to\pi_1(M),$$

induced from the injection $i: M-B\hookrightarrow M$. We assume that $\widehat{Q}$ has a finite presentation as follows:

$$\widehat{Q}=\langle\beta_1, \cdots, \beta_t|\square=1, \cdots, \square=1 \text{ (some relations) }\rangle.$$

Let $\theta: \pi_1(M-B) \rightarrow B_n$ be the braid monodromy representation of the continuous fiber bundle $\mu|_{E-S-\mu^{-1}(B)}$ in Lemma 1, where B_n is the n-th braid group:

$$B_n = \langle \sigma_1, \cdots, \sigma_{n-1} | [\sigma_i, \sigma_j] = 1(|i-j| \geq 2),\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} (i=1, \cdots, n-2) \rangle.$$

We define a homomorphism $\varphi: B_n \rightarrow Aut(\pi_1(F))$ as follows:

$$\begin{cases} \varphi(\sigma_j)(\gamma_j) = \gamma_j^{-1} \gamma_{j+1} \gamma_j \\ \varphi(\sigma_j)(\gamma_{j+1}) = \gamma_j \\ \varphi(\sigma_j)(\gamma_k) = \gamma_k (\text{if } k \neq j, j+1). \end{cases}$$

Then we have the following theorem of Zariski-Van Kampen type:

Main Theorem. *If there is a continuous section ξ of $\mu: E \rightarrow M$ such that $\xi(M) \cap S = \phi$, then*

$$\pi_1(E-S) \cong \langle \gamma_1, \cdots, \gamma_n | \gamma_j = \varphi(\theta(\beta_k))(\gamma_j)(1 \leq j \leq n, 1 \leq k \leq t) \rangle \rtimes \pi_1(M).$$

(a semi-direct product)

Here $\beta_1, \cdots, \beta_k$ generate the kernel of the homomorphism $j_*: \pi_1(M-B) \rightarrow \pi_1(M)$ and $\gamma_1, \cdots, \gamma_n$ generate the image of the homomorphism $i_*: \pi_1(F) \rightarrow \pi_1(E-S-\mu^{-1}(B))$, where i_* is induced from injection $i: F \hookrightarrow E-Q-\mu^{-1}(B)$.

REMARK. In Main Theorem, the relations $\gamma_j = \varphi(\theta(\beta_k))(\gamma_j)$ are same as the usual monodromy relations, so it is not essential to facter the homomorphism

$$\pi_1(M-B) \longrightarrow Aut(\pi_1(F))$$

through the braid group.

Corollary. *Under the same assumptions in Main Theorem, assume moreover that M is simply connected $(i.e. \pi_1(M) = \{1\})$, then*

$$\pi_1(E-S) \cong \langle \gamma_1, \cdots, \gamma_n | \gamma_j = \varphi(\theta(\beta_k))(\gamma_j)(1 \leq j \leq n, 1 \leq k \leq t) \rangle.$$

3. Proof of Main Theorem. Since $\mu: E-S-\mu^{-1}(B) \rightarrow M-B$ is a continuous fiber bundle, there is the following exact sequence:

$$\begin{aligned} \cdots \longrightarrow \pi_2(F) \xrightarrow{i_*} \pi_2(E-S-\mu^{-1}(B)) \underset{\xi_*}{\overset{\mu_*}{\rightleftarrows}} \pi_2(M-B) \xrightarrow{\triangle} \\ \rightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(E-S-\mu^{-1}(B)) \xrightarrow{\mu_*} \pi_1(M-B) \rightarrow \\ \rightarrow \pi_0(F) \rightarrow \cdots \text{ (exact)}, \end{aligned}$$

where μ_* and ξ_* are the homomorphisms induced by μ and ξ respectively.

$\mu_* \circ \xi_* = id|_{\pi_2(M-B)}$, since $\mu \circ \xi = id|_{M-B}$. Therefore we have $\triangle = \triangle \circ \mu_* \circ \xi_* = 0$. On the other hand $\pi_0(F) = \{1\}$, since F is connected. Hence we have the following exact sequence:

$$(1)\qquad 1\to\pi_1(F)\xrightarrow{i_*}\pi_1(E-S-\mu^{-1}(B))\underset{\xi_*}{\overset{\mu_*}{\rightleftarrows}}\pi_1(M-B)\to 1.$$

We denote $i_*\pi_1(F)$ by K, $\xi_*\pi_1(M-B)$ by H and $\pi_1(E-S-\mu^{-1}(B))$ by G. The short exact sequence means that:

$$G\cong K\rtimes H \text{ (a semi-direct product).}$$

Now let $B=B_1\cup\cdots\cup B_l$ be the irreducible decomposition of B and α_j be the meridian of B_j (see Figure 3).

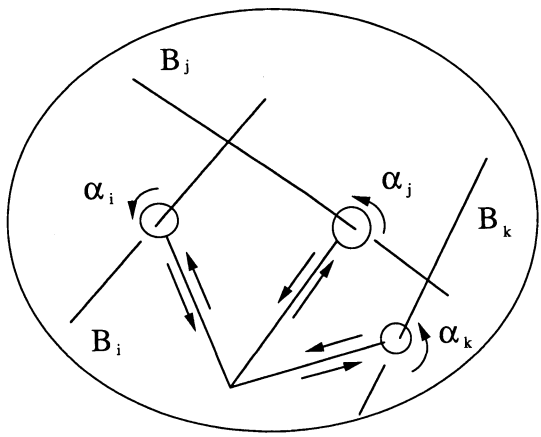

Figure 3

REMARK. Here we assume that B has a finite irreducible decomposition for simplicity. But even if B has an infinite irreducible decomposition the following argument is the same.

From a theorem of Van Kampen [4] (see also Namba [2] Cor.1.2.8), we have the following exact sequence:

$$1\to\langle\!\langle\alpha_1,\cdots,\alpha_l\rangle\!\rangle^{\pi_1(M-B)}\to\pi_1(M-B)\to\pi_1(M)\to 1$$

where $\widehat{Q}=\langle\!\langle\alpha_1,\dots,\alpha_l\rangle\!\rangle^{\pi_1(M-B)}$ is the smallest normal subgroup of $\pi_1(M-B)$ which contains $\alpha_1,\cdots,\alpha_l$.

$\mu^{-1}(B)$ is a hypersurface of E, which has the irreducible decomposition

$$\mu^{-1}(B)=\mu^{-1}(B_1)\cup\cdots\cup\mu^{-1}(B_l).$$

$\xi_*(\alpha_j)$ is a meridian of $\mu^{-1}(B_j)$, for μ: $E\to M$ is a line bundle and so $d\mu$: $T_pM\to T_{\mu(p)}M$ is surjective. Then, from the theorem of Van Kampen again, we have the folloing exact sequence:

$$1\to\langle\!\langle\xi_*(\alpha_1),\dots,\xi_*(\alpha_l)\rangle\!\rangle^{G}\to G\to\pi_1(E-S)\to 1,$$

where $\ll \xi_*(\alpha_1), \ldots, \xi_*(\alpha_l) \gg^G$ is the smallest normal subgroup of G which contains $\xi_*(\alpha_1), \ldots, \xi_*(\alpha_l)$.

We denote $\ll \xi_*(\alpha_1), \ldots, \xi_*(\alpha_l) \gg^G$ by N, $\xi_*(\ll \alpha_1, \ldots, \alpha_l \gg^{\pi_1(M-B)})$ by Q and KN by R.

Then we can easily check that

(2) $$N \cap H = Q \text{ and } R \cap NH = N.$$

Consider the natural exact sequence

$$1 \to R/N \to G/N \to G/R \to 1.$$

Note that, by (1) and (2),

$$G/R = KH/R = (KN)(NH)/R = R(NH)/R \cong (NH)/(R \cap (NH)) = (NH)/N.$$

Hence, we have the exact sequence

(3) $$1 \to R/N \to G/N \xrightarrow{f} (NH)/N \to 1.$$

The homomorphism $g: (NH)/N \to G/N$ defined by

$$g: nh(modN) \mapsto h(modN)(n \in N, h \in H)$$

is well-defined and satisfies $f \circ g =$ the identity. Hence the exact sequence (3) splits, so

$$G/N \cong (R/N) \rtimes (NH/N) \text{ (a semi-direct product).}$$

We can easily check that

$$K \cap N = \ll a^{-1} q a q^{-1} | a \in K, q \in Q \gg^K,$$

where $\ll a^{-1} q a q^{-1} | a \in K, q \in Q \gg^K$ is the smallest normal subgroup of K which contains $\{a^{-1} q a q^{-1} | a \in K, q \in Q\}$. Furthermore, note that if K and Q are respectively generated by $\{a_1, \ldots, a_n\}$ and $\{q_1, \ldots, q_t\}$, then

$$K \cap N = \ll a_j^{-1} q_k a_j q_k^{-1} | 1 \leq j \leq n, 1 \leq k \leq t \gg^K.$$

We assume that $\ll \alpha_1, \ldots, \alpha_l \gg^{\pi_1(M-B)}$ has a finite presentation as follows:

$$\ll \alpha_1, \ldots, \alpha_l \gg^{\pi_1(M-B)} = < \beta_1, \ldots, \beta_t | \square = 1, \ldots, \square = 1 \text{ (some relations)} >.$$

Since $K = i_* \pi_1(F)$ is isomorphic to the n-th free group $< \gamma_1, \ldots, \gamma_n >$, we have:

$$K \cap N \cong \ll \gamma_j^{-1} \xi_*(\beta_k) \gamma_j \xi_*(\beta_k)^{-1} | 1 \leq j \leq n, 1 \leq k \leq t \gg^K.$$

Thus,

$$K/(K \cap N) \cong \langle \gamma_1, \ldots, \gamma_n | \gamma_j^{-1} \xi_*(\beta_k) \gamma_j \xi_*(\beta_k)^{-1} = 1 (1 \leq j \leq n, 1 \leq k \leq t) \rangle.$$

Since G/N is isomorphic to $\pi_1(E-S)$, R/N is isomorphic to $K/(K \cap N)$ and $NH/N \cong H/(N \cap H) = H/Q$ is isomorphic to $\pi_1(M)$, we have:

$$\pi_1(E-S)\cong(K/(K\cap N))\rtimes\pi_1(M)\text{ (a semi-direct product).}$$

Now useing φ and θ (defined in section 2), we have:

$$\xi_*(\beta_k)\gamma_j\xi_*(\beta_k)^{-1}=\varphi(\theta(\beta_k))(\gamma_j)\text{ (see Figure 4).}$$

Hence,

$$K/(K\cap N)\cong\langle\gamma_1,\ \dots,\ \gamma_n|\gamma_j=\varphi(\theta(\beta_k))(\gamma_j)(1\leq j\leq n,\ 1\leq k\leq t)\rangle.$$

This completes the proof of Main Theorem.

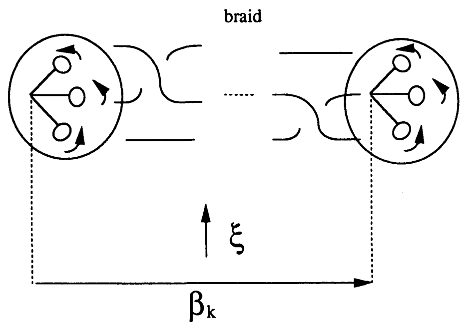

Figure 4

4. Proof of Lemma 1. (Due to M.Sakuma): For a given point $q\in M-B$, we can take a neiborhood U of q such that

(i) $\mu'^{-1}(U) \underset{homeomorphic}{\approx} \amalg_{i=1}^{n}\tilde{U}_i(\mu:\ \tilde{U}_i\overset{\sim}{\to}U)$ We write $\mu^{-1}(q)\cap\tilde{U}_i=\{\tilde{q}_i\}$.

(ii) The following diagram is commutative.

$$\begin{array}{l}\mu^{-1}(U)\overset{\sim}{\to}U\times\boldsymbol{C}\\ \quad\mu\searrow\curvearrowleft\nearrow P_1\text{ (where }P_1:\ (p,\ z)\mapsto p)\\ \qquad U\end{array}$$

Here we define a map $h_i:\ \tilde{U}_i\to\boldsymbol{C}$ as follows:

$$h_i:\ \tilde{U}_i\to\mu^{-1}(U)\cong U\times\boldsymbol{C}\underset{projection}{\to}\boldsymbol{C}.$$

Then we can write $\tilde{U}_i$ as follows:

$$\tilde{U}_i=\{(x,\ h_i(x)\}\in U\times\boldsymbol{C}|x\in U\}.$$

We write $z_j=h_i(\tilde{q}_i)$, then there exists a positive number $\varepsilon>0$ such that
(1) $\mathrm{Im}h_i\subset\mathrm{Int}(D_\varepsilon(z_i))$, where $D_\varepsilon(z_i)$ is an ε-disk whose center is z_i and *Int* $(D_\varepsilon(z_i))$ is the interior of $D_\varepsilon(z_i)$.
(2) $D_\varepsilon(z_1)$, ..., $D_\varepsilon(z_n)$ are disjoint each other.
From Lemma 2 bellow, there exists a fiber preserving homeomorphism Φ such

that $\Phi(\tilde{U}_i)=U\times\{x\}$.

$$\begin{tikzcd} \mu^{-1}(U) \arrow[r, phantom, "\cong"] \arrow[dr, "\mu"'] & U\times \boldsymbol{C} \arrow[r, "\Phi"] \arrow[d, "P_1"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & U\times \boldsymbol{C} \arrow[dl, "P_1"] \\ & U & \end{tikzcd}$$

So we can take local coordinates of $\mu: E-S-\mu^{-1}(B)\to M-B$. This shows Lemma 1.

Lemma 2. *Let D be an ε-disk of $\boldsymbol{C}$ whose center is the origin. Let U be the neiborhood of q as above. Let $h: U\to Int(D)$ be a continuous map such that $h(q)=0$, where $Int(D)$ is the interior of D. Put $\tilde{U}=\{(x, h(x))\in U\times Int(D)|x\in U\}\subset U\times Int(D)$. Then there exists a homeomorphism $\Psi: U\times D\xrightarrow{\sim} U\times D$ such that*

(i) $\Psi(\tilde{U})=U\times\{0\}$.

(ii) Ψ *is fiber preserving. (i.e. the folloing diagram is commutative.)*

$$\begin{tikzcd} U\times D \arrow[rr, "\Psi"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & U\times D \arrow[dl] \\ & U & \end{tikzcd}$$

(iii) $\Psi|_{U\times\partial D}: U\times\partial D\to U\times\partial D$ *is the identity map.*

Proof of Lemma 2. First we define a homeomorphism $H_x: D\to D$ for each point $x\in U$ as follows:

(i) $H_x(h(x))=0$.

(ii) $H_x|_{\partial D} \quad =id|_{\partial D}$.

(iii) H_x is extended to D with radial extention (see Figure 5).

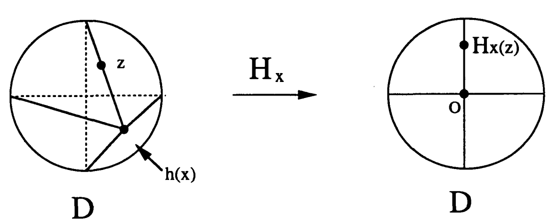

Figure 5

Second we define a homeomorphism $\Psi: U\times D\xrightarrow{\sim} U\times D$ as follows:

$$\Psi(x, z)=\Psi(x, Hx(z)).$$

Ψ satisfies the above conditions. $\qquad$ (*q.e.d.*)

5. Case of Trivial Line Bundle. In Main Theorem, we assumed the existence of a continuous section ξ such that $\xi(M)\cap S=\phi$. In the case of the trivial line bundle we can prove the following proposition:

Proposition 1. *Let M be a connected complex manifold and μ: $E\rightarrow M$ be a trivial line bundle on M(i, e, $E=M\times\boldsymbol{C}$ and $\mu(p, z)=p$ for every point $(p, z)\in M\times\boldsymbol{C}$). Let $f_{\cdot}, \ldots, f_n$ be holomorphic functions on M and S be the hypersurface of E defined by*

$$S=\{(p, z)\in E|z^{n}+f_{1}(p)z^{n-1}+\cdots+f_{n}(p)=0\}.$$

Then there is a continuous section ξ of μ: $E\rightarrow M$ such that $\xi(M)\cap S=\phi$.

Proof. We define a continuous function h: $M\rightarrow\boldsymbol{C}$ by

$$h(p)=|f_{1}(p)|+\cdots+|f_{n}(p)|+1.$$

We define a section ξ: $M\rightarrow E$ by

$$\xi(p)=(p, h(p)).$$

One can easily see that this section ξ of μ satisfies $\xi(M)\cap S=\phi$. In fact, if there is a point $p\in M$ such that $\xi(p)\in S$, then

$$\{h(p)\}^{n}+f_{1}(p)\{h(p)\}^{n-1}+\cdots+f_{n}(p)=0.$$

Since $h(p)\geq 1$

$$1=\frac{f_{1}(p)}{h(p)}-\frac{f_{2}(p)}{\{h(p)\}^{2}}-\cdots-\frac{f_{n}(p)}{\{h(p)\}^{n}}.$$

Hence

$$1\leq\frac{|f_{1}(p)|}{h(p)}+\frac{|f_{2}(p)|}{\{h(p)\}^{2}}+\cdots+\frac{|f_{n}(p)|}{\{h(p)\}^{n}}.$$

Since $\{h(p)\}^{k}\geq h(p)(k=1, 2, \ldots)$,

$$\begin{aligned}1&\leq\frac{|f_{1}(p)|}{h(p)}+\frac{|f_{2}(p)|}{h(p)}+\cdots+\frac{|f_{n}(p)|}{h(p)}\\&=\frac{|f_{1}(p)|+\cdots+|f_{n}(p)|}{|f_{1}(p)|+\cdots+|f_{n}(p)|+1}<1.\end{aligned}$$

A contradiction. (*q.e.d.*)

Let μ: $\boldsymbol{C}^{m+1}\rightarrow\boldsymbol{C}^{m}$ be the trivial line bundle on $\boldsymbol{C}^{m}$ defined by

$$\mu:(z_{1}, \ldots, z_{m}, z_{m+1})\rightarrow(z_{1}, \ldots, z_{m}).$$

Let S be the hypersurface of $\boldsymbol{C}^{m+1}$ defined by

$$S=\{(z_1, ..., z_m, z_{m+1})\in \boldsymbol{C}^{m+1}|z_{m+1}{}^n+f_1(z)z_{m+1}{}^{n-1}+\cdots+f_n(z)=0\}\cdots(1),$$

where $z=(z_1, ..., z_m)$ and $f_1(z), ..., f_n(z)$ are polynomials.

By Corollary to Main Theorem and Proposition 1, we have

Theorem 1. *Let S be the hypersurface of $\boldsymbol{C}^{m+1}$ defined by* (1). *Then,*

$$\pi_1(\boldsymbol{C}^{m+1}-S)\cong\langle\gamma_1, ..., \gamma_n|\gamma_j=\varphi(\theta(\beta_k))(\gamma_i), (1\leq j\leq n, 1\leq k\leq t)\rangle.$$

Furthermore, let $(X_0: X_1: \cdots: X_{m+1})$ be homogeneous coordinatss of $\boldsymbol{P}^{m+1}$ such that $(X_1/X_0, \cdots, X_{m+1}/X_0)=(z_1, ..., z_{m+1})\in \boldsymbol{C}^{m+1}$ and $\bar{S}$ be the closure of S in $\boldsymbol{P}^{m+1}$. Then we have the following theorem of Zariski:

Theorem 2(Zariski [5]).
Suppose that $p_\infty=(0: \cdots: 0: 1)$ is not contained in $\bar{S}$. Then

$$\pi_1(\boldsymbol{P}^{m+1}-\bar{S})$$
$$\cong\langle\gamma_1, ..., \gamma_n|\gamma_n\gamma_{n-1}...\gamma_1=1, \gamma_j=\varphi(\theta(\beta_k))(\gamma_j), (1\leq j\leq n, 1\leq k\leq t)\rangle.$$

Proof. Let H_∞ be the hypersurface of$\boldsymbol{P}^{m+1}$ defined by $H_\infty=\{X_0=0\}$, (*i.e.* hyperplane at infinity) and α be a meridian of H_∞ in $\boldsymbol{P}^{m+1}-\bar{S}-H_\infty$ (see Figure 6).

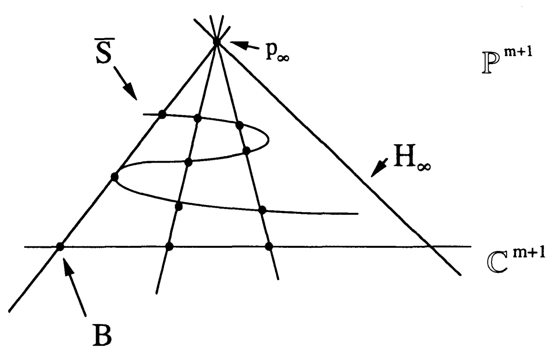

Figure 6

From the theorem of Van Kampen [4], we have the following exact sequence:

$$1\to\langle\!\langle\alpha\rangle\!\rangle^{\pi_1(\boldsymbol{C}^{m+1}-S)}\to\pi_1(\boldsymbol{C}^{m+1}-S)\to\pi_1(\boldsymbol{P}^{m+1}-\bar{S})\to 1 \text{ (exact)}.$$

We can take α as $(\gamma_n\gamma_{n-1}\cdots\gamma_1)^{-1}$ in $\boldsymbol{C}^{m+1}-S$ (see Figure 7).
Thus,

$$\pi_1(\boldsymbol{P}^{m+1}-\bar{S})\cong\pi_1(\boldsymbol{C}^{m+1}-S)/\langle\!\langle\gamma_n\gamma_{n-1}\cdots\gamma_1\rangle\!\rangle^{\pi_1(\boldsymbol{C}^{m+1}-S)}.$$

This shows Theorem 2.

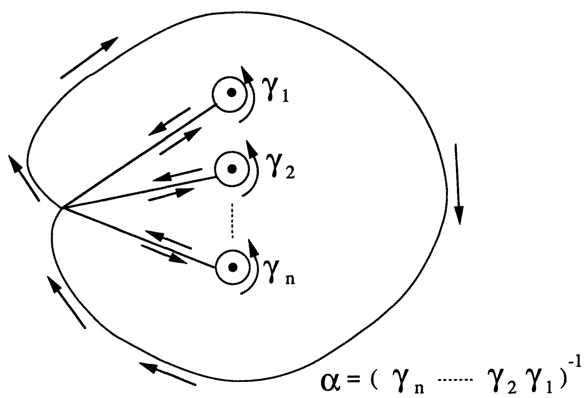

Figure 7

REMARK. A similar theorem to Theorem 1 holds for $\mu: \boldsymbol{B}^m(\varepsilon)\times \boldsymbol{B}^1(\varepsilon')\rightarrow \boldsymbol{B}^m(\varepsilon)$, where $\boldsymbol{B}^m(\varepsilon)$ is a m-dimensional complex ball; $\boldsymbol{B}^m(\varepsilon)=\{(z_1, \ldots, z_m)\in \boldsymbol{C}^m \| z_1|^2+\cdots+|z_m|^2<\varepsilon^2\}$. In this case, the existence of continuous section with a similar conditions to Theorem 1 is obvious.

6. Calculations of Fundamental Groups of Finite Branched Coverings

EXAMPLE 1.

Let X be the surface in $\boldsymbol{C}^3$ defined by

$$X=\{\lambda, x, y)\in \boldsymbol{C}^3 | y^2=x(x-1)(x-\lambda)\}.$$

X has two isolated singular points at (0,0,0) and (1,1,0). Hence X is normal. Let $\pi: X\rightarrow \boldsymbol{C}^2$ be the projection map defined by

$$\pi(\lambda, x, y)=(\lambda, x).$$

Then π is a double branched covering of $\boldsymbol{C}^2$. The branch loucus S of π is a curve in $\boldsymbol{C}^2$ and is written as:

$$S=\{(\lambda, x)\in \boldsymbol{C}^2 | x(x-1)(x-\lambda)=0\}.$$

According to Theorem 1, we can calculate $\pi_1(\boldsymbol{C}^2-S)$. Let $\mu: \boldsymbol{C}^2\rightarrow \boldsymbol{C}$ be the trivial line bundle on $\boldsymbol{C}$ defined by

$$\mu(\lambda, x)=\lambda.$$

The branch locus B of μ is $\{0, 1\}\subset \boldsymbol{C}$ and $\pi_1(\boldsymbol{C}-B)$ is isomorphic to the free group $\langle\beta_1, \beta_2\rangle$, where β_1 and β_2 are its generators and can be considered as the

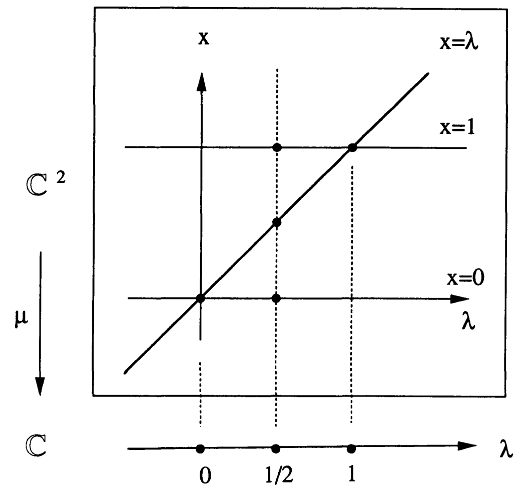

Figure 8

meridians of $\{0\}$ and $\{1\}$, respectively. We may take $q_0=\frac{1}{2}$ as a reference point of $\pi_1(\boldsymbol{C}-B)$. In this case the standerd fiber F of $\mu|_{C-S-\mu^{-1}(B)}$ is $\boldsymbol{C}-\{\text{3-points}\}$. We define γ_1, γ_2 and γ_3 as the meridians of $\left(\frac{1}{2}, 1\right)$,$\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$, respectively. The image of β_1 and β_2 by $\theta:\ \pi_1(\boldsymbol{C}-B)\rightarrow B_3$ are σ_1^2 and σ_2^2, respectively. Then we have

$$\begin{aligned}\pi_1(\boldsymbol{C}^2-S)&\cong\langle\gamma_1,\ \gamma_2,\ \gamma_3|\gamma_j=\varphi(\theta(\beta_k))(\gamma_j),\ j=1,\ 2,\ 3,\ k=1,\ 2\rangle\\ &\cong\langle\gamma_1,\ \gamma_2,\ \gamma_3|\gamma_2\gamma_3=\gamma_3\gamma_2,\ \gamma_1\gamma_2=\gamma_2\gamma_1\rangle.\end{aligned}$$

By using the Reidemeister- Schreier method (*c.f*. Rolfsen [3] P.315-P.316), we can calculate $\pi_1(\mathrm{Reg}X)$, where $\mathrm{Reg}X$ is the set of regular points of X. Since $\pi_1(\boldsymbol{C}^2-B)$ is generated by three elements and since π is a double branched covering, we take the 3-th free group F_3 and the 5-th free group F_5. As in Figure 9, we take their generators $\{\gamma_1,\ \gamma_2,\ \gamma_3\}$ and $\{b_1,\ b_2,\ b_3,\ b_4,\ b_5\}$, respectively, where$\pi^{-1}(\gamma_1)=\{x_1,\ x_2\}$, $\pi^{-1}(\gamma_2)=\{y_1,\ y_2\}$, $\pi^{-1}(\gamma_3)=\{z_1,\ z_2\}$, and

$$\begin{aligned}b_1&=y_1x_1^{-1}\\ b_2&=x_1y_2x_2^{-1}x_1^{-1}\\ b_3&=z_1x_1^{-1}\\ b_4&=x_1z_2x_2^{-1}x_1^{-1}\\ b_5&=x_1x_2.\end{aligned}$$

Then we transfer the relation of $\pi_1(\boldsymbol{C}^2-S)\gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1}=1$ in the words of F_5:

$$y_1z_2y_2^{-1}z_1^{-1}=b_1b_4b_5b_5^{-1}b_2^{-1}b_3^{-1}=1$$

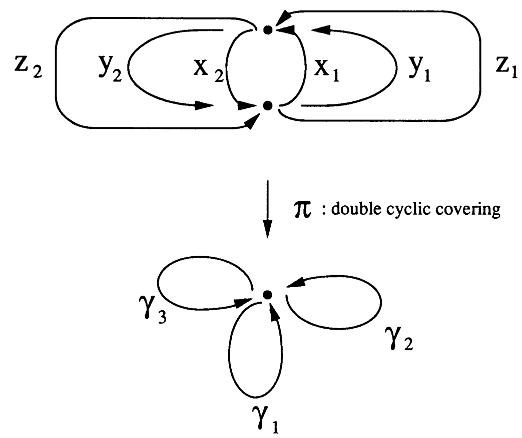

Figure 9

$$x_1y_2z_1y_1^{-1}z_2^{-1}x_1^{-1}=b_2b_5b_3b_1^{-1}b_5^{-1}b_4^{-1}=1.$$

In a similar way, we transfer the relation $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}=1$ in the words of F_5:

$$x_1y_2x_2^{-1}y_1^{-1}=b_2b_5b_5^{-1}b_1^{-1}=1$$
$$x_1x_2y_1x_1^{-1}y_2^{-1}x_1^{-1}=b_5b_1b_5^{-1}b_2^{-1}=1.$$

We also transfer the relations $\gamma_1^2=1$, $\gamma_2^2=1$, $\gamma_3^2=1$ since the ramification index of each irreducible component of S is equal to 2:

$$x_1x_2=b_5=1$$
$$y_1y_2=b_1b_2b_5=1$$
$$z_1z_2=b_3b_4b_5=1.$$

Putting $\alpha_1=b_1$ and $\alpha_2=b_4$, we have

$$\begin{aligned}\pi_1(\boldsymbol{Reg}\ X)&\cong\langle\alpha_1,\ \alpha_2|\alpha_1^2=1,\ (\alpha_1\alpha_2)^2=1\rangle\\ &\cong(\boldsymbol{Z}/2\boldsymbol{Z})*(\boldsymbol{Z}/2\boldsymbol{Z})\text{(free product)}.\end{aligned}$$

EXAMPLE 2.

Let X be the hypersurfaces of $\boldsymbol{C}^4$ defined by

$$X=\{(x,\ y,\ z,\ w)\in\boldsymbol{C}^4|w^n=z^2-xy^2\}(n\geq 2).$$

The singular locus of X is the line $\{(x,\ y,\ z,\ w)\in X|y=z=w=0\}$. Hence X is normal. Let $\pi:\ X\to\boldsymbol{C}^3$ be the projection map defined by:

$$\pi(x,\ y,\ z,\ w)=(x,\ y,\ z).$$

Then π is a cyclic branched covering of $\boldsymbol{C}^3$. The branch locus S of π is a surface in $\boldsymbol{C}^3$ and is written as:

$$S=\{(x, y, z)\in \boldsymbol{C}^3 | z^2-xy^2=0\} \text{ (the Cartan umbrella).}$$

According to Theorem 1, we can calculate $\pi_1(\boldsymbol{C}^3-S)$. The result is

$$\pi_1(\boldsymbol{C}^3-S)\cong\langle\gamma\rangle(\text{the free group}).$$

From the Reidemeister-Schreier method again, we have:

$\pi_1(RegX)\cong\{i.e.\ RegX$ is simply connected).

EXAMPLE 3.

Let X be the hypersurface of $\boldsymbol{C}^{m+2}$ defined by

$$X=\{(z_1, \ldots, z_{m+2})\in \boldsymbol{C}^{m+2} | z_{m+2}^2+z_{m+1}^2+g(z_1, \ldots, z_m)=0\},$$

where g is a polynomial which is not constant. The singular locus of X is at most $(m-1)$-dimensional. Hence X is normal. Let $\pi: X\to \boldsymbol{C}^{m+1}$ be the projection map defined by:

$$\pi(z_1, \ldots, z_{m+1}, z_{m+2})=(z_1, \ldots, z_{m+1}).$$

Then π is a branched covering of $\boldsymbol{C}^{m+1}$. The branch locus S of π is a hypersurface in $\boldsymbol{C}^{m+1}$ and is written as:

$$S=\{z_{m+1}^2+g(z_1, \ldots, z_m)=0\}.$$

By Theorem 1, $\pi_1(\boldsymbol{C}^{m+1}-S)$ can be written as:

$$\pi_1(\boldsymbol{C}^{m+1}-S)\cong\langle\gamma_1, \gamma_2 | \square=1, \ldots, \square=1\rangle.$$

From the Reidemeister-Schreier method again, we have:

$$\pi_1(RegX)\cong\begin{cases}\{1\} \text{ or} \\ \boldsymbol{Z}/q\boldsymbol{Z}(\exists\, q\in \boldsymbol{Z}) \text{ or} \\ \boldsymbol{Z}\end{cases}$$

($i.e.$ $\pi_1(RegX)$ is isomorphic to a cyclic group).

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Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560, Japan