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UNIFORM ESTIMATES FOR FUNDAMENTAL SOLUTIONS ASSOCIATED WITH NON-LOCAL DIRICHLET FORMS

Dedicated to Professor Masatoshi Fukushima for his 60th birthday

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0. Introduction

The generator of a Markov process of jump type is a pseudo-differential operator of non-local type. If the pseudo-differential operator satisfies a certain regularity condition, the fundamental solution of the parabolic equation associated with the operator can be constructed by the general theory of pseudo-differential operators, and properties of the solution are investigated by the theory ([8] Chap.7 §4, [9], [11]). However, in the case where the elements of the operator are discontinuous, we need another theory.

Any symmetric stable process is associated with a Dirichlet form of non-local type. In Komatsu [7] we proved that, if a Dirichlet form is bounded from above and below by Dirichlet forms associated with stable processes of the same index, there exists a strong Feller process associated with the Dirichlet form. We shall call the strong Feller process a stable type process. More specifically, we obtained some uniform estimates for the transition functions of stable type processes which are fundamental solutions of parabolic equations in the weak sense associated with Dirichlet forms of non-local type. We note that Carlen-Kusuoka-Stroock [3] studied upper bounds for transition functions in a more ganeral context.

In this paper, we still work with the transition functions of stable type processes and present, among others, a lower estimate and a uniform Hölder estimate for them, which we were unable to obtain in the previous paper [7]. Our lower estimate is almost the same as the one naturally expected from the Aronson estimate in [1] and [2]. These results can be proved through some improvements of the proof in [7]. Actually we employ a wider class of stable type processes than [7] and prove that the uniform estimates of their transition functions similar to those in [7] remain valid for this enlarged class. Finally we examine those examples where the Dirichlet forms are expressed as integrals of bilinear forms involving pseudo-differential operators. This type of forms has been considered by Jacob ([4], [5], [6]). An example of this type will indicate the necessity of the employment of the present enlarged class of stable type processes.

1. Notations and theorems

The Dirichlet form of non-local type associated with a measure $K(t,dx,dy)$ can be written in the form

$$\mathscr{E}_t(f,g)=\iint(f(x)-f(y))(g(x)-g(y))K(t,dx,dy). \tag{1.1}$$

We shall consider only the case where the measure K has the form

$$K(t,dx,dy)=\frac{1}{2}k(t,x,y)|x-y|^{-d-\alpha}dxdy,$$

where $0<\alpha<2$, $(t,x,y)\in \boldsymbol{R}_+\times \boldsymbol{R}^d\times \boldsymbol{R}^d$ and $k(t,x,y)$ is a measurable function satisfying $k(t,x,y)=k(t,y,x)$ and

$$\lim_{s\to t}\sup_{|x|+|y|<N}|k(s,x,y)-k(t,x,y)|=0 \tag{1,2}$$

for any t and N. Let $\boldsymbol{K}^\gamma=\boldsymbol{K}^\gamma[c_1,c_2,c_3]$ denote the class of functions k satisfying the above mentioned properties and that, for all $(t,x,y)\in \boldsymbol{R}_+\times \boldsymbol{R}^d\times \boldsymbol{R}^d$,

$$c_1\le k(t,x,y)\le c_2+c_3t^{-\gamma/\alpha}|x-y|^\gamma, \tag{1.3}$$

where $0<\gamma\le\alpha/2$ and c_1,c_2,c_3 are positive constants.

A function $u_t(x)=u(t,x)$ is said to be a weak solution of the parabolic equation

$$(d/dt)(u_t,\cdot)_{L^2}=-\mathscr{E}_t(u_t,\cdot) \tag{1.4}$$

associated with the Dirichlet form $\mathscr{E}_t$ if

$$\sup_{s<\tau<t}(u_\tau,u_\tau)_{L^2}+\int_s^t\mathscr{E}_\tau(u_\tau,u_\tau)d\tau<\infty,$$

$$(u_t,f)_{L^2}-(u_s,f)_{L^2}+\int_s^t\mathscr{E}_\tau(u_\tau,f)d\tau=0$$

for any $0<s<t$ and any test function f on $\boldsymbol{R}^d$, where $(\cdot,\cdot)_{L^2}$ denotes the usual inner product of the Hilbert space $L^2(\boldsymbol{R}^d,dx)$.

Theorem 1. *Let $k(t,x,y)$ be a function in a class $\boldsymbol{K}^\gamma$ and $\mathscr{E}_t(\cdot,\cdot)$ denote the Dirichlet form associated with the function k. Then there exists uniquely a function $S(s,x;t,y)$ such that $S(s,x;t,y)=S(s,y;t,x)>0$ and, for any ϕ in L^2, the function*

$$u_t(y)=\int \phi(x)S(s,x;t,y)dx$$

is a weak solution of the parabolic equation $(d/dt)(u_t,\cdot)_{L^2}=\mathscr{E}_t(u_t,\cdot)$ *associated with the Dirichlet form* $\mathscr{E}_t$ *satisfying the initial condition*

$$\|u_t-\phi\|_{L^2}\to 0 \quad \text{as } t\downarrow s.$$

This function $S(s,x;t,y)$ *is continuous on the set* $\{(s,x,t,y);\ s<t,\ x\in \boldsymbol{R}^d$ *and* $y\in \boldsymbol{R}^d\}$.

The function $S(s,x;t,y)$ in Theorem 1 is called the fundamental solution of equation (1.4). Under the additional assumption that $(\partial/\partial x)^\mu(\partial/\partial y)^\nu k(t,x,y)$ are bounded and continuous on $\boldsymbol{R}_+\times\boldsymbol{R}^d\times\boldsymbol{R}^d$ for all $\mu,\ \nu\in\boldsymbol{Z}^d_+$, Theorem 1 is proved by the theory of pseudo-differential operators (cf. [8]). And in this case functions $(\partial/\partial x)^\mu(\partial/\partial y)^\nu S(s,x;t,y)$ are continuous on $\{(s,x,t,y);\ s<t,\ x\in\boldsymbol{R}^d$ and $y\in\boldsymbol{R}^d\}$. This fact is a base of our arguments. Theorem 1 in the general case, however, is a consequence of other theorems in this section.

We shall fix positive constants α, γ, c_1, c_2 and c_3, and use the convention of letting c's stand for positive constants depending only on these constants. Each c may denote a constant different from other c's. Let k be a function in the class $\boldsymbol{K}^\gamma[c_1,c_2,c_3]$. Without calling attention on each occasion, let $\mathscr{E}_t$ denote the Dirichlet form associated with the function k and $S(s,x;t,y)$, the fundamental solution of the parabolic equation associated with $\mathscr{E}_t$. Now set

$$T_t(y)=T(t,y)=S(0,0;t,y). \tag{1.5}$$

The purpose of this paper is to obtain a lower estimate and a continuity estimate for the function $T_t(y)$ depending only on constants α, γ, c_1, c_2 and c_3. The following theorem will be proved easily.

Theorem 2. *Let* $k\in\boldsymbol{K}^\gamma$. *Then*

$$T_t(y)+\int_t^\infty \mathscr{E}_t(T_\tau,T_\tau)d\tau\le ct^{-d/\alpha}. \tag{1.6}$$

In the case $0<\gamma<\alpha/2$, choose and fix a constant β satisfying $\alpha/2<\beta<(\alpha-\gamma)\wedge 1$, and define $r(\sigma)=\sigma^\beta$. In the case $\gamma=\alpha/2$, let $r(\sigma)$ denote the function defined by

$$r(\sigma)=\sigma^{\alpha/2}/(1+\kappa(\log\sigma)^2),\quad \kappa=(\alpha\wedge(2-\alpha))^2/6. \tag{1.7}$$

Then the function $r(\sigma)$ is a concave function on $\boldsymbol{R}_+$. We claim no more upperestimate of the function $T_t(y)$ than Theorem 2. However, the following moment bound plays the role of an upper estimate in the proof of the cotinuity estimate in this paper.

Theorem 3. *Let $k\in \boldsymbol{K}^{\gamma}$. Then*

$$\int r(t^{-1/\alpha}|y|)T_t(y)dy\leq c. \tag{1.8}$$

In view of the Aronson estimate [2], it would be expected that the function $T_t(y)$ satisfies the lower estimate

$$T_t(y)\geq ct^{-d/\alpha}\exp[-ct^{-1}|y|^{\alpha}].$$

Though the above estimate is nothing but a conjecture, the following rather loose lower estimate can be proved. Define a function $\Psi(\sigma)$ by

$$\Psi(\sigma)=\exp[-\sigma^{\alpha}\log(e+\sigma)]. \tag{1.9}$$

Theorem 4. *Let $k\in \boldsymbol{K}^{\gamma}$. Then*

$$T_t(y)\geq ct^{-d/\alpha}\Psi(t^{-1/\alpha}|y|)^c>0. \tag{1.10}$$

It needs no explanation to see that, if there is a strictly positive function $\Psi_0(\sigma)$ on $\boldsymbol{R}_+$ which is decreasing and depends only on α, γ, c_1, c_2 and c_3 such that

$$\int S(0,x_1;t,y)\wedge S(0,x_2;t,y)dy\geq \Psi_0(t^{-1/\alpha}|x_1-x_2|) \tag{1.11}$$

and if the moment bound (1.8) with $r(\sigma)=\sigma^{\beta}$, $0<\beta<1$, is satisfied, then the Hölder continuity estimate can be proved by the Nash method [10]. Since overlap estimate (1.11) is immediate from Theorem 4, we have the following theorem.

Theorem 5. *If $\gamma<\alpha/2$ and $k\in \boldsymbol{K}^{\gamma}$, then*

$$|T_\tau(\eta)-T_t(y)|\leq cs^{-d/\alpha}\left(\frac{(\tau-t)+|\eta-y|^{\alpha}}{t-s}\right)^c \tag{1.12}$$

for all $0<s<t<\tau$ and $y,\eta\in \boldsymbol{R}^d$.

From the point of view of pseudo-differential operator theory, it seems significant to investigate the case $\gamma=\alpha/2$ (see Section 4). Since moment bound (1.8) is obtained only for the function $r(\sigma)$ given by (1.7), we cannot prove the Hölder continuity estimate in a similar way to the proof of the above theorem. But the function $T_t(y)$ satisfies a certain continuity estimate in this case. Note that moment bound (1.8) is quite similar to the moment bound in [7] (Theorem 1), where the function $r(\sigma)$ is defined by

$$r(\sigma)=\sigma^{\alpha/2}/(1+c(\log\sigma)^2)^{1/2}$$

in place of (1.7). In the present case, we can work with the functions

(1.13) $$J(\sigma)=\sigma(1+(\log\sigma)^2),\quad \hat{J}(\sigma)=\sigma(1+(\log\sigma)^2)^{-1}$$

in place of functions $J(\sigma)$ and $\hat{J}(\sigma)$ in [7], §4. Since the functions $J(\sigma)$ and $\hat{J}(\sigma)$ given by (1.13) satisfy inequalities

$$cr(\sigma)\le\hat{J}(\sigma^{\alpha/2})\le cr(\sigma),$$

$$J(\sigma\sigma')\le cJ(\sigma)J(\sigma'),\qquad \hat{J}(\sigma\sigma')\le cJ(\sigma)\hat{J}(\sigma'),$$

$$c\sigma\le\hat{J}(J(\sigma)),\quad J(J(\sigma))\le c\sigma(1+(\log\sigma)^4),$$

it is easy to check that the following theorem can be proved in almost the same way as that in [7], §4. Define a function $\Phi(\sigma)$ by

(1.14) $$\Phi(\sigma)=\begin{cases}\exp\left[-\dfrac{\log(1/\sigma)}{\log\log(1/\sigma)}\right] & (0<\sigma<e^{-e})\\ e^{-e} & (e^{-e}\le\sigma).\end{cases}$$

Theorem 6. *If $\gamma=\alpha/2$ and $k\in\boldsymbol{K}^\gamma$, then*

(1.15) $$|T_\tau(\eta)-T_t(y)|\le cs^{-d/\alpha}\Phi\left(\frac{(\tau-t)+|\eta-y|^\alpha}{t-s}\right)^c$$

for all $0<s<t<\tau$ and $y,\eta\in\boldsymbol{R}^d$.

As was mentioned before, these theorems are proved at first under the additional assumption that $(\partial/\partial x)^\mu(\partial/\partial y)^\nu k(t,x,y)$ are bounded and continuous for all $\mu,\nu\in\boldsymbol{Z}^d_+$, and the assumption is removed afterward by making use of these proved theorems. We shall explain briefly about it, for it was shown in [7], §5. Let $\{\delta_m(x)\}$ be a sequence of non-negative test functions on $\boldsymbol{R}^d$ such that the support of δ_m decreases to $\{0\}$ as $m\to\infty$ and $\int\delta_m(x)dx=1$. Let $k\in\boldsymbol{K}^\gamma$ and define

$$k_m(t,x,y)=\iint k(t,[\xi]_m,[\eta]_m)\delta_m(x-\xi)\delta_m(y-\eta)d\xi d\eta,$$

where $[\xi]_m=\xi I_{(|\xi|<m)}$ and $[\eta]_m=\eta I_{(|\eta|<m)}$. From condition (1.2), each function k_m satisfies the above mentioned additional assumption, $k_m\in\boldsymbol{K}^\gamma$ and

(1.16) $$\lim_{m\to\infty}\sup_{s<\tau<t}\iint_{|x|+|y|<N}(k_m(\tau,x,y)-k(\tau,x,y))^2dxdy=0$$

for any $s<t$ and N. Let $\mathscr{E}_{m,t}(\cdot,\cdot)$ be the Dirichlet form associated with the function k_m and $S_m(s,x;t,y)$ be the fundamental solution of the parabolic equation associated with $\mathscr{E}_{m,t}$. We see from Theorem 2 and Theorem 5 in the case $\gamma<\alpha/2$ or Theorem 6 in the case $\gamma=\alpha/2$ that the functions $\{S_m(s,x;t,y)\}$ are uniformly bounded and equi-continuous on any compact subset of $\{(s,x,t,y);\ s<t,\ x\in \boldsymbol{R}^d \text{ and } y\in \boldsymbol{R}^d\}=D$. From the Ascoli-Arzera theorem, choosing a subsequence $\{m(n)\}\subset\{m\}$ if necessary, we may suppose that $\{S_n\}$ converges to a certain function $S=S(s,x;t,y)$ as $n\to\infty$ locally uniformly on the set D. Then from Theorem 3

$$\int S(s,x;t,y)dy=1, \qquad S(s,x;t,y)\to\delta(y-x) \text{ as } t\downarrow s.$$

Using (1.16) and Theorem 1, it can be shown that, if $\phi\in L^2$, then the function

$$u_t(y)=\int\phi(x)S(s,x;t,y)dx$$

is a weak solution of the parabolic equation associated with $\mathscr{E}_t$. Since the weak solution to the Cauchy problem for the parabolic equation is uniquely determined, we see that the function $S(s,x;t,y)$ is the fundamental solution of the parabolic equation associated with $\mathscr{E}_t$. Therefore Theorem 1 remains valid without the additional assumption. Theorem 2 also remains valid from the Fatou lemma. Obviously Theorem 3, 4, 5 and 6 still hold without the additional assumption.

Here we shall remark that the function $S(s,x;t,y)$ is a strong Feller transition function. In fact, by making use of Theorem 3, for any $|x|>2N$,

$$\begin{aligned}\int_{|y|<N} S(s,x;t,y)dy &\le \int_{|y-x|>|x|/2} S(s,x;t,y)dy\\ &\le r(|x|/2(t-s)^{1/\alpha})^{-1}\int r(|y-x|/(t-s)^{1/\alpha})S(s,x;t,y)dy\\ &\le c\, r(|x|/2(t-s)^{1/\alpha})^{-1}.\end{aligned}$$

Therefore the transition function $S(s,x;t,y)$ maps each bounded measurable function on $\boldsymbol{R}^d$ with compact support to a continuous function vanishing at the infinity point.

Finally we shall consider the case where the function $k(t,x,y)=k(x,y)$ is independent of the time parameter t and it satisfies the inequality

$$c_1\le k(x,y)\le c_2+c_3|x-y|^\gamma \tag{1.17}$$

for all $x,y\in\boldsymbol{R}^d$, where $0<\gamma\le\alpha/2$ and c_1, c_2 and c_3 are positive constants. Let τ be a positive constant and set

$$k^\tau(t,x,y)=(((2-t/\tau)\vee 0)\wedge 1)k(x,y)+(((t/\tau-1)\vee 0)\wedge 1)c_1.$$

Then $k^\tau(t,x,y)=k(x,y)$ for $t\le\tau$ and $k^\tau\in \boldsymbol{K}^\gamma[c_1,c_2,c_3']$ with $c_3'=c_3(2\tau)^{\gamma/\alpha}$. Let $S^\tau(s,x;t,y)$ be the fundamental solution connected with the function $k^\tau(t,x,y)$. In place of theorems stated above, we shall consider modified theorems in which the parameter set is restricted to the bounded interval $[0,\tau]$ and the class $\boldsymbol{K}^\gamma[c_1,c_2,c_3]$ is not the original one but the set of symmetric functions $k(x,y)$ satisfying (1.17). Applying the original theorems to functions k^τ and S^τ, we see that the modified theorems also hold good.

2. Upper and moment estimates

In this section we shall prove Theorem 2 and Theorem 3 assuming that $(\partial/\partial x)^\mu(\partial/\partial y)^\nu k(t,x,y)$ are bounded and continuous on $\boldsymbol{R}_+\times\boldsymbol{R}^d\times\boldsymbol{R}^d$ for any $\mu,\nu\in\boldsymbol{Z}^d_+$. Let A_t be the operator defined by

$$A_tf(x)=\int(f(x+z)-f(x)-[z]_1\cdot\partial f(x))k(t,x,x)|z|^{-d-\alpha}dz$$
$$+\int(f(x+z)-f(x))(k(t,x,x+z)-k(t,x,x))|z|^{-d-\alpha}dz$$

for smooth bounded function f on $\boldsymbol{R}^d$, where $[z]_1=zI_{(|z|\le 1)}$ and $\partial=\partial_x=((\partial/\partial x_1),\cdots,(\partial/\partial x_d))$. Then we have

$$\mathscr{E}_t(f,g)=-(A_tf,g)_{L^2}. \tag{2.1}$$

It is known that functions $T_t(y)$ and $(\partial/\partial y_j)T_t(y)$ are continuously differentiable on $(0,\infty)\times\boldsymbol{R}^d$ and $(\partial/\partial t)T_t(y)=A_tT_ty$, morever

$$\sup_{s\le\tau\le t}\int(T_t(y)^2+|\partial T_t(y)|^2)dy<\infty$$

for any $0<s<t$.

Throughout this section let $\rho(x)$ be a smooth function such that $0\le\rho(x)\le 1$, $\rho(x)=1$ for $|x|\le 1$ and $\rho(x)=0$ for $|x|\ge 2$, and $\rho(x)=\rho(-x)$. Set

$$\langle\xi\rangle=\left(\int(1-e^{-i\xi\cdot z}\rho(z))|z|^{-d-\alpha}dz\right)^{1/\alpha}.$$

Then we have $c\le\langle\xi\rangle/(1+|\xi|)\le c$ and

$$\iint|f(x)-f(y)|^2\rho(x-y)|x-y|^{-d-\alpha}dxdy=c\int\langle\xi\rangle^\alpha|\mathscr{F}f(\xi)|^2d\xi \tag{2.2}$$

for any function f on $\boldsymbol{R}^d$, where $\mathscr{F}$ denotes the Fourier transform. Define $E(t)=(T_t,T_t)_{L^2}$ for $t>0$. It is easy to show that

$$(2.3) \qquad (d/dt)E(t) = -2\mathscr{E}_t(T_t, T_t).$$

Lemma 2.1. *Assume that the function k is non-negative and satisfies the inequality $k(t,x,y) \geq c_1$ on the set $\{(t,x,y);\ |x-y|<2\}$. Then there exists a constant C depending only on α and c_1 such that $E(t) \leq Ct^{-d/\alpha}$.*

Proof. From the assumption and (2.2) we have

$$-(d/dt)E(t) = 2\mathscr{E}_t(T_t, T_t) \geq c\int \langle\xi\rangle^{\alpha}|\mathscr{F}T_t(\xi)|^2 d\xi.$$

For any $\lambda > 0$,

$$\begin{aligned} E(t) &= (2\pi)^{-d}\int |\mathscr{F}T_t(\xi)|^2 d\xi \\ &\leq c\int I_{(\langle\xi\rangle \leq \lambda)} d\xi + c\int I_{(\langle\xi\rangle \geq \lambda)}|\mathscr{F}T_t(\xi)|^2 d\xi \\ &\leq c\lambda^d + c\lambda^{-\alpha}\int \langle\xi\rangle^{\alpha}|\mathscr{F}T_t(\xi)|^2 d\xi. \end{aligned}$$

Choosing the best constant λ, we have

$$\int \langle\xi\rangle^{\alpha}|\mathscr{F}T_t(\xi)|^2 d\xi \geq cE(t)^{1+\alpha/d}.$$

Using these inequalities,

$$(d/dt)(E(t)^{-\alpha/d}) = -\frac{\alpha}{d}E(t)^{-1-\alpha/d}(d/dt)E(t) \geq c.$$

Since $E(+0)^{-1} = 0$, we have the inequality $E(t)^{-\alpha/d} \geq ct$. q.e.d.

Theorem 1 is immediate from Lemma 2.1. The lemma implies that

$$\|S(s,x;t,\cdot)\|_{L^2} \leq c(t-s)^{-d/2\alpha}.$$

Therefore

$$\begin{aligned} T_t(y) &= \int S(0,0;t/2,x)S(t/2,x;t,y)dx \\ &\leq \|S(0,0;t/2,\cdot)\|_{L^2}\|S(t/2,y;t,\cdot)\|_{L^2} \leq ct^{-d/\alpha}. \end{aligned}$$

From (2.3) and Lemma 2.1

$$\int_t^\infty \mathscr{E}_t(T_t,T_t)d\tau = E(t)/2 \leq ct^{-d/\alpha}.$$

Hence we obtain estimate (1.6).

For a while, let $f_n(x)$ be the convolution of functions $(|x| \wedge n)^\delta$ and $\rho(x)$ for a fixed constant δ, $0<\delta<\alpha$. We see that the second order derivatives of the function f_n are bounded and that

$$|(\partial/\partial x_j)f_n(x+z)| \leq cf_n(x) \qquad (|z|<1),$$
$$f_n(x+z) \leq c(f_n(x)+|z|^\delta).$$

Then

$$\begin{aligned}
A_tf_n(x) &= k(t,x,x)\int_{|z|\leq 1}(f_n(x+z)-f_n(x)-z\cdot\partial f_n(x))|z|^{-d-\alpha}dz \\
&\quad + \int_{|z|\leq 1}(f_n(x+z)-(f_n(x))(k(t,x,x+z)-k(t,x,x))|z|^{-d-\alpha}dz \\
&\quad + \int_{|z|>1}(f_n(x+z)-(f_n(x))k(t,x,x+z)|z|^{-d-\alpha}dz \\
&\leq c_2c\int_{|z|\leq 1}|z|^{2-d-\alpha}dz \\
&\quad + (\sup_y|\partial_yk(t,x,y)|)\,c\int_{|z|\leq 1}|z|^{2-d-\alpha}dz \\
&\quad + (\sup_y|k(t,x,y)|)\,c\int_{|z|>1}(f_n(x)+|z|^\delta)|z|^{-d-\alpha}dz \\
&\leq a+af_n(x),
\end{aligned}$$

where a is a constant independent of n. Since $f_n(x)=n^\delta$ for $|x|>n+2$, we see that

$$\begin{aligned}
\int f_n(x)T_t(x)dx &= f_n(0)+\int_0^t(\int A_sf_n(x)\cdot T_s(x)dx)ds \\
&\leq c+\int_0^t(a+a\int f_n(x)T_s(x)dx)ds.
\end{aligned}$$

This inequality implies that

$$\int f_n(x)T_t(x)dx \leq ce^{at}.$$

Letting $n\to\infty$, we see that if $0<\delta<\alpha$, then

(2.4)
$$\sup_{0\le\tau\le t}\int|y|^{\delta}T_t(y)dy<\infty.$$

Recall that the concave function $r(\sigma)$ is defined in the case $0<\gamma<\alpha/2$ by $r(\sigma)=\sigma^{\beta}$ with a fixed constant β satisfying $\alpha/2<\beta<(\alpha-\gamma)\wedge 1$, and in the case $\gamma=\alpha/2$ by (1.7). Define a function $M(t)$ by

(2.5)
$$M(t)=\int r(|y|)T_t(y)dy.$$

The following lemma is essential to prove Theorem 3.

Lemma 2.2. (i) *In the case* $0<\gamma<\alpha/2$ *and* $k\in\boldsymbol{K}^{\gamma}$,

(2.6)
$$(d/dt)M(t)\le ct^{\beta/\alpha-1/2}\mathscr{E}_t(T_t,\log T_t)^{1/2}+ct^{\beta/\alpha-1}.$$

(ii) *In the case* $\gamma=\alpha/2$ *and* $k\in\boldsymbol{K}^{\gamma}$,

(2.7)
$$(d/dt)M(t)\le c\mathscr{E}_t(T_t,\log T_t)^{1/2}+ct^{-1/2}.$$

Proof. Let $\rho_n(x)=\rho(x/n)$ and $r_n(x)=r(|x|)\rho_n(x)$. Then

$$M(t)-M(s)=-\lim_{n\to\infty}\int_s^t\mathscr{E}_t(r_n,T_\tau)d\tau.$$

Let $[X]_+$ denote the positive part of X. Set $b=t^{1/\alpha}$. Since

$$([1-T_t(x)/T_t(y)]_+)^2\le[1-T_t(x)/T_t(y)]_+\log(T_t(y)/T_t(x))$$

and $k\in\boldsymbol{K}^{\gamma}[c_1,c_2,c_3]$,

$$\begin{aligned}
&|\mathscr{E}_t(r_n,T_t)|/2\\
&\le\iint|r_n(x)-r_n(y)|[1-T_t(x)/T_t(y)]_+T_t(y)K(t,dx,dy)\\
&\le\Big(\iint_{|z|\le b}|r_n(y-z)-r_n(y)|^2T_t(y)K\Big)^{1/2}\mathscr{E}_t(T_t,\log T_t)^{1/2}\\
&\quad+\iint_{|z|>b}|r_n(y-z)-r_n(y)|T_t(y)K\\
&\le c\,(I_{n,1})^{1/2}\mathscr{E}_t(T_t,\log T_t)^{1/2}+cI_{n,2},
\end{aligned}$$

where

$$I_{n,1}=\iint_{|z|\le b}|r_n(y-z)-r_n(y)|^2|z|^{-d-\alpha}T_t(y)dydz,$$

$$I_{n,2}=\iint_{|z|>b}|r_n(y-z)-r_n(y)||z|^{-d-\alpha+\gamma}b^{-\gamma}T_t(y)dydz,$$

Since the function $r(\sigma)$ is concave,

$$I_{n,1}\leq 2\iint_{|z|\leq b}(r(|y-z|)-r(|y|))^2|z|^{-d-\alpha}T_t(y)dydz$$
$$+2\iint_{|z|\leq b}|r(|y|)^2(\rho_n(y-z)-\rho_n(y))^2|z|^{-d-\alpha}T_t(y)dydz$$
$$\leq 2\int_{|z|\leq b}r(|z|)^2|z|^{-d-\alpha}dz+2J_{n,1}\int r(|y|)^2|y|^{-\alpha}T_t(y)dy,$$
$$I_{n,2}\leq b^{-\gamma}\iint_{|z|>b}|r(|y-z|)-r(|y|)|\ |z|^{-d-\alpha+\gamma}T_t(y)dydz$$
$$+b^{-\gamma}\iint_{|z|>b}r(|z|)|\rho_n(y-z)-\rho_n(y)|\ |z|^{-d-\alpha+\gamma}T_t(y)dydz$$
$$\leq b^{-\gamma}\Big(\int_{|z|>b}r(|z|)|z|^{-d-\alpha+\gamma}dz+J^{\varepsilon}_{n,2}\int r(|y|)|y|^{-\varepsilon}T_t(y)dy\Big),$$

where

$$J_{n,1}=\sup_y\Big(|y|^{\alpha}\int_{|z|\leq b}(\rho_n(y-z)-\rho_n(y))^2|z|^{-d-\alpha}dz\Big),$$
$$J^{\varepsilon}_{n,2}=\sup_y\Big(|y|^{\varepsilon}\int_{|z|>b}|\rho_n(y-z)-\rho_n(y)|\ |z|^{-d-\alpha+\gamma}dz\Big).$$

We see that $J_{n,1}\leq cn^{\alpha-2}$. Using (2.4) we have

$$\varlimsup_{n\to\infty}I_{n,1}\leq 2\int_{|z|\leq b}r(|z|)^2|z|^{-d-\alpha}dz.$$

Choose the constant ε so as to satisfy $0<\varepsilon<(\alpha-\gamma)\wedge 1$. Then

$$J^{\varepsilon}_{n,2}=n^{\gamma+\varepsilon-\alpha}\sup_y\Big(|y|^{\varepsilon}\int_{|z|>b/n}|\rho(y-z)-\rho(y)|\ |z|^{\gamma-d-\alpha}dz\Big)$$
$$\leq n^{\gamma+\varepsilon-\alpha}\Big\{4\int_{|z|>b/n}|\rho(y-z)-\rho(y)|\ |z|^{\gamma-d-\alpha}dz$$
$$+\sup_{|y|>4}\Big(|y|^{\varepsilon}\int|\rho(y-z)-\rho(y)|\ |z|^{\gamma-d-\alpha}dz\Big)\Big\}$$

$$\leq n^{\gamma+\varepsilon-\alpha}\{c+\sup_x|\partial\rho(x)|\int_{b/n<|z|\leq 1}|z|^{\gamma+1-d-\alpha}dz$$

$$+\sup_{|y|>4}(|y|^{\varepsilon}\int_{|z|>|y|/2}|z|^{\gamma-d-\alpha}dz)\}$$

$$\leq cn^{\varepsilon-1}\vee n^{\gamma+\varepsilon-\alpha}.$$

This and (2,4) imply that

$$\overline{\lim_{n\to\infty}}I_{n,2}\leq b^{-\gamma}\int_{|z|>b}r(|z|)|z|^{-d-\alpha+\gamma}dz.$$

Therefore we have

$$|(d/dt)M(t)|\leq c(\int_{|z|\leq b}r(|z|)^2|z|^{-d-\alpha}dz)^{1/2}\mathscr{E}_t(T_t,\log T_t)^{1/2}$$

$$+cb^{-\gamma}\int_{|z|>b}r(|z|)|z|^{\gamma-d-\alpha}dz.$$

Inequality (2.6) is immediately obtained from the above inequality. Since

$$\int r(|z|)^2|z|^{-d-\alpha}dz+\int r(|z|)|z|^{-d-\alpha/2}dz<\infty, \tag{2.8}$$

we have (2.7). q.e.d.

Let $Q(t)$ be the entropy

$$Q(t)=-\int T_t(y)\log T_t(y)dy \tag{2.9}$$

of the probability density function $T_t(y)$. Since $-\log T_t\geq 1-T_t$, $Q(t)\geq 1-E(t)>-\infty$. From (2.4) and the inequality $-T\log T\leq(a-1)T+e^{-a}$ we see that

$$Q(t)\leq\int(|y|^{\alpha/2}-1)T_t(y)dy+\int\exp[-|y|^{\alpha/2}]dy<\infty.$$

There is a general inequality that if $T(x)$ is a probability density function on $\boldsymbol{R}^d$ and $\beta,\beta'>0$, then

$$\int(|x|^{\beta}\wedge|x|^{\beta'})T(x)dx\geq(ce^{Q})^{\beta/d}\wedge(ce^{Q})^{\beta'/d},$$

where $Q=-\int T(x)\log T(x)dx$ (cf. [7], Lemma 2.2). Applying this ineguality to functions $T_t(y)$ and $r(|y|)$, we have

(2.10) $$M(t) \geq c \exp[cQ(t)] \wedge \exp[cQ(t)].$$

Let $0<s<t$. Then, for each $\delta>0$,

$$-\int T_t \log(T_t+\delta)dy + \int T_s \log(T_s+\delta)dy$$
$$= \int_s^t (\mathscr{E}_\tau(T_\tau, \log(T_\tau+\delta)) + \mathscr{E}_\tau(T_\tau, T_\tau/(T_\tau+\delta)))d\tau$$
$$\geq \int_s^t \mathscr{E}_\tau(T_\tau, \log(T_\tau+\delta))d\tau.$$

Since $0 \leq \mathscr{E}_\tau(T_\tau, \log(T_\tau+\delta)) \uparrow \mathscr{E}_\tau(T_\tau, \log T_\tau)$ as $\delta \downarrow 0$,, we see that

(2.11) $$Q(t)-Q(s) \geq \int_s^t \mathscr{E}_\tau(T_\tau, \log T_\tau)d\tau \qquad (0<s<t).$$

This does not always imply the inequality $(d/dt)Q(t) \geq \mathscr{E}_t(T_t, \log T_t)$.

Lemma 2.3. *Let $k \in \boldsymbol{K}^\gamma$. Then $M(1) \leq c$.*

Proof. By Theorem 2 there is a constant b, depending only on α and c_1, such that $T_t(y) \leq e^b t^{-d/\alpha}$. Define functions $g(t)$ and $h(t)$ by

$$g(t) = Q(t) - (d/\alpha)\log t + b,$$
$$h(t) = \int_1^t \mathscr{E}_\tau(T_\tau, \log T_\tau)d\tau - (d/\alpha)\log t.$$

Then the function $g(t)$ is non-negative, and from (2.11) we see that

$$g(t)-g(s) \geq h(t)-h(s) \qquad (0<s<t).$$

Suppose that $\gamma<\alpha/2$. From (2.6),

$$(d/dt)M(t) \leq ct^{\beta/\alpha-1/2}((d/dt)h(t)+d/t\alpha)^{1/2} + ct^{\beta/\alpha-1}$$
$$= t^{\beta/\alpha-1}(c(1+(t\alpha/d)(d/dt)h(t))^{1/2}+c)$$
$$\leq t^{\beta/\alpha-1}(c+ct(d/dt)h(t)).$$

For any $0<\varepsilon<1$, by the integration by parts,

$$M(1)-M(\varepsilon) \leq \int_\varepsilon^1 (ct^{\beta/\alpha-1} + ct^{\beta/\alpha}(d/dt)h(t))dt$$
$$\leq c + c(\varepsilon^{\beta/\alpha}(h(1)-h(\varepsilon)) + (\beta/\alpha)\int_\varepsilon^1 t^{\beta/\alpha-1}(h(1)-h(t))dt)$$

$$\leq c+c(\varepsilon^{\beta/\alpha}g(1)+(\beta/\alpha)\int_{\varepsilon}^{1}t^{\beta/\alpha-1}(g(1)-g(t))dt)$$

$$\leq c+c\,g(1)(\varepsilon^{\beta/\alpha}+(\beta/\alpha)\int_{\varepsilon}^{1}t^{\beta/\alpha-1}dt)=c+c\,g(1).$$

Since $M(+0)=0$, it follows that $M(1)\leq c+c\,g(1)$. In the case $\gamma=\alpha/2$ it can be similarly obtained that $M(1)\leq c+c\,g(1)$. On the other hand, from inequality (2.10) and the relation $g(1)=Q(1)+b\geq 0$ we have

$$c\exp[c\,g(1)]\leq c\exp[cQ(1)]\leq M(1).$$

Hence estimates $g(1)\leq c$ and $M(1)\leq c$ follows from the inequality $c+c\,g(1)\geq M(1)\geq c\exp[c\,g(1)]$. q.e.d.

The canonical scale change is necessary to show Theorem 3 from Lemma 2.3. For $\lambda>0$, let $\mathscr{E}_t(\lambda|\cdot,\cdot)$ be the Dirichlet form defined by

$$\mathscr{E}_t(\lambda|f,g)=\iint(f(x)-f(y))(g(x)-g(y))K(\lambda|t,dx,dy), \tag{2.12}$$

where $K(\lambda|t,dx,dy)=k(\lambda^{\alpha}t,\lambda x,\lambda y)|x-y|^{-d-\alpha}dxdy$. Then the function

$$S(\lambda|s,x;t,y)=\lambda^{d}S(\lambda^{\alpha}s,\lambda x;\lambda^{\alpha}t,\lambda y) \tag{2.13}$$

is the fundamental solution of the parabolic equation

$$(d/dt)(u_t,\cdot)_{L^2}=-\mathscr{E}_t(\lambda|u_t,\cdot).$$

Set

$$T(\lambda|t,y)=\lambda^{d}T(\lambda^{\alpha}t,\lambda y)=S(\lambda|0,0;t,y). \tag{2.14}$$

Note that $k(\lambda^{\alpha}t,\lambda x,\lambda y)\in\boldsymbol{K}^{\gamma}$ if $k(t,x,y)\in\boldsymbol{K}^{\gamma}$ for any $\lambda>0$ and that

$$\int r(t^{-1/\alpha}|y|)T_t(y)dy=\int r(|y|)T(t^{1/\alpha}|1,y)dy.$$

From Lemm 2.3 the right hand side of the above equality is limited by a constant independent of t, and this implies Theorem 3.

3. The lower estimate

In this section we shall prove Theorem 4 assuming that the function $k(t,x,y)$ and its derivatives of every order with respect to x and y are bounded and continuous on $\boldsymbol{R}_+ \times \boldsymbol{R}^d \times \boldsymbol{R}^d$. An estimate for the overlap of fundamental solutions will give the lower estimate of the fundamental solution. The proof is quite similar to that in [7], §3, but the result is considerably sharp.

For given function $k(t,x,y) \in \boldsymbol{K}^\gamma$, let $\tilde{k}(t,x,y) = k(t, t^{1/\alpha}x, t^{1/\alpha}y)$ and

$$(3.1) \qquad \tilde{\mathscr{E}}_t(\cdot,\cdot) = \mathscr{E}_1(t^{1/\alpha}|\cdot,\cdot), \quad U_t(y) = T(t^{1/\alpha}|1,y),$$

these are the same ones given by (2.12) and (2.14) taking $\lambda = t^{1/\alpha}$. Then the equation $-(d/dt)(T_t,f)_{L^2} = \mathscr{E}_t(T_t,f)$ is equal to the equation

$$(3.2) \qquad -t((\partial/\partial t)U_t,f)_{L^2} = \frac{1}{\alpha}(U_t, y\cdot\partial f(y))_{L^2} + \tilde{\mathscr{E}}_t(U_t,f).$$

Throughout this section let $P(y)$ denote the probability density function

$$(3.3) \qquad P(y) = C(1+|y|)^{-d-\alpha}/\log(e+|y|).$$

Define for $0<\delta<1$

$$(3.3) \qquad G_\delta(t) = -\int P(y)\log(U_t(y)+\delta)dy.$$

From (1.6), $\log\delta < \log(U_t+\delta) < U_t < c$, therefore $-c < G_\delta(t) < \log\delta^{-1}$.

Lemma 3.1.

$$(3.4) \qquad t(d/dt)G_\delta(t) \le c + cG_\delta(t) + \tilde{\mathscr{E}}_t(U_t, P/(U_t+\delta))$$

Proof. Applying (3.2) for function $f(y) = P(y)/(U_t(y)+\delta)$,

$$\begin{aligned} t(d/dt)G_\delta &= -t((\partial/\partial t)U_t,\ P/(U_t+\delta))_{L^2} \\ &= -\frac{1}{\alpha}(\partial\cdot(yU_t), P/(U_t+\delta))_{L^2} + \tilde{\mathscr{E}}_t(U_t, P/(U_t+\delta)) \\ &= -\frac{d}{\alpha}(U_t, P/(U_t+\delta))_{L^2} - \frac{1}{\alpha}(y\cdot\partial\log(U_t+\delta), P)_{L^2} + \tilde{\mathscr{E}}_t(U_t, P/(U_t+\delta)) \\ &\le -\frac{1}{\alpha}(y\cdot\partial\log(U_t+\delta), P)_{L^2} + \tilde{\mathscr{E}}_t(U_t, P/(U_t+\delta)) \\ &= -\frac{d}{\alpha}G_\delta + \frac{1}{\alpha}(y\cdot\partial P, \log(U_t+\delta))_{L^2} + \tilde{\mathscr{E}}_t(U_t, P/(U_t+\delta)). \end{aligned}$$

Note that $y\cdot\partial P = (y\cdot\partial\log P)P$ and

$$y\cdot\partial\log P = y\cdot(-(d+\alpha)/(1+|y|) - 1/((e+|y|)\log(e+|y|)))(y/|y|)$$

$$= -d-\alpha-1+F(y),$$

where

$$F(y)=(d+\alpha)/(1+|y|)+(\log(1+|y|/e)+e/(e+|y|))/\log(e+|y|).$$

Since $F(y)$ is a positive bounded function,

$$t(d/dt)G_\delta \le \frac{\alpha+1}{\alpha}G_\delta+\frac{1}{\alpha}\int F(y)P(y)\log(U_t(y)+\delta)dy+\tilde{\mathscr{E}}_t(U_t,P/(U_t+\delta))$$

$$\le \frac{\alpha+1}{\alpha}G_\delta+c+\tilde{\mathscr{E}}_t(U_t,P/(U_t+\delta)). \qquad \text{q.e.d.}$$

Now fix $t>0$ and set

$$2\theta(x,y)=\log P(x)-\log P(y),$$
$$2\omega(x,y)=\log(U_t(x)+\delta)-\log(U_t(y)+\delta).$$

Then

$$\begin{aligned}
&(U_t(x)-U_t(y))(P(x)/(U_t(x)+\delta)-P(y)/(U_t(y)+\delta))\\
&=(e^{2\theta-2\omega}-1)(e^{2\omega}-1)P(y)\\
&=4\,\mathrm{sh}(\theta-\omega)\mathrm{sh}\,\omega\,(P(x)P(y))^{1/2}\\
&=2(P(x)+P(y))\mathrm{sh}(\theta-\omega)\mathrm{sh}\,\omega/\mathrm{ch}\,\theta.\\
&\le 2(P(x)+P(y))((\mathrm{th}\,\theta)/\theta)(\theta-\omega)\omega\\
&\le (P(x)+P(y))((\mathrm{th}\,\theta)/\theta)(\theta^2-\omega^2),
\end{aligned}$$

because of inequalities $\mathrm{sh}(\theta-\omega)\mathrm{sh}\,\omega \le ((\mathrm{sh}\,\theta)/\theta)(\theta-\omega)\omega$ and $2(\theta-\omega)\omega \le \theta^2-\omega^2$. The following inequality is essential for the overlap estimate.

Lemma 3.2.

$$\tilde{\mathscr{E}}_t(U_t,P/(U_t+\delta)) \le c-c\int P(y)(\log(U_t(y)+\delta)+G_\delta(t))^2dy \tag{3.5}$$

Proof. For certain positive constants c_1, c_2 and c_3, the function k belongs to the class $\boldsymbol{K}^\gamma[c_1,c_2,c_3]$, and this implies that $c_1 \le \tilde{k}(t,x,y) \le c_2+c_3|x-y|^\gamma$. Since $|\theta(x,y)| \le c\log(1+|x-y|) \le c|x-y|$,

$$\iint P(y)\theta(x,y)^2(c_2+c_3|x-y|^\gamma)|x-y|^{-d-\alpha}dxdy<c.$$

Therefore

$$\begin{aligned}
&\tilde{\mathscr{E}}_t(U_t, P/(U_t+\delta)) \\
&\quad \leq \iint P(y)((\mathrm{th}\,\theta)/\theta)(\theta^2-\omega^2)\tilde{k}|x-y|^{-d-\alpha}dxdy \\
&\quad \leq c - c_1\iint P(y)((\mathrm{th}\,\theta)/\theta)\omega^2|x-y|^{-d-\alpha}dxdy.
\end{aligned}$$

It is easy to see that $(\mathrm{th}\,\theta(x,y))/\theta(x,y) \geq c(\log(e+|x-y|))^{-1}$, so that

$$((\mathrm{th}\,\theta(x,y))/\theta(x,y))|x-y|^{-d-\alpha} \geq cP(x-y).$$

From this we have

$$\begin{aligned}
\tilde{\mathscr{E}}_t(U_t, P/(U_t+\delta)) &\leq c - c\iint P(y)\omega(x,y)^2 P(x-y)dxdy \\
&\leq c - c\iint_{|x|>|y|} P(y)\omega(x,y)^2 P(x)dxdy
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int P(y)(\log(U_t(y)+\delta)+G_\delta(t))^2 dy \\
&\quad = \int P(y)(2\int \omega(x,y)P(x)dx)^2 dy \\
&\quad \leq 4\int P(y)(\int \omega(x,y)^2 P(x)dx)dy \\
&\quad = 8\iint_{|x|>|y|} P(y)\omega(x,y)^2 P(x)dxdy.
\end{aligned}$$

Combining these inequalities, we obtain (3.5). q.e.d.

From Theorem 2 and Theorem 3 there are positive constants a and b depending only on α, γ, c_1, c_2 and c_3 such that

$$U_t(y) \leq a, \qquad \int r(|y|)U_t(y)dy \leq \frac{1}{4}r(b).$$

This and preceding two lemmas will lead to the following lemma.

Lemma 3.3. *There is a positive constant c_0 depending only on α, γ, c_1, c_2 and c_3 such that $G_\delta(t) \leq c_0$ for all $t>0$ and δ, $0<\delta<1$.*

Proof. Define a function $h_\delta(u,v)$, $0<u$ and $-\infty<v<\infty$, by

$$h_\delta(u,v)=\frac{1}{u}(\log(u+\delta)+v)^2.$$

The function $h_\delta(\cdot,v)$ is decreasing on $[e^{2-v},\infty)$. Therefore

$$\begin{aligned}&\int P(y)(\log(U_t(y)+\delta)+G_\delta(t))^2dy\\&\quad=\int P(y)U_t(y)h_\delta(U_t(y),G_\delta(t))dy\\&\quad\geq h_\delta(a,G_\delta(t))\int P(y)U_t(y)I_{(U_t(y)>\varepsilon)}dy\end{aligned}$$

as long as $\varepsilon\geq\exp[2-G_\delta(t)]$. Let

$$c_4=(4\int I_{(|x|<b)}dx)^{-1}.$$

Then

$$\begin{aligned}&\int_{|x|<b}U_t(y)I_{(U_t(y)>c_4)}dy\\&\quad\geq\int_{|x|<b}U_t(y)dy-\int_{|x|<b}c_4dy=\frac{3}{4}-\int_{|x|\geq b}U_t(y)dy\\&\quad\geq\frac{3}{4}-r(b)^{-1}\int r(|y|)U_t(y)dy\geq\frac{1}{2}.\end{aligned}$$

If $c_4\geq\exp[2-G_\delta(t)]$, then

$$\begin{aligned}&\int P(y)(\log(U_t(y)+\delta)+G_\delta(t))^2dy\\&\quad\geq h_\delta(a,G_\delta(t))\int_{|x|<b}P(y)U_t(y)I_{(U_t(y)>c_4)}dy\\&\quad\geq\frac{1}{2}(\inf_{|x|<b}P(x))\,h_\delta(a,G_\delta(t))\geq(c+cG_\delta(t))^2.\end{aligned}$$

This inequality and (3.4) and (3.5) imply that there are positive constants c_5 and c_6 depending only on α, γ, c_1, c_2 and c_3 such that

$$t(d/dt)G_\delta(t)\leq c_5{}^2-c_6{}^2G_\delta(t)^2 \tag{3.6}$$

as long as $G_\delta(t)\geq 2+\log(1/c_4)$. It is easy to show from this differential inequality that

$$G_\delta(t)\leq(2+\log(1/c_4))\vee(c_5/c_6)=c_0. \qquad \text{q.e.d.}$$

Let $w(\sigma)$ be a function on $\boldsymbol{R}_+$ defined by

$$w(|x_1-x_2|)=\int P(x-x_1)\wedge P(x-x_2)dx. \tag{3.7}$$

In the case $d\geq 2$,

$$\begin{aligned}w(2\sigma)&=2\int_\sigma^\infty\int_{\boldsymbol{R}^{d-1}}P((s,\xi))dsd\xi\\&\geq c\int_\sigma^\infty\int_{\boldsymbol{R}^{d-1}}(1+s^2+|\xi|^2)^{-(d+\alpha)/2}(\log(e+s^2+|\xi|^2))^{-1}dsd\xi.\end{aligned}$$

Substituting the variable ξ by $(1+s^2)^{1/2}\zeta$ and using the inequality that

$$\log(e+s^2+|\zeta|^2+s^2|\zeta|^2)\leq 2\log(e+s^2)\log(e+|\zeta|^2),$$

the function $w(2\sigma)$ is estimated from below in the following manner.

$$\begin{aligned}w(2\sigma)&\geq c\int_\sigma^\infty(1+s^2)^{-(\alpha+1)/2}(\log(e+s^2))^{-1}ds\\&\quad\times\int_{\boldsymbol{R}^{d-1}}(1+|\zeta|^2)^{-(d+\alpha)/2}(\log(e+|\zeta|^2))^{-1}d\zeta\\&=c\int_0^\infty(1+(\sigma+t)^2)^{-(\alpha+1)/2}(\log(e+(\sigma+t)^2))^{-1}dt\\&\geq c\int_0^\infty(1+\sigma^2+t^2)^{-(\alpha+1)/2}(\log(e+\sigma^2+t^2))^{-1}dt\\&\geq c((1+\sigma^2)^{\alpha/2}\log(e+\sigma^2))^{-1}\\&\quad\times\int_0^\infty(1+\tau^2)^{-(\alpha+1)/2}(\log(e+\tau^2))^{-1}d\tau\\&=c((1+\sigma^2)^{\alpha/2}\log(e+\sigma^2))^{-1}.\end{aligned}$$

Hence it is obtained that

$$w(\sigma)\geq c((1+\sigma)^\alpha\log(e+\sigma))^{-1}. \tag{3.8}$$

The same lower estimate for the function $w(\sigma)$ is easily obtained in the case

$d=1$. Let $\Psi(\sigma)$ be the function defined by (1.9). Then (3.8) is eguivalent to the lower setimate $\exp[-1/w(\sigma)]\geq c\Psi(\sigma)$.

Lemma 3.4. *Let $S^{(i)}(s,x;t,y), i=1$ or 2, be fundamental solutions for Dirichlet forms which are associated with functions $k^{(i)}(t,x,y)$ belonging to the same class $\boldsymbol{K}^{\gamma}[c_1,c_2,c_3]$. Then*

$$\int S^{(1)}(0,x_1;t,y)\wedge S^{(2)}(0,x_2;t,y)dy\geq c\Psi(t^{-1/\alpha}|x_1-x_2|)^c. \tag{3.9}$$

Proof. Set $U^{(i)}(t,y)=t^{d/\alpha}S^{(i)}(0,t^{1/\alpha}x_i;t,t^{1/\alpha}y)$. Then it suffices to prove that

$$\int \min_i U^{(i)}(t,y)dy\geq c\Psi(|x_1-x_2|)^c.$$

Using the inequality

$$a_1b_1+a_2b_2\leq(a_1\vee a_2)(b_1\vee b_2)+(a_1\wedge a_2)(b_1\wedge b_2),$$

we have from Lemma 3.3 that

$$\begin{aligned}
&\int \min_i P(y-x_i)\cdot\min_i \log(U^{(i)}(t,y)+\delta)dy\\
&\geq\sum_i\int P(y-x_i)\log(U^{(i)}(t,y)+\delta)dy\\
&\quad-\int\max_i P(y-x_i)\cdot\max_i \log(U^{(i)}(t,y)+\delta)dy\\
&\geq -2c_0-\int\max_i P(y-x_i)\cdot(U^{(1)}(t,y)+U^{(2)}(t,y))dy\\
&\geq -2c_0-2C,
\end{aligned}$$

where c_0 is the constant in Lemma 3.3 and C, in (3.3). On the other hand

$$\begin{aligned}
&\int \min_i P(y-x_i)\cdot\min_i \log(U^{(i)}(t,y)+\delta)dy\\
&\leq\int\min_i P(y-x_i)\cdot(\log\delta+\log(1+U^{(1)}\wedge U^{(2)})/\delta))dy\\
&\leq\log\delta\cdot w(|x_1-x_2|)+(C/\delta)\int\min_i U^{(i)}(t,y)dy.
\end{aligned}$$

Therefore

$$C\int\min_i U^{(i)}(t,y)dy\geq\delta(-\log\delta\cdot w(|x_1-x_2|)-2(c_0+C)).$$

Set $q=2(c_0+C)$, $\sigma=|x_1-x_2|$ and $w=w(\sigma)$. Since

$$\max_{s>0} e^{-s}(sw-q)=w\exp[-q/w-1],$$

it follows from (3.8) that

$$\begin{aligned}&\int \min_i U^{(i)}(t,y)dy\\ &\geq c((1+\sigma)^\alpha \log(e+\sigma))^{-1}\exp[-c(1+\sigma)^\alpha \log(e+\sigma)]\\ &\geq c\exp[-c(1+\sigma)^\alpha \log(e+\sigma)]\geq c\Psi(\sigma)^c.\end{aligned}$$

q.e.d.

The above estimate for the overlap of fundamental solutions will be equivalent to the lower estimate stated in section 1 as Theorem 4. It is obvious that estimate (1.10) implies estimate (3.9). And so, we shall show the inverse implication.

Let

$$\begin{aligned}U^{(1)}(t,x)&=t^{d/\alpha}S(0,0;t,t^{1/\alpha}x),\\ U^{(2)}(t,x)&=t^{d/\alpha}S(t,t^{1/\alpha}(t^{-1/\alpha}y);2t,t^{1/\alpha}x).\end{aligned}$$

Then

$$\begin{aligned}T(2t,y)&=\int S(0,0;t,x)S(t,y;2t,x)dx\\ &=t^{-d/\alpha}\int U^{(1)}(t,x)\cdot U^{(2)}(t,x)dx.\end{aligned}$$

Let R be a positive constant depending only on α, γ, c_1, c_2 and c_3 such that

$$\int r(|y|)U^{(i)}(t,y)dy\leq R \qquad (i=1,2).$$

There are positive constants B and ε such that $r(\sigma)>B\sigma^\varepsilon$ for all $\sigma\geq r^{-1}(R)$. Let $[X]_+$ denote $X\vee 0$. For any $\lambda>0$ we have

$$\begin{aligned}t^{d/\alpha}T(2t,y)&\geq\int_{|x|<\lambda} U^{(1)}\cdot U^{(2)}dx\\ &\geq\Big(\int_{|x|<\lambda} U^{(1)}\wedge U^{(2)}dx\Big)^2\Big/\Big(\int_{|x|<\lambda} 1dx\Big)\\ &\geq c\lambda^{-d}\Big(\int U^{(1)}\wedge U^{(2)}dx-\int_{|x|\geq\lambda} U^{(1)}\wedge U^{(1)}dx\Big)^2\end{aligned}$$

$$\geq c\lambda^{-d}[\int U^{(1)} \wedge U^{(2)}dx - r(\lambda)^{-1}\int r(|x|)U^{(1)} \wedge U^{(2)}dx]_+{}^2$$

$$\geq c\lambda^{-d}[\int U^{(1)} \wedge U^{(2)}dx - (R/B)\lambda^{-\varepsilon}]_+{}^2.$$

Since for any $v>0$

$$\sup_{\lambda} \lambda^{-d}([v-(R/B)\lambda^{-\varepsilon}]_+)^2 = cv^{d/\varepsilon+2},$$

it follows from Lemma 3.3 that

$$1 t^{d/\alpha}T(2t,y) \geq c(\int U^{(1)} \wedge U^{(2)}dx)^{d/\varepsilon+2} \geq c\Psi(t^{-1/\alpha}|y|)^c.$$

Hence we obtain Theorem 4.

4. Examples

The Hölder continuity of the fundamental solution is not obtained in the case $k \in \boldsymbol{K}^{\alpha/2}$. The necessity for studyng the case $\gamma=\alpha/2$ in condition (1.3) or (1.17), however, arises when we want to consider Dirichlet forms determined by pseudo-differential operators.

Let $\hat{k}(x)=(k_j(x))_{1\leq j\leq N}$ be an $\boldsymbol{R}^N$-valued smooth homogeneous function on $\boldsymbol{R}^d\setminus\{0\}$ with index $-d-\alpha/2$, and $a(x)=(a_{ij}(x))_{1\leq i,j\leq N}$ be a symmetric real matrices valued function on $\boldsymbol{R}^d$. Let $p_j(\xi)$ denote the Fourier transform of the function $k_j(x)$ in the sense of distribution. Then the function $p_j(\xi)$ is homogeneous with index $\alpha/2$ and the pseudo-differential operator $p_j(D)$ can be written in the form

$$(4.1) \qquad p_j(D)f(x) = \int (f(y)-f(x))k_j(x-y)dy$$

$$= \lim_{\varepsilon\downarrow 0}(k_j^\varepsilon * f(x) - m_j^\varepsilon f(x)),$$

where $k_j^\varepsilon(x)=k_j(x)I_{(|x|>\varepsilon)}$ and $m_j^\varepsilon=\int k_j^\varepsilon(x)dx$. Let $p_j(D)^*$ be the adjoint operator of $p_j(D)$. Then

$$(4.2) \qquad p_j(D)^*f(x) = \int (f(y)-f(x))k_j(y-x)dy$$

$$= \lim_{\varepsilon\downarrow 0}(\tilde{k}_j^\varepsilon * f(x) - m_j^\varepsilon f(x)),$$

where $\tilde{k}_j^\varepsilon(x)=k_j^\varepsilon(-x)$. Assume that $a_{ij}(x)$ and its first order derivatives are bounded for any i and j. We shall consider the bilinear form

$$\mathscr{B}(f,g)=\Sigma_{i,j=1}^N \int p_i(D)f(x)\cdot a_{ij}(x)\cdot p_j(D)g(x)dx \tag{4.3}$$

$$=\int \langle p(D)f(x)\cdot a(x),\ p(D)f(x)\rangle dx,$$

where $p(D)=(p_1(D),\cdots,p_N(D))$ and $\langle u,v\rangle=\Sigma_{j=1}^N u_j v_j$ for $u=(u_1,\cdots,u_N)$ and $v=(v_1,\cdots,v_N)$. The Dirichlet problem related to such bilinear forms had been studied in [4]. It is not known the concrete condition for $a(x)$ under which the bilinear form (4.3) becomes a Dirichlet form.

We shall write the bilinear form in a form like (1.1) and derive a necessary condition for it to be a Dirichlet form. Let $p_j^\varepsilon(D)$ and $p_j^\varepsilon(D)^*$ be operators given by

$$p_j^\varepsilon(D)f=k_j^\varepsilon*f-m_j^\varepsilon f, \qquad p_j^\varepsilon(D)^*f=\tilde{k}_j^\varepsilon*f-m_j^\varepsilon f.$$

Then

$$\begin{aligned}
&p_j^\varepsilon(D)^*a_{ij}(x)p_i^\varepsilon(D)f(x)\\
&=\tilde{k}_j^\varepsilon*(a_{ij}(k_i^\varepsilon*f))-m_j^\varepsilon a_{ij}(k_i^\varepsilon*f)\\
&\quad -m_i^\varepsilon\tilde{k}_j^\varepsilon*(a_{ij}f)+m_j^\varepsilon m_i^\varepsilon a_{ij}f\\
&=\iint f(y)k_i^\varepsilon(z-y)a_{ij}(z)k_j^\varepsilon(z-x)dydz\\
&\quad -\int f(y)k_i^\varepsilon(x-y)a_{ij}(x)\Big(\int k_j^\varepsilon(z-x)dz\Big)dy\\
&\quad -\int f(y)\Big(\int k_i^\varepsilon(z-y)dz\Big)a_{ij}(y)k_j^\varepsilon(y-x)dy\\
&\quad +f(x)\Big(\int k_i^\varepsilon(x-y)dy\Big)a_{ij}(x)\Big(\int k_j^\varepsilon(z-x)dz\Big)\\
&=\iint (f(y)-f(x))\{k_i^\varepsilon(z-y)a_{ij}(z)k_j^\varepsilon(z-x)\\
&\quad -k_i^\varepsilon(x-y)a_{ij}(x)k_j^\varepsilon(z-x)-k_i^\varepsilon(z-y)a_{ij}(y)k_j^\varepsilon(y-x)\}dydz.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Sigma_{i,j=1}^N p_j(D)^*a_{ij}(x)p_i(D)f(x)\\
&=\iint (f(y)-f(x))\{\langle \hat{k}(z-y)a(z),\hat{k}(z-x)\rangle
\end{aligned}$$

$$-\langle\hat{k}(x-y)a(x),\hat{k}(z-x)\rangle-\langle\hat{k}(z-y)a(y),\hat{k}(y-x)\rangle\}dydz$$

$$=\int(f(x)-f(y))[\int\{\langle\hat{k}(x-y)a(z),\hat{k}(z-x)\rangle$$

$$+\langle\hat{k}(z-y)a(z),\hat{k}(y-x)\rangle-\langle\hat{k}(z-y)a(z),\hat{k}(z-x)\rangle\}dz$$

$$-\langle\hat{k}(x-y),\int\hat{k}(z-x)(a(z)-a(x))dz\rangle$$

$$-\langle\int\hat{k}(z-y)(a(z)-a(y))dz,\hat{k}(y-x)\rangle]dy.$$

Let $\theta=\theta(x-y)=(x-y)/|x-y|$ and set

$$h_0(x,y)=2^{d+\alpha}\Sigma_{i,j=1}^N\int a_{ij}\left(\frac{x+y}{2}+\frac{|x-y|}{2}\zeta\right)[-k_i(\zeta+\theta)k_j(\zeta-\theta)+k_i(2\theta)k_j(\zeta-\theta)+k_i(\zeta+\theta)k_j(-2\theta)]d\zeta, \tag{4.4}$$

$$h_1(x,y)=-|x-y|^{\alpha/2}[\langle\hat{k}(\theta),p(D)^*a(x)\rangle+\langle p(D)^*a(y),\hat{k}(-\theta)\rangle] \tag{4.5}$$

where $p(D)^*=(p_1(D)^*,\cdots,p_N(D)^*)$. Then we see that $h_0(x,y)=h_0(y,x)$ and $h_1(x,y)=h_1(y,x)$. Let $h(x,y)=h_0(x,y)+h_1(x,y)$. Then

$$\Sigma_{i,j=1}^N p_j(D)^*a_{ij}(x)p_i(D)f(x)$$
$$=\int(f(x)-f(y))h(x,y)|x-y|^{-d-\alpha}dy,$$

and hence

$$\mathscr{B}(f,g)=\int g(x)(\Sigma_{i,j=1}^N p_j(D)^*a_{ij}(x)p_i(D)f(x))dx \tag{4.6}$$
$$=\iint(f(x)-f(y))(g(x)-g(y))h(x,y)/2|x-y|^{d+\alpha}dxdy.$$

If the function $h(x,y)$ is non-negative, the bilinear form $\mathscr{B}(\cdot,\cdot)$ is a Diriclet form. But it is not so easy to find simple conditions which imply that $h(x,y)\geq 0$ for all x and y. Let $|\phi|_S$ denote the maximum norm of the continuous function ϕ on S^{d-1}. For $|\theta|=1$,

$$|k_i(\zeta+\theta)k_j(\zeta-\theta)-k_i(2\theta)k_j(\zeta-\theta)-k_i(\zeta+\theta)k_j(-2\theta)|$$
$$\leq c(|k_i|_S|k_j|_S+|\partial k_i|_S|k_j|_S+|k_i|_S|\partial k_j|_S)$$
$$\times\{|\zeta|^{-d-\alpha/2}I_{(|\zeta|>2)}+I_{(|\zeta|\leq 2,|\zeta-\theta|\geq 1/2,|\zeta+\theta|\geq 1/2)}$$

$$+|\zeta-\theta|^{1-d-\alpha/2}I_{(|\zeta-\theta|<1/2)}+|\zeta+\theta|^{1-d-\alpha/2}I_{(|\zeta+\theta|<1/2)}\}.$$

Therefore

$$|h_0(x,y)|\le c\,\Sigma_{i,j=1}^N \sup_z |a_{ij}(z)|\ |k_i|_S(|k_j|_S+|\partial k_j|_S)<\infty. \tag{4.7}$$

It is obvious that

$$|h_1(x,y)|\le c\,\Sigma_{i,j=1}^N \sup_z(|a_{ij}(z)|+|\partial a_{ij}(z)|)\,|k_i|_S\cdot|x-y|^{\alpha/2}. \tag{4.8}$$

Now we shall introduce the following definition. Let N be a positive integer and $Q=(Q_{ij})_{1\le i,j\le N}$ be a non-negative definite matrix. A mapping $a(x)=(a_{ij}(x))_{1\le i,j\le N}$ from $\boldsymbol{R}^d$ to the space of non-negative definite matrices is said to belong to the class $A[N,Q]$ if $a_{ij}(x)$ and its first order derivatives are bounded for any i and j and if

$$\Sigma_{i,j=1}^N\int|a_{ij}(x)-Q_{ij}|dx<\infty. \tag{4.9}$$

Theorem 7. *Assume that homogeneous functions $k_j(x)$, $1\le j\le N$, with index $-d-\alpha/2$ are smooth on S^{d-1} and that the space $\boldsymbol{R}^N$ is generated by vectors $\{(k_1(\omega),\cdots,k_N(\omega));\ \omega\in S^{d-1}\}$. Let $p_j(D)$ and $p_j(D)^*$ denote pseudo-differential operators given by* (4.1) *and* (4.2). *If $a(x)$ belongs to a class $A[N,Q]$ and the bilinear form $\mathscr{B}(\cdot,\cdot)$ defined by* (4.3) *is a Dirichlet form, then*

$$p(D)^*a(x)=(p_1(D)^*,\cdots,p_N(D)^*)a(x)=0. \tag{4.10}$$

Proof. Since $\mathscr{B}(\cdot,\cdot)$ is a Dirichlet form, we see that $h(x,y)=h_0(x,y)+h_1(x,y)\ge 0$ for all x and y, where h_0 and h_1 are functions defined by (4.4) and (4.5). It follows from (4.7) that $-h_1(x,y)\le c$ for all x and y.
Therefore

$$\begin{aligned}0\ge\overline{\lim_{r\to\infty}}\,&[\langle \hat{k}((x-r\omega)/|x-r\omega|),p(D)^*a(x)\rangle\\ &+\langle p(D)^*a(y),\hat{k}((r\omega-x)/|x-r\omega|)\rangle]\\ =\langle\hat{k}(-\omega),&\,p(D)^*a(x)\rangle+\overline{\lim_{r\to\infty}}\langle p(D)^*a(r\omega),\hat{k}(\omega)\rangle,\end{aligned}$$

for any $\omega\in S^{d-1}$, where $\hat{k}=(k_1,\cdots,k_N)$. Since ∂a_{ij} are bounded, condition (4.9) implies that

$$\lim_{r\to\infty}\sup_{|\omega|=1}|a_{ij}(r\omega)-Q_{ij}|=0$$

so that $|p(D)^*a(r\omega)|$ tends to 0 as $r\to\infty$. Hence we see that

$$\langle \hat{k}(-\omega), p(D)^*a(x)\rangle \le 0$$

for any $\omega\in S^{d-1}$ and $x\in \boldsymbol{R}^d$. Let $\rho(x)$ be the same function as in Section 2. From condition (4.9),

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int \rho(x/n)p(D)^*a(x)dx\right| \\
&= \lim_{n\to\infty}\left|\int \hat{k}(z)\left(\int (a(x)-Q)(\rho((x-z)/n)-\rho(x/n))dx\right)dz\right| \\
&\le \lim_{n\to\infty}\iint |\hat{k}(z)(a(x)-Q)|(1\wedge(c|z|/n))\,dxdz=0,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\int \rho(x/n)\langle \hat{k}(-\omega), p(D)^*a(x)\rangle dx=0.$$

Since $\langle \hat{k}(-\omega), p(D)^*a(x)\rangle \le 0$ everywhere, it must be that

$$\langle \hat{k}(-\omega), p(D)^*a(x)\rangle = 0$$

for any $\omega\in S^{d-1}$ and $x\in \boldsymbol{R}^d$. From the assumption that $\boldsymbol{R}^N$ is generated by vectors $\{\hat{k}(\omega);\ \omega\in S^{d-1}\}$, we have condition (4.10). q.e.d.

EXAMPLE 1. Let us consider the case where $N=2$,

$$p_1(\xi)=(\xi_1^2+2\xi_2^2)^{\alpha/4}, \qquad p_2(\xi)=(2\xi_1^2+\xi_2^2)^{\alpha/4}$$

and $a(x)=(a_{ij}(x))_{1\le i,j\le 2}$ with $a_{12}(x)=a_{21}(x)=\phi(x)$. If ϕ is a tempered function on $\boldsymbol{R}^2$, and if

$$p(D)^*a(x)=(p_1(D)a_{11}+p_2(D)\phi,\ p_1(D)\phi+p_2(D)a_{22})=0,$$

then

$$\begin{aligned}
a_{11}(x)&=-\mathscr{F}^{-1}[(p_2(\xi)/p_1(\xi))\mathscr{F}\phi(\xi)](x)+Q_{11},\\
a_{22}(x)&=-\mathscr{F}^{-1}[(p_1(\xi)/p_2(\xi))\mathscr{F}\phi(\xi)](x)+Q_{22}
\end{aligned}$$

for some constants Q_{11} and Q_{22}. Obviously, condition (4.9) is satisfied. For sufficiently large Q_{11} and Q_{22}, the bilinear form $\mathscr{B}(f,g)$ associated with $p_1(D)$, $p_2(D)$ and $a(x)$ becomes a Dirichlet form.

EXAMPLE 2. Let us consider one dimensional case. If $a(x)$ is a tempered function, not identically 0, then condition (4.10) is not satisfied for the pseudo-differential operator $p(D)=|D|^{\alpha/2}$. Therefore, no matter how large the constant Q is, the form

$$\mathscr{B}_Q(f,g)=\int p(D)f(x)\cdot(a(x)+Q)p(D)g(x)dx$$

does not become a Dirichlet form. If constants Q and λ are sufficiently large, the bilinear form

$$\mathscr{B}_Q^\lambda(f,g)=\mathscr{B}_Q(f,g)+\lambda(|D|^{\alpha/4}f,\ |D|^{\alpha/4}g)_{L^2}$$

is a Dirichlet form, because this can be written in the form

$$\mathscr{B}_Q^\lambda(f,g)=\iint(f(x)-f(y))(g(x)-g(y))$$
$$\times\{h(x,y)+c_1(\alpha)Q+c_2(\alpha)\lambda|x-y|^{\alpha/2}\}|x-y|^{-1-\alpha}dxdy$$

where $c_1(\alpha)$ and $c_2(\alpha)$ are positive constants depending only on α. If Q and λ are sufficiently large, the function

$$k(x,y)=h(x,y)+c_1(\alpha)Q+c_2(\alpha)\lambda|x-y|^{\alpha/2}$$

satisfies condition (1.17) for $\gamma=\alpha/2$, but not for $\gamma<\alpha/2$. This is the main reason why we consider the case $k\in \boldsymbol{K}^{\alpha/2}$.

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