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HEAT KERNEL AND SINGULAR VARIATION OF DOMAINS

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1. Introduction

We consider a bounded region M in $\boldsymbol{R}^n$ ($n=2$ or 3) whose boundary is smooth. Let w be a fixed point in M. By $B(\varepsilon;w)$ we denote a ball of radius ε with the center w. We put $M_\varepsilon=M\backslash\overline{B(\varepsilon;w)}$.

Let $U(x,y,t)$ ($U^{(\varepsilon)}(x,y,t)$; respectively) be the heat kernel in M (M_ε ; respectively) with the Dirichlet condition on its boundary ∂M (∂M_ε; respectively). That is, it satisfies

$$(1.1)\qquad \begin{cases} (\partial_t-\Delta_x)U(x,y,t)=0 & x,y\in M,\quad t>0 \\ U(x,y,t)=0 & x\in\partial M,\quad y\in M,\quad t>0 \\ \lim\limits_{t\to 0}U(x,y,t)=\delta(x-y) & x,y\in M \end{cases}$$

$$(1.1)\qquad \begin{cases} (\partial_t-\Delta_x)U^{(\varepsilon)}(x,y,t)=0 & x,y\in M_\varepsilon,\quad t>0 \\ U^{(\varepsilon)}(x,y,t)=0 & x\in\partial M_\varepsilon,\quad y\in M_\varepsilon,\quad t>0 \\ \lim\limits_{t\to 0}U^{(\varepsilon)}(x,y,t)=\delta(x-y) & x,y\in M_\varepsilon \end{cases}$$

We put

$$(1.2)\qquad (U_tf)(x)=\int_M U(x,y,t)f(y)dy,\qquad f\in L^p(M)$$

and

$$(1.3)\qquad (U_t^{(\varepsilon)}f)(x)=\int_{M_\varepsilon} U^{(\varepsilon)}(x,y,t)f(y)dy,\qquad f\in L^p(M_\varepsilon).$$

Then, U_tf and $U_t^{(\varepsilon)}f$ satisfy the following.

$$\begin{cases} (\partial_t-\Delta_x)(U_tf)(x)=0 \qquad x\in M,\quad t>0 \end{cases}$$

$$\begin{cases} (U_t f)(x)=0 & x\in\partial M, \quad t>0 \\ \|U_t f-f\|_{L^p(M)}\to 0 & \text{as } t\to 0 \end{cases}$$

$$\begin{cases} (\partial_t-\Delta_x)(U_t^{(\varepsilon)}f)(x)=0 & x\in M_\varepsilon, \quad t>0 \\ (U_t^{(\varepsilon)}f)(x)=0 & x\in\partial M_\varepsilon, \quad t>0 \\ \|U_t^{(\varepsilon)}f-f\|_{L^p(M_\varepsilon)}\to 0 & \text{as } t\to 0 \end{cases}$$

We want to construct an approximate kernel of $U^{(\varepsilon)}(x,y,t)$ by using $U(x,y,t)$. We put

$$V^{(\varepsilon)}(x,y,t)=U(x,y,t)-L_n(\varepsilon)\int_0^t U(x,w,\tau)U(w,y,t-\tau)d\tau, \tag{1.4}$$

where

$$L_n(\varepsilon)=\begin{cases} -2\pi(\log\varepsilon)^{-1} & (\text{if } n=2) \\ 4\pi\varepsilon & (\text{if } n=3). \end{cases}$$

and we put

$$(V_t^{(\varepsilon)}f)(x)=\int_M V^{(\varepsilon)}(x,y,t)f(y)dy, \qquad f\in L^p(M).$$

Let T and T_ε be operators on M and M_ε, respectively. Then, $\|T\|_p$, $\|T_\varepsilon\|_{p,\varepsilon}$ denotes the operator norm on $L^p(M)$, $L^p(M_\varepsilon)$, respectively. Let f and f_ε be functions on M and M_ε, respectively. Then, $\|f\|_p$, $\|f_\varepsilon\|_{p,\varepsilon}$ denotes the norm on $L^p(M)$, $L^p(M_\varepsilon)$, respectively.

Let χ_ε denote the characteristic function of M_ε. Then, we have the following Theorems 1 and 2.

Theorem 1. *Assume that $n=2$. Then, there exists a constant C_t, which may depend on t but which is independent of ε such that*

$$\|U_t^{(\varepsilon)}-\chi_\varepsilon V_t^{(\varepsilon)}\chi_\varepsilon\|_{p,\varepsilon} \le \begin{cases} C_t\varepsilon^{1/p}|\log\varepsilon|^{-1} & \text{if } p\in(2,\infty) \\ C_t\varepsilon^{(1-s)/2}|\log\varepsilon|^{-1} & \text{if } p=2 \\ C_t\varepsilon^{1-(1/p)}|\log\varepsilon|^{-1} & \text{if } p\in(1,2) \end{cases}$$

hold. Here $s\in(0,1)$ is an arbitrary fixed constant.

Theorem 2. *Assume that $n=3$. Then, there exists a constant C_t independent* of ε *such that*

$$\| U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon \|_{p,\varepsilon} \leq \begin{cases} C_t \varepsilon^{1+(2/p)} & \text{if } p \in (3,\infty) \\ C_t \varepsilon^{(5-s)/3} & \text{if } p \in [3/2, 3] \\ C_t \varepsilon^{3-(2/p)} & \text{if } p \in (1, 3/2) \end{cases}$$

hold. Here $s \in (0,1)$ *is an arbitrary fixed constant.*

REMARK. Thus, by Theorems 1 and 2, we know that

$$-L_n(\varepsilon)\chi_\varepsilon(x)\chi_\varepsilon(y)\int_0^t U(x,w,\tau)U(w,y,t-\tau)d\tau$$

gives a main asymptotic term of the difference between $U^{(\varepsilon)}(x,y,t)$ and $U(x,y,t)$.

The Hadamard variation of the heat kernel was discussed in [2]. And we have various papers on singular variation of domain. See, for example, [3], [4], [5], [6].

We give the proof of Theorems 1 and 2 in section 2 and section 3, respectively. In Appendix we give some properties of $U(x,y,t)$ and $U^{(\varepsilon)}(x,y,t)$. Following an usual custom, we use the same letter C in inegualities which are independent of ε.

2. Proof of Theorem 1

Throughout this section we assume that $n=2$. We put

$$U(x,y,t) = W(x,y,t) + S(x,y,t), \tag{2.1}$$

where

$$W(x,y,t) = (4\pi t)^{-n/2}\exp(-|x-y|^2/4t). \tag{2.2}$$

We write $B(\varepsilon;w) = B_\varepsilon$. Without loss of generality we may assume that $w=0$.

We take arbitrary $f \in L^p(M_\varepsilon)$. Let $\hat{f}$ be an extension of f to M which is 0 on B_ε. At first we want to estimate $|(V_t^{(\varepsilon)}\hat{f})(x)|_{|x\in\partial B_\varepsilon}$. By (2.2), we have

$$\begin{aligned} &\int_0^t W(x,w,\tau)d\tau \mid x \in \partial B_\varepsilon \\ &= \int_0^t (4\pi\tau)^{-1}\exp(-\varepsilon^2/4\tau)d\tau \\ &= (4\pi)^{-1}\int_{\varepsilon^2/4t}^{\infty} s^{-1}e^{-s}ds \end{aligned} \tag{2.3}$$

$$=(4\pi)^{-1}(e^{-s}\log s)\Big|_{s=\varepsilon^2/4t}^{s=\infty}+\int_{\varepsilon^2/4t}^{\infty}e^{-s}(\log s)ds$$
$$=(4\pi)^{-1}(-2\log\varepsilon+R(\varepsilon,t)),$$

where

$$R(\varepsilon,t)=2(1-\exp(-\varepsilon^2/4t))\log\varepsilon+(\exp(-\varepsilon^2/4t))\log(4t)$$
$$+\int_{\varepsilon^2/4t}^{\infty}e^{-s}(\log s)ds.$$

Let γ be the Euler constant. Then,

$$\gamma=-\int_0^{\infty}e^{-s}(\log s)ds.$$

Thus, we have

$$R(\varepsilon,t)=2(1-\exp(-\varepsilon^2/4t))\log\varepsilon \tag{2.4}$$
$$+(\exp(-\varepsilon^2/4t))\log(4t)$$
$$-\gamma-\int_0^{\varepsilon^2/4t}e^{-s}(\log s)ds$$
$$=-\gamma+\log(4t)+\int_0^{\varepsilon^2/4t}e^{-s}\log(\varepsilon^2/(4ts))ds.$$

Since $0\le\log(\varepsilon^2/(4ts))\le 2(\varepsilon^2/(4ts))^{1/2}=\varepsilon(ts)^{-1/2}$ hold for any $s\in(0,\ \varepsilon^2/4t)$, we have

$$\Big|\int_0^{\varepsilon^2/4t}e^{-s}\log(\varepsilon^2/(4ts))ds\Big| \tag{2.5}$$
$$\le\varepsilon t^{-1/2}\int_0^{\varepsilon^2/4t}s^{-1/2}e^{-s}ds$$
$$\le\varepsilon t^{-1/2}\int_0^{\infty}s^{-1/2}e^{-s}ds=\pi^{1/2}\varepsilon t^{-1/2}.$$

It is easy to see that $|\log t|\le 2(t+t^{-1/2})$ holds for any $t\in(0,\infty)$. Thus, by (2.3), (2.4) and (2.5), we get

$$\int_0^t W(x,w,\tau)d\tau_{|x\in\partial B_\varepsilon}=-(2\pi)^{-1}\log\varepsilon+R(\varepsilon,t), \tag{2.6}$$

where

$$|R(\varepsilon,t)| \le C(|\log t| + t^{-1/2} + 1)$$
$$\le C(t + t^{-1/2} + 1)$$

hold for any sufficiently small $\varepsilon > 0$.

On the other hand, since

$$U(w,y,t-\tau) - U(w,y,t) = \int_0^\tau \frac{\partial}{\partial s} U(w,y,t-s)ds$$
$$= -\int_0^\tau \frac{\partial}{\partial t} U(w,y,t-s)ds$$

hold for $\tau \in (0,t)$ and $y \in M_\varepsilon$, we see that

$$\int_0^t W(x,w,\tau)U(w,y,t-\tau)d\tau$$
$$=(\int_0^t W(x,w,\tau)d\tau)U(w,y,t)$$
$$-\int_0^t W(x,w,\tau)(\int_0^\tau \frac{\partial}{\partial t} U(w,y,t-s)ds)d\tau.$$

Thus, we have

$$\int_{M_\varepsilon}(\int_0^t W(x,w,\tau)U(w,y,t-\tau)d\tau)f(y)dy$$
$$=(\int_0^t W(x,w,\tau)d\tau)\int_{M_\varepsilon} U(w,y,t)f(y)dy$$
$$-\int_0^t W(x,w,\tau)(\int_0^\tau(\int_{M_\varepsilon}\frac{\partial}{\partial t} U(w,y,t-s)f(y)dy)ds)d\tau$$
$$=(\int_0^t W(x,w,\tau)d\tau)(U_t f)(w) - \int_0^t W(x,w,\tau)(\int_0^\tau \frac{\partial}{\partial t}(U_{t-s}f)(w)ds)d\tau$$

for $x \in M$. Combining this equality with (1.2), (1.4) and (2.1), we can easily see

$$(V_t^{(\varepsilon)} f)(x) \tag{2.7}$$
$$=\int_{M_\varepsilon} U(x,y,t)f(y)dy$$
$$-L_n(\varepsilon)\int_{M_\varepsilon}(\int_0^t W(x,w,\tau)U(w,y,t-\tau)d\tau)f(y)dy$$

$$
\begin{aligned}
&-L_n(\varepsilon)\int_{M_\varepsilon}\Big(\int_0^t S(x,w,\tau)U(w,y,t-\tau)d\tau\Big)f(y)dy \\
&=(U_t\hat{f})(x)-L_n(\varepsilon)\Big(\int_0^t W(x,w,\tau)d\tau\Big)(U_t\hat{f})(w) \\
&\quad+L_n(\varepsilon)\int_0^t W(x,w,\tau)\Big(\int_0^\tau \frac{\partial}{\partial t}(U_{t-s}\hat{f})(w)ds\Big)d\tau \\
&\quad-L_n(\varepsilon)\int_0^t S(x,w,\tau)(U_{t-\tau}\hat{f})(w)d\tau
\end{aligned}
$$

for $x\in M$.

We recall that $L_n(\varepsilon)=-2\pi(\log\varepsilon)^{-1}$ for $n=2$. Thus, by (2.6) and (2.7), we have

$$
(2.8)\qquad (V_t^{(\varepsilon)}\hat{f})(x)_{|x\in\partial B_\varepsilon}=\sum_{i=1}^{3} I_i(\varepsilon,t),
$$

where

$$
\begin{aligned}
I_1(\varepsilon,t)&=(U_t\hat{f})(x)_{|x\in\partial B_\varepsilon}-(U_t\hat{f})(w) \\
I_2(\varepsilon,t)&=-2\pi(\log\varepsilon)^{-1}\int_0^t W(x,w,\tau)\Big(\int_0^\tau \frac{\partial}{\partial t}(U_{t-s}\hat{f})(w)ds\Big)d\tau \\
I_3(\varepsilon,t)&=2\pi(\log\varepsilon)^{-1}\Big(R(\varepsilon,t)(U_t\hat{f})(w)+\int_0^t S(x,w,\tau)(U_{t-\tau}\hat{f})(w)d\tau\Big)
\end{aligned}
$$

for $x\in\partial B_\varepsilon$.

Notice that $S(x,w,\tau)$ is uniformly bounded for $x\in M$ and $\tau\in[0,t]$. Thus, by (2.6) and Lemma A.3 in Appendix,

$$
\begin{aligned}
(2.9)\qquad &|R(\varepsilon,t)(U_t\hat{f})(w)| \\
&\le C|R(\varepsilon,t)|t^{-1/p}\|\hat{f}\|_p\le Ct^{-1/p}(t+t^{-1/2}+1)\|f\|_{p,\varepsilon}
\end{aligned}
$$

and

$$
\begin{aligned}
(2.10)\qquad &\Big|\int_0^t S(x,w,\tau)(U_{t-\tau}\hat{f})(w)d\tau\Big| \\
&\le C\int_0^t |(U_{t-\tau}\hat{f})(w)|d\tau \\
&\le C\|\hat{f}\|_p\int_0^t (t-\tau)^{-1/p}d\tau\le Ct^{1-(1/p)}\|f\|_{p,\varepsilon}
\end{aligned}
$$

hold for $p>1$ and $x\in M$. Therefore, by (2.8), (2.9) and (2.10), we have

$$|I_3(\varepsilon,t)|\leq C|\log\varepsilon|^{-1}t^{-1/p}(t+t^{-1/2}+1)\|f\|_{p,\varepsilon} \tag{2.11}$$

for $p>1$. The same calculation as above yields

$$\begin{aligned}
&\left|\int_0^\tau\frac{\partial}{\partial t}(U_{t-s}\hat{f})(w)ds\right| \\
&\leq C\|\hat{f}\|_p\int_0^\tau(t-s)^{-1-(1/p)}ds \\
&\leq C\|f\|_{p,\varepsilon}((t-\tau)^{-1/p}-t^{-1/p}) \\
&\leq C\|f\|_{p,\varepsilon}t^{-1/p}\tau^{1/p}(t-\tau)^{-1/p}
\end{aligned} \tag{2.12}$$

for $p>1$, $\tau\in(0,t)$ and

$$\begin{aligned}
|I_1(\varepsilon,t)|&=\varepsilon|\nabla_x(U_t\hat{f})(w+\theta(x-w))|_{|x\in\partial B_\varepsilon} \\
&\leq C\varepsilon t^{-(1/p)-(1/2)}\|f\|_{p,\varepsilon}
\end{aligned} \tag{2.13}$$

for $p>1$, where $\theta\in(0,1)$ denotes some constant. Furhermore, by (2.2), (2.8) and (2.12),

$$\begin{aligned}
&|I_2(\varepsilon,t)| \\
&\leq C|\log\varepsilon|^{-1}t^{-1/p}\|f\|_{p,\varepsilon}\int_0^t\tau^{(1/p)-1}(t-\tau)^{-1/p}\exp(-\varepsilon^2/4\tau)d\tau \\
&\leq C|\log\varepsilon|^{-1}t^{-1/p}\|f\|_{p,\varepsilon}\int_0^t\tau^{(1/p)-1}(t-\tau)^{-1/p}d\tau \\
&= C|\log\varepsilon|^{-1}t^{-1/p}\|f\|_{p,\varepsilon}\int_0^1 s^{(1/p)-1}(1-s)^{-1/p}ds \\
&\leq C|\log\varepsilon|^{-1}t^{-1/p}\|f\|_{p,\varepsilon}
\end{aligned} \tag{2.14}$$

hold for $p>1$.

Summing up (2.8), (2.11), (2.13) and (2.14), we get the following.

Proposition 2.1. *Fix $p>1$ and $t>0$. Then, there exists a constant C independent of ε, t such that*

$$|(V_t^{(\varepsilon)}\hat{f})(x)|_{x\in\partial B_\varepsilon}\leq Ct^{-1/p}(t+t^{-1/2}+1)|\log\varepsilon|^{-1}\|f\|_{p,\varepsilon}$$

holds for any $f\in L^p(M_\varepsilon)$.

Now we are in a position to prove Theorem 1. We put $v(x,t)=(U_t^{(\varepsilon)}f)(x)-(V_t^{(\varepsilon)}\hat{f})(x)$. Then $v(x,t)$ satisfies the following.

$$\text{(2.15)}\qquad \begin{cases} (\partial_t-\Delta_x)v(x,t)=0 & x\in M_\varepsilon,\quad t>0 \\ v(x,t)=0 & x\in\partial M,\quad t>0 \\ v(x,t)=-(V_t^{(\varepsilon)}\hat{f})(x) & x\in\partial B_\varepsilon,\quad t>0 \\ \lim_{t\to 0} v(x,t)=0 & a.a.\quad x\in M_\varepsilon. \end{cases}$$

By the maximum principle we have

$$\sup_{x\in M_\varepsilon}|v(x,t)|\le\sup_{x\in\partial M_\varepsilon}|v(x,t)|\le\sup_{x\in\partial B_\varepsilon}|(V_t^{(\varepsilon)}\hat{f})(x)|.$$

Thus, by Proposition 2.1,

$$\text{(2.16)}\qquad \begin{aligned} &\|U_t^{(\varepsilon)}f-\chi_\varepsilon V_t^{(\varepsilon)}\hat{f}\|_{\infty,\varepsilon} \\ &=\|v(\cdot,t)\|_{\infty,\varepsilon}\le Ct^{-1/p}(t+t^{-1/2}+1)|\log\varepsilon|^{-1}\|f\|_{p,\varepsilon} \end{aligned}$$

hold for $p>1$.

On the other hand, by $(1.1)_\varepsilon$ and (2.15), $v(x,t)$ is explicitly represented as follows.

$$\text{(2.17)}\qquad v(x,t)=\int_0^t\Big(\int_{\partial B_\varepsilon}(V_\tau^{(\varepsilon)}\hat{f})(z)\frac{\partial U^{(\varepsilon)}}{\partial\nu_z}(x,z,t-\tau)d\sigma_z\Big)d\tau$$

Here $\partial/\partial\nu_z$ denotes the derivative along the exterior normal direction with respect to M_ε. Thus, by (2.17), Proposition 2.1 and Lemma A.5 in Appendix, we have

$$\text{(2.18)}\qquad \begin{aligned} &\|U_t^{(\varepsilon)}f-\chi_\varepsilon V_t^{(\varepsilon)}\hat{f}\|_{1,\varepsilon} \\ &=\|v(\cdot,t)\|_{1,\varepsilon} \\ &\le\int_0^t\Big(\sup_{x\in\partial B_\varepsilon}|(V_\tau^{(\varepsilon)}\hat{f})(z)|\Big)\Big(\int_{\partial B_\varepsilon}\Big(\int_{M_\varepsilon}\Big|\frac{\partial U^{(\varepsilon)}}{\partial\nu_z}(x,z,t-\tau)\Big|dx\Big)d\sigma_z\Big)d\tau \\ &\le C\varepsilon|\log\varepsilon|^{-1}\|f\|_{p,\varepsilon}\int_0^t\tau^{-1/p}(\tau+\tau^{-1/2}+1)(t-\tau)^{-1/2}d\tau \\ &\le C\varepsilon|\log\varepsilon|^{-1}\|f\|_{p,\varepsilon}t^{(1/2)-(1/p)}(t+t^{-1/2}+1) \end{aligned}$$

for $p>2$.

We fix $p>2$ and $t>0$. Then, by (2.16), (2.18) and the interpolation inequality, we see

$$\text{(2.19)}\qquad \|U_t^{(\varepsilon)}f-\chi_\varepsilon V_t^{(\varepsilon)}\hat{f}\|_{p,\varepsilon}$$

$$\leq \| U_t^{(\varepsilon)} f - \chi_\varepsilon V_t^{(\varepsilon)} \hat{f} \|_{1,\varepsilon}^{1/p} \| U_t^{(\varepsilon)} f - \chi_\varepsilon V_t^{(\varepsilon)} \hat{f} \|_{\infty,\varepsilon}^{1-(1/p)}$$
$$\leq C_t \varepsilon^{1/p} |\log \varepsilon|^{-1} \| f \|_{p,\varepsilon}.$$

Therefore we get the following.

Proposition 2.2. *Fix $p>2$ and $t>0$. Then, there exists a constant C_t independent of ε such that*

$$\| U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon \|_{p,\varepsilon} \leq C_t \varepsilon^{1/p} |\log \varepsilon|^{-1}$$

holds.

From (1.4) and Lemma A.2 in Appendix, we can see that $(V_t^{(\varepsilon)})^* = V_t^{(\varepsilon)}$ and $(U_t^{(\varepsilon)})^* = U_t^{(\varepsilon)}$. Thus, by the duality argument,

$$\| U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon \|_{p',\varepsilon} = \| U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon \|_{p,\varepsilon} \tag{2.20}$$

holds for any $p \in (2,\infty)$, where $p' = (1-1/p)^{-1}$. Furthermore, by Proposition 2.2, (2.20) and the Riesz-Thorin interpolation theorem,

$$\| U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon \|_{2,\varepsilon} \leq \| U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon \|_{p,\varepsilon} \tag{2.21}$$

holds for any $p \in (2,\infty)$.

From Proposition 2.2, (2.20) and (2.21), we can easily get Theorem 1.

3. Proof of Theorem 2

Throughout this section we assume that $n=3$. We recall (2.2). Then

$$\int_0^t W(x,w,\tau) d\tau_{|x \in \partial B_\varepsilon}$$
$$= \int_0^t (4\pi\tau)^{-3/2} \exp(-\varepsilon^2/4\tau) d\tau$$
$$= 4^{-1} \pi^{-3/2} \varepsilon^{-1} \int_{\varepsilon^2/4t}^\infty s^{-1/2} e^{-s} ds.$$

Since

$$\int_0^\infty s^{-1/2} e^{-s} ds = \pi^{1/2}$$

and

$$|\int_0^{\varepsilon^2/4t} s^{-1/2}e^{-s}ds| \leq \int_0^{\varepsilon^2/4t} s^{-1/2}ds = \varepsilon t^{-1/2}$$

hold, we have

$$\int_0^t W(x,w,\tau)d\tau_{|x\in\partial B_\varepsilon} = (4\pi\varepsilon)^{-1} + R_1(\varepsilon,t), \tag{3.1}$$

where

$$|R_1(\varepsilon,t)| \leq 4^{-1}\pi^{-3/2}t^{-1/2}.$$

We fix an arbitrary $f \in L^p(M_\varepsilon)$. We recall (2.7) and $L_n(\varepsilon) = 4\pi\varepsilon$ for $n=3$. Thus, by (2.7) and (3.1), we have

$$(V_t^{(\varepsilon)}\hat{f})(x)_{|x\in\partial B_\varepsilon} = \sum_{i=4}^{6} I_i(\varepsilon,t), \tag{3.2}$$

where

$$I_4(\varepsilon,t) = (U_t\hat{f})(x)_{|x\in\partial B_\varepsilon} - (U_t\hat{f})(w)$$

$$I_5(\varepsilon,t) = 4\pi\varepsilon \int_0^t W(x,w,\tau)(\int_0^\tau \frac{\partial}{\partial t}(U_{t-s}\hat{f})(w)ds)d\tau$$

$$I_6(\varepsilon,t) = -4\pi\varepsilon(R_1(\varepsilon,t)(U_t\hat{f})(w) + \int_0^t S(x,w,\tau)(U_{t-\tau}\hat{f})(w)d\tau)$$

for $x \in \partial B_\varepsilon$. By (3.1) and Lemma A.3 in Appendix,

$$\begin{aligned} &|I_6(\varepsilon,t)| \\ &\leq C\varepsilon(t^{-1/2}|(U_t\hat{f})(w)| + \int_0^t |(U_{t-\tau}\hat{f})(w)|d\tau) \\ &\leq C\varepsilon(t^{-(1/2)-(3/2p)}\|\hat{f}\|_p + \|\hat{f}\|_p \int_0^t (t-\tau)^{-3/2p}d\tau) \\ &\leq C\varepsilon t^{-3/2p}(t+t^{-1/2})\|f\|_{p,\varepsilon} \qquad (p>3/2), \end{aligned} \tag{3.3}$$

$$\begin{aligned} |I_4(\varepsilon,t)| &= \varepsilon|\nabla_x(U_t\hat{f})(w+\theta(x-w))|_{|x\in\partial B_\varepsilon} \\ &\leq C\varepsilon t^{-(3/2p)-(1/2)}\|f\|_{p,\varepsilon} \qquad (p>1) \end{aligned} \tag{3.4}$$

and

$$|\int_0^\tau \frac{\partial}{\partial t}(U_{t-s}\hat{f})(w)ds| \leq C\|f\|_{p,\varepsilon}\int_0^\tau (t-s)^{-1-(3/2p)}ds \qquad (p>1) \tag{3.5}$$

hold for $\tau\in(0,t)$, where $\theta\in(0,1)$ denotes some constant.

Next we want to estimate $I_5(\varepsilon,t)$. By (2.2), (3.2) and (3.5), we see

$$|I_5(\varepsilon,t)|\leq C\varepsilon\|f\|_{p,\varepsilon}I_7(\varepsilon,t), \tag{3.6}$$

where

$$I_7(\varepsilon,t)=\int_0^t\tau^{-3/2}\exp(-\varepsilon^2/4\tau)\Big(\int_0^\tau(t-s)^{-1-(3/2p)}ds\Big)d\tau.$$

Since

$$\begin{aligned}I_7(\varepsilon,t)&=\iint_{0\leq s\leq\tau\leq t}\tau^{-3/2}\exp(-\varepsilon^2/4\tau)(t-s)^{-1-(3/2p)}dsd\tau\\&=\int_0^t(t-s)^{-1-(3/2p)}\Big(\int_s^t\tau^{-3/2}\exp(-\varepsilon^2/4\tau)d\tau\Big)ds\end{aligned}$$

and

$$\begin{aligned}&\int_s^t\tau^{-3/2}\exp(-\varepsilon^2/4\tau)d\tau\\&=2\varepsilon^{-1}\int_{\varepsilon^2/4t}^{\varepsilon^2/4s}r^{-1/2}e^{-r}dr\\&\leq2\varepsilon^{-1}\int_{\varepsilon^2/4t}^{\varepsilon^2/4s}r^{-1/2}dr\\&=2(st)^{-1/2}(t^{1/2}-s^{1/2})\leq2(st)^{-1/2}(t-s)^{1/2}\end{aligned}$$

hold for $s\in(0,t)$, we have

$$\begin{aligned}I_7(\varepsilon,t)&\leq2t^{-1/2}\int_0^ts^{-1/2}(t-s)^{-(1/2)-(3/2p)}ds\\&=2t^{-(1/2)-(3/2p)}\int_0^1\tau^{-1/2}(1-\tau)^{-(1/2)-(3/2p)}d\tau\\&\leq Ct^{-(1/2)-(3/2p)}\end{aligned}$$

for $p>3$. Combining this inequality with (3.6), we get

$$|I_5(\varepsilon,t)|\leq C\varepsilon t^{-(1/2)-(3/2p)}\|f\|_{p,\varepsilon} \tag{3.7}$$

for $p>3$.

Summing up (3.2), (3.3), (3.4) and (3.7), we can get the following.

Proposition 3.1. *Fix $p>3$ and $t>0$. Then there exists a constant C independent of ε, t such that*

$$|(V_t^{(\varepsilon)}\hat{f})(x)|_{|x\in\partial B_\varepsilon}\leq C\varepsilon t^{-3/2p}(t+t^{-1/2})\|f\|_{p,\varepsilon}$$

holds for any $f\in L^p(M_\varepsilon)$.

By Proposition 3.1, Lemma A.5 in Appendix and the same argument as in section 2, we have

$$\|U_t^{(\varepsilon)}f-\chi_\varepsilon V_t^{(\varepsilon)}\hat{f}\|_{\infty,\varepsilon}\leq \sup_{x\in\partial B_\varepsilon}|(V_t^{(\varepsilon)}\hat{f})(x)| \leq C\varepsilon t^{-3/2p}(t+t^{-1/2})\|f\|_{p,\varepsilon} \tag{3.8}$$

and

$$\begin{aligned}\|U_t^{(\varepsilon)}f-\chi_\varepsilon V_t^{(\varepsilon)}\hat{f}\|_{1,\varepsilon} &\leq \int_0^t \sup_{x\in\partial B_\varepsilon}|(V_\tau^{(\varepsilon)}\hat{f})(z)|)(\int_{\partial B_\varepsilon}(\int_{M_\varepsilon}|\frac{\partial U^{(\varepsilon)}}{\partial \nu_z}(x,z,t-\tau)|dx)d\sigma_z)d\tau \\ &\leq C\varepsilon^3\|f\|_{p,\varepsilon}\int_0^t \tau^{-3/2p}(\tau+\tau^{-1/2})(t-\tau)^{-1/2}d\tau \\ &= C\varepsilon^3\|f\|_{p,\varepsilon}t^{(1/2)-(3/2p)}\int_0^1 s^{-3/2p}(ts+(ts)^{-1/2})(1-s)^{-1/2}ds \\ &\leq C\varepsilon^3 t^{(1/2)-(3/2p)}(t+t^{-1/2})\|f\|_{p,\varepsilon}\end{aligned} \tag{3.9}$$

for $p>3$.

From (3.8), (3.9) and the interpolation inequality (see (2.19)), we get the following.

Proposition 3.2. *Fix $p>3$ and $t>0$. Then there exists a constant C_t independent of ε such that*

$$\|U_t^{(\varepsilon)}-\chi_\varepsilon V_t^{(\varepsilon)}\chi_\varepsilon\|_{p,\varepsilon}\leq C_t\varepsilon^{1+(2/p)}$$

holds.

Furthermore, by the duality argument and the Riesz-Thorin interpolation theorem, we have (2.20) for any $p\in(3,\infty)$ and

$$\|U_t^{(\varepsilon)}-\chi_\varepsilon V_t^{(\varepsilon)}\chi_\varepsilon\|_{r,\varepsilon}\leq \|U_t^{(\varepsilon)}-\chi_\varepsilon V_t^{(\varepsilon)}\chi_\varepsilon\|_{p,\varepsilon} \tag{3.10}$$

for any $p\in(3,\infty)$ and $r\in[3\backslash 2, 3]$.

From Proposition 3.2, (2.20) and (3.10), we can easily get Theorem 2.

4. Appendix

Let M, M_ε, $U(x,y,t)$, $U^{(\varepsilon)}(x,y,t)$ be as in Introduction. See Friendman [1] for the fundamental properties of the heat kernel. We have the following.

Lemma 1.1. *There exists a constant C independent of x,y,t such that*

$$0 \le U(x,y,t) \le Ct^{-n/2}\exp(-|x-y|^2/Ct) \tag{A.1}$$

$$\left|\frac{\partial U}{\partial x_i}(x,y,t)\right| \le Ct^{-(n+1)/2}\exp(-|x-y|^2/Ct) \qquad (1 \le i \le n) \tag{A.2}$$

$$\left|\frac{\partial U}{\partial t}(x,y,t)\right| \le Ct^{-(n+2)/2}\exp(-|x-y|^2/Ct) \tag{A.3}$$

hold for $x,\ y \in \bar{M}$, $t>0$.

Lemma A.2. *We have*

$$U(x,y,t) = U(y,x,t) \qquad x,\ y \in \bar{M},\ t>0$$

and

$$U^{(\varepsilon)}(x,y,t) = U^{(\varepsilon)}(y,x,t) \qquad x,\ y \in \bar{M}_\varepsilon,\ t>0.$$

Let U_t be as in (1.2). Then we have the following.

Lemma A.3. *Fix $p \in (1,\infty)$. Then there exists a constant C independent of t such that*

$$\sup_{x\in\bar{M}} |(U_t f)(x)| \le Ct^{-n/2p}\|f\|_p \tag{A.4}$$

$$\sup_{x\in\bar{M}} \left|\frac{\partial}{\partial x_i}(U_t f)(x)\right| \le Ct^{-(n/2p)-(1/2)}\|f\|_p \qquad (1 \le i \le n) \tag{A.5}$$

$$\sup_{x\in\bar{M}} \left|\frac{\partial}{\partial t}(U_t f)(x)\right| \le Ct^{-(n/2p)-1}\|f\|_p \tag{A.6}$$

hold for $f \in L^p(M)$ and $t>0$.

Proof. We take an arabitrary $x \in \bar{M}$. Then, by (1.2), (A.1) and using the transformation of co-ordinates : $y = x + (Ct)^{1/2}z$, we have

$$
\begin{aligned}
|(U_t f)(x)| &\leq Ct^{-n/2}\int_M \exp(-|x-y|^2/Ct)|f(y)|dy \\
&\leq Ct^{-n/2}\|f\|_p\Big(\int_M \exp(-|x-y|^2/Ct)dy\Big)^{1/p'} \\
&\leq Ct^{-(n/2)+(n/2p')}\|f\|_p\Big(\int_{R^n} \exp(-|z|^2)dz\Big)^{1/p'} \\
&\leq Ct^{-n/2p}\|f\|_p,
\end{aligned}
$$

where $(1/p)+(1/p')=1$.

Therefore we get (A.4). By the same argument as above, we get (A.5) and (A.6) from (A.2) and (A.3), respectively.

q.e.d

By $B(r;w)$ we denote a ball of radius $r>0$ with the center w. And we write $B_r=B(r;w)$ as before.

Lemma A.4. *There exists a constant C independent of ε, x, t such that*

$$
0\leq -\frac{\partial U^{(\varepsilon)}}{\partial v_z}(x,z,t)\leq Ct^{-(n+1)/2}\exp(-|x-z|^2/Ct) \tag{A.7}
$$

hold for $z\in\partial B_\varepsilon$, $x\in M_\varepsilon$ and $t>0$.
Here $\partial/\partial v_z$ denotes the derivative along the exterior normal direction with respect to M_ε.

Proof. Let $F^{(r)}(x,y,t)$ be the fundamental solution of the heat equation in $R^n\backslash\bar{B}_r$ under the Dirichlet condition on ∂B_r. Then we have the following identity.

$$
F^{(\varepsilon)}(x,y,t)=F^{(1)}(\varepsilon^{-1}x,\varepsilon^{-1}y,\varepsilon^{-2}t)\varepsilon^{-n} \qquad x,\ y\in R^n\backslash B_\varepsilon,\ t>0
$$

Thus,

$$
\frac{\partial F^{(\varepsilon)}}{\partial y_i}(x,y,t)=\Big(\frac{\partial F^{(1)}}{\partial y_i}\Big)(\varepsilon^{-1}x,\varepsilon^{-1}y,\varepsilon^{-2}t)\varepsilon^{-(n+1)} \qquad (1\leq i\leq n)
$$

holds for x, $y\in R^n\backslash B_\varepsilon$, $t>0$.
It is well known that there exists a constant C such that

$$
|F^{(1)}(x,y,t)|\leq Ct^{-n/2}\exp(-|x-y|^2/Ct)
$$

$$
\Big|\frac{\partial F^{(1)}}{\partial y_i}(x,y,t)\Big|\leq Ct^{-(n+1)/2}\exp(-|x-y|^2/Ct) \qquad (1\leq i\leq n)
$$

hold for $x,\ y \in R^n \backslash B_1,\ t>0$. Thus, we get

$$|\frac{\partial F^{(\varepsilon)}}{\partial y_i}(x,y,t)| \leq Ct^{-(n+1)/2}\exp(-|x-y|^2/Ct) \qquad (1 \leq i \leq n) \tag{A.8}$$

for $x,\ y \in R^n \backslash B_\varepsilon,\ t>0$.
It should be remarked that the constant C in (A.8) does not depend on ε.
Let $H^{(\varepsilon)}(x,y,t)=F^{(\varepsilon)}(x,y,t)-U^{(\varepsilon)}(x,y,t)$. Then, $H^{(\varepsilon)}(x,y,t)$ satisfies the following.

$$\begin{cases} (\partial_t-\Delta_x)H^{(\varepsilon)}(x,y,t)=0 & x,\ y \in M_\varepsilon,\ t>0 \\ H^{(\varepsilon)}(x,y,t)=F^{(\varepsilon)}(x,y,t) \geq 0 & x \in \partial M,\ y \in M_\varepsilon,\ t>0 \\ H^{(\varepsilon)}(x,y,t)=0 & x \in \partial B_\varepsilon,\ y \in M_\varepsilon,\ t>0 \\ \lim_{t \to 0} H^{(\varepsilon)}(x,y,t)=0 & x,\ y \in M_\varepsilon \end{cases}$$

By the maximum principle, $H^{(\varepsilon)}(x,y,t) \geq 0$ holds for $x,y \in \bar{M}_\varepsilon,\ t>0$. Therefore,

$$0 \leq U^{(\varepsilon)}(x,y,t) \leq F^{(\varepsilon)}(x,y,t) \tag{A.9}$$

hold for $x,y \in \bar{M},\ t>0$.

We fix an arbitrary $z \in \partial B_\varepsilon$. Then, $\nu_z=-(z-w)/|z-w|$ denotes the exterior normal unit vector at $z \in \partial B_\varepsilon$ with respect to M_ε. We recall that

$$U^{(\varepsilon)}(x,z,t)=F^{(\varepsilon)}(x,z,t)=0 \tag{A.10}$$

for $x \in M_\varepsilon,\ t>0$. Therefore, by (A.9) and (A.10),

$$\begin{aligned} 0 &\leq (U^{(\varepsilon)}(x,z-h\nu_z,t)-U^{(\varepsilon)}(x,z,t))/h \\ &\leq (F^{(\varepsilon)}(x,z-h\nu_z,t)-F^{(\varepsilon)}(x,z,t))/h \end{aligned} \tag{A.11}$$

hold for $x \in M_\varepsilon,\ t>0$ and any sufficiently small $h>0$. Letting $h \downarrow 0$ in (A.11), we have

$$0 \leq -\frac{\partial U^{(\varepsilon)}}{\partial \nu_z}(x,z,t) \leq -\frac{\partial F^{(\varepsilon)}}{\partial \nu_z}(x,z,t)$$

for $x \in M_\varepsilon,\ t>0$.
Combining this inequality with (A.8),

$$\begin{aligned} 0 &\leq -\frac{\partial U^{(\varepsilon)}}{\partial \nu_z}(x,z,t) \leq |(\nabla_z F^{(\varepsilon)})(x,z,t)| \\ &\leq Ct^{-(n+1)/2}\exp(-|x-z|^2/Ct) \end{aligned}$$

hold for $x \in M_\varepsilon$, $t>0$.

Therefore we get (A.7).

q.e.d.

Now we have teh following.

Lemma A.5. *There exists a constant C independent of ε, t such that*

$$\int_{\partial B_\varepsilon}\left(\int_{M_\varepsilon}\left|\frac{\partial U^{(\varepsilon)}}{\partial \nu_z}(x,z,t)\right|dx\right)d\sigma_z \leq C\varepsilon^{n-1}t^{-1/2} \tag{A.12}$$

holds for $t>0$.

Proof. We fix an arbitrary $z \in \partial B_\varepsilon$. Then, by (A.7) and using the transformation of co-ordinates : $x = z + (Ct)^{1/2}y$,

$$\begin{aligned}\int_{M_\varepsilon}\left|\frac{\partial U^{(\varepsilon)}}{\partial \nu_z}(x,z,t)\right|dx &\leq Ct^{-(n+1)/2}\int_{M_\varepsilon}\exp(-|x-z|^2/Ct)dx \\ &\leq Ct^{-1/2}\int_{R^n}\exp(-|y|^2)dy \\ &\leq Ct^{-1/2}\end{aligned}$$

hold for $t>0$. Here C denotes some different positive constants independent of ε, t. Integrating this inequality on ∂B_ε, we can immediately get (A.12).

q.e.d.

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