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GEOMETRY OF SCROLLS

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1. Introduction

In this paper, we study topology and geometry of immersed curves in the plane $\boldsymbol{R}^2$ (or preferably in the sphere $\boldsymbol{R}^2 \cup \{\infty\}$).

From topological point of view, we distinguish curves by *geotopy* ([6], [2]). Two curves are said to be *geotopic* if there is a diffeomorphism between neighborhoods of the curves which takes one to the other. Geotopy preserves information on intersections. If we restrict ourselves to *normal* curves ([8]), i.e., curves whose self-intersections are transvers double points, the intersection information is represented by a *Gauss word.* A Gauss word is simply a sequence of labels of crossing points with signs. Conversely, a Gauss word determines a geotopy type of curves.

On the geometric side, we look at *vertices* of a curve. A *vertex* is a stationary point of the curvature. It is well-known that a vertex is a concept which belongs to Möbius geometry. That is, it is invariant not only under Euclidean motions, but also under inversions. We assume that curves have only finitely many vertices, none of which are located at crossings (cf. Theorem 2.5). Then a curve is divided into finitely many vertex-free curves. Since a vertex-free curve on the plane has no self-intersections (Kneser, see[5]), topological complexity of the original curve then comes from intersections of these vertex-free pieces. As a basic case, we investigate intersections of two vertex-free curves.

Figure 1.1

Figure 1.1 shows typical vertex-free curves. From their appearances, we sometimes refer to a vertex-free curve as a *scroll.* In Figure 1.1, we recognize that one is a scroll with increasing curvature and the other with decreasing curvature. Note that these monotonicity properties of curvature are independent of choice of the orientations of a curve. Thus (non-oriented) vertex-free curves fall into two classes, one with increasing curvature, the other with decreasing curvature. This rather trivial observation will be useful in the proof of our main results. The main results are Theorems 4.8, 4.9 and 5.5. Theorem 4.8 determines how two scrolls of different classes can intersect. This situation appears as a special case in a closed curve with exactly two vertices. Theorem 4.9 characterizes geotopy types of closed curves with two vertices. In other words, we can determine all topological types of closed curves which have at least four vertices. In this sense, Theorem 4.9 is a 4-vertex theorem. Theorem 5.5 treats intersections of two scrolls, both of which have increasing (or decreasing) curvature.

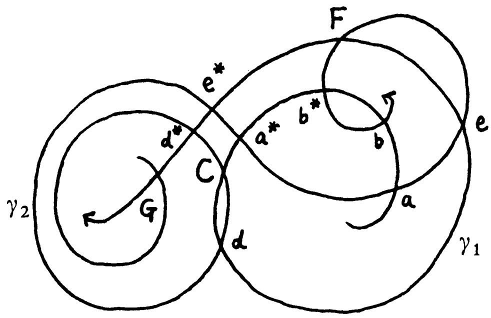


Figure 1.2

We briefly illustrate Theorem 4.8. The key idea is "$*$-pairing" of crossings. In Figure 1.2, we see two oriented scrolls γ_1 and γ_2. γ_1 and γ_2 have decreasing and increasing curvature respectively. A crossing point p is said to be *positive* (resp. *negative*), if p is a positive (resp. negative) crossing as regarded as a point on γ_1, namely, if γ_2 crosses γ_1 at p from the left to the right (resp. from the right to the left) of γ_1. Suppose p is a positive crossing. If another crossing q is such that among all crossing points in the past of p along γ_2, q is the nearest future point of p on γ_1, then we put $p^*=q$. This $*$-pairing does not apply to all crossings. In Figure 1.2, we have assigned capital letters to the crossings which are excluded from $*$-pairing. The crossings on γ_1 read as

$$\gamma_1 : abb^*a^*CdeFe^*d^*G.$$

We use the following abbreviations for certain types of subwords:

$$[a_1,a_2,\cdots,a_k]=a_1a_2\cdots a_kb_k\cdots b_2b_1$$

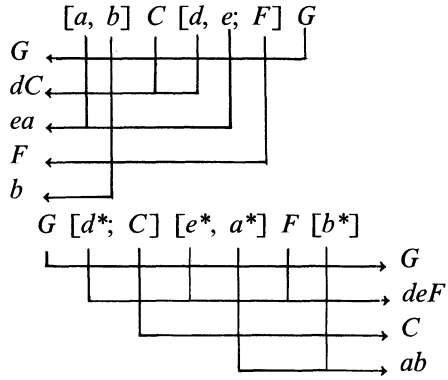
and

$$[a_1,a_2,\cdots,a_k;x]=a_1a_2\cdots a_kxb_k\cdots b_2b_1,$$

where $\{a_i,b_i\}$'s are $*$-pairs. Then the intersection sequence is rewritten as

$$\gamma_1 : [a,b]C[d,e;F]G.$$

In order to describe how γ_1 and γ_2 intersect, we need to know how the intersection sequence of γ_1 shold be in general, and how the intersection sequence of γ_2 is obtained from that of γ_1. Theorem 4.8 answers these questions. To answer the second question, we prove that the geotopy type is determined only by the intersection sequence of γ_1 with $*$-data. Once this is proved, it is easy to have explicit ways of transforming intersections. Here is a method:



The rule is to pick up heads of groups made by brackets until a capital letter. The above diagrams may be enough to explain the rule. As is clear from Figure 1.2, the intersection sequence of γ_2 is

$$\gamma_2: Gd^*Cde^*a^*aeFb^*b = G[d^*;C][e^*,a^*]F[b^*].$$

In the case of a closed curve with two vertices, which is thought of as two vertex-free curves, it is shown that subwords of $[a_1,a_2,\cdots,a_k]$-type never appear, and consequently $*$-pairing is uniquely determined by signs of crossings (Theorem 4.9). It should be mentioned that Jackson [3] has also obtained a structure theorem for closed 2-vertex curves. Our theorem extends Jackson's, and gives a complete description of topology of closed 2-vertex curves.

If both γ_1 and γ_2 have increasing curvature, they can intersect more flexibly. There is however a law quite similar to the case of scrolls of different kinds (Theorem 5.5).

The proofs are based on some geometric lemmas — Lemmas 3.1 and 4.5. In particular, Lemma 3.1 turns out an useful tool, where a simple loop, which we shall call a *shell* following Umehara [7], plays an important role. As an application of Lemma 3.1, we also have a 6-vertex theorem (Theorem 3.5.). Lemma 3.1 is most efficient when it is combined with technique of rounding out a corner of piecewise smooth curve (Proposition 2.3). With these altogether, we can determine the minimal number of vertices for a given geotopy type with small number of crossings. We append a table of geotopy types of closed curves in the sphere and their minimal numbers of vertices, where curves with up to 5 crossings are listed.

The authors would like to thank G. Cairns for kindly sending his papers including [2].

2. Deformation of curves and the number of vertices

We begin with some terminologies used throughout the paper. A smooth regular curve $c: I \to \boldsymbol{R}^2$ is said to have a *vertex* at $t \in I$ if the derivative of the curvature function vanishes at t. A vertex is said to be *maximal* (resp. *minimal*) if the curvature takes local maximam (resp. local minimum) at the vertex ([7]). By *honest* vertices we mean maximal or minimal vertices ([5]). A regular curve $c:[a,b] \to \boldsymbol{R}^2$ is called a *shell* at p if $p=c(a)=c(b)$ and $c|_{(a,b)}$ has no self-intersection ([7]). A shell $c:[a,b] \to \boldsymbol{R}^2$ is said to be *positive* (resp. *negative*) if the velocity vector $c'(a)$ points to the left (resp. right) of $c'(b)$.

In this section, we give some useful ways of modifiying curves, and a kind of transversality theorem which will be suitable for the study of vertices. For that purpose we need careful controll of vertices, which is derived from the following lemma.

Lemma 2.1. *Let $c:[a,b] \to \boldsymbol{R}^2$ be a C^1 regular curve which is of class C^∞ except at $t_0 \in (a,b)$, and $\kappa:[a,b] \backslash \{t_0\} \to \boldsymbol{R}^2$ be its curvature. Suppose*
(i) $\kappa' > 0$ (resp. $\kappa' < 0$) *on* $[a,b] \backslash \{t_0\}$ *and*
(ii) $\lim_{t \to t_0 - 0} \kappa(t) \leq \lim_{t \to t_0 + 0} \kappa(t)$ (resp. $\lim_{t \to t_0 - 0} \kappa(t) \geq \lim_{t \to t_0 + 0} \kappa(t)$).
Then for any sufficiently small $\varepsilon > 0$, there is a C^∞ regular curve $\tilde{c}:[a,b] \to \boldsymbol{R}^2$ close to c in C^1 such that $\tilde{c}(t) = c(t)$ for t with $|t - t_0| > \varepsilon$ and that $\tilde{\kappa}' > 0$ (resp. $\tilde{\kappa}' < 0$) *everywhere.*

Proof. We prove the lemma in the case when $\kappa' > 0$ on $[a,b] \backslash \{t_0\}$. Note that we may assume $\kappa > 0$ on $[a,b] \backslash \{t_0\}$ (use Möbius transformation if necessary). Put $\kappa_\pm = \lim_{t \to t_0 \pm 0} \kappa(t)$ and $\lambda = \kappa_-^{-1} - \kappa_+^{-1}$. Let $\hat{c}:[a, b+\lambda] \to \boldsymbol{R}^2$ be the evolute of the curve c defined by

$$\hat{c}(t) = \begin{cases} c(\hat{t}) + \kappa(\hat{t})^{-1} v(\hat{t}) & \text{if } t < t_0 \text{ or } t > t_0 + \lambda \\ c(t_0) - (t - t_0 - 1/\kappa_-) v(t_0) & \text{if } t_0 \leq t \leq t_0 + \lambda, \end{cases}$$

where v is the (left-pointing) unit normal vector to c and $\hat{t} = t$ (resp. $t - \lambda$) if $t < t_0$ (resp. $t > t_0 + \lambda$) (see Figure 2.1). We have

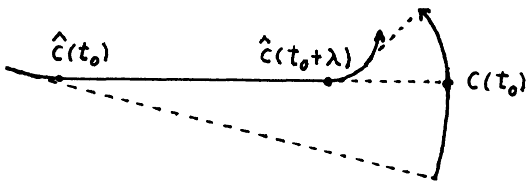
$$\hat{c}'(t) = \begin{cases} (1/\kappa)'(\hat{t}) v(\hat{t}) & \text{if } t < t_0 \text{ or } t > t_0 + \lambda \\ -v(t_0) & \text{if } t_0 < t < t_0 + \lambda. \end{cases}$$

From this we see that assumptions of the lemma imply that the curve $\hat{c}$ can be reparametrized as a C^1 regular curve, with the following curvature property:

$$\hat{\kappa}(t) = \begin{cases} |c'(\hat{t})| \kappa(\hat{t})^3 / \kappa'(\hat{t}) > 0 & \text{if } t < t_0 \text{ or } t > t_0 + \lambda \\ 0 & \text{if } t_0 < t < t_0 + \lambda. \end{cases}$$

It is then easy to see that we can modify $\hat{c}$ to get a C^∞ regular curve $\hat{c}:[a, b+\lambda] \to \boldsymbol{R}^2$

with strictly positive curvature such that $\hat{c}=\hat{c}$ outside small neighborhood of $[t_0, t_0+\lambda]$, and the length of $\hat{c}$ = the length of $\hat{c}$. The desired curve $\tilde{c}$ is obtained by taking an involute of $\hat{c}$. □



Figre 2.1

Corollary 2.2. *If a smooth curve $c:[a,b]\to \boldsymbol{R}^2$ has a sinlge non-honest vertex at $t_0 \in (a,b)$, then we have a vertex-free curve by deforming c slightly near $t=t_0$.*

Proposition 2.3. *Let $c:[a,b]\to \boldsymbol{R}^2$ be a curve such that $c_- := c|_{[a,t_0]}$ and $c_+ := c|_{[t_0,b]}$ are smooth regular vertex-free curves, and θ the signed angle of $c'_+(t_0)$ relative to $c'_-(t_0)$. Let $\kappa_\pm$ denote the curvature of $c_\pm$. Suppose $\theta \in (-\pi,\pi)$ and $\theta \neq 0$. Then we have a smooth regular curve $\tilde{c}:[a,b]\to \boldsymbol{R}^2$ close to c which differs from c only near t_0 and satisfies the following conditions:*
(i) If either $\theta>0$, $\kappa'_->0$ and $\kappa'_+<0$ or $\theta<0$, $\kappa'_-<0$ and $\kappa'_+>0$ then $\tilde{c}$ has exactly one vertex.
(ii) If either $\kappa'_->0$ and $\kappa'_+>0$, or $\kappa'_-<0$ and $\kappa'_+<0$, then $\tilde{c}$ has exactly two vertices.
(iii) If either $\theta>0$, $\kappa'_-<0$ and $\kappa'_+>0$ or $\theta<0$, $\kappa'_->0$ and $\kappa'_+<0$ then $\tilde{c}$ has exactly three vertices.

Proof. (i): Put a parabola as shown in Figure 2.2(i) (this figure explains the case $\theta>0$). The parabola is tangent to c_- at $P=c(t_0-\varepsilon_1)$ and to c_+ at $Q=c(t_0+\varepsilon_2)$, where ε_1 and ε_2 are small positive numbers. The vertex of the parabola is between P and Q. Replace $c|_{[t_0-\varepsilon_1,t_0+\varepsilon_2]}$ by a piece of the parabola, and we have a C^1-curve (the curve accompanied by a broken line in the figure). Then modify this C^1-curve around P and Q using Lemma 2.1, and we get the desired curve.

(ii): This time put a parabola as shown in Figure 2.2(ii) (this figure explains the case $\kappa'_\pm>0$ and $\theta>0$). Modify the curve indicated by a broken line around P and Q using Lemma 2.1 and (i) respectively.

(iii): The proof is similar. See Figure 2.2(iii). □

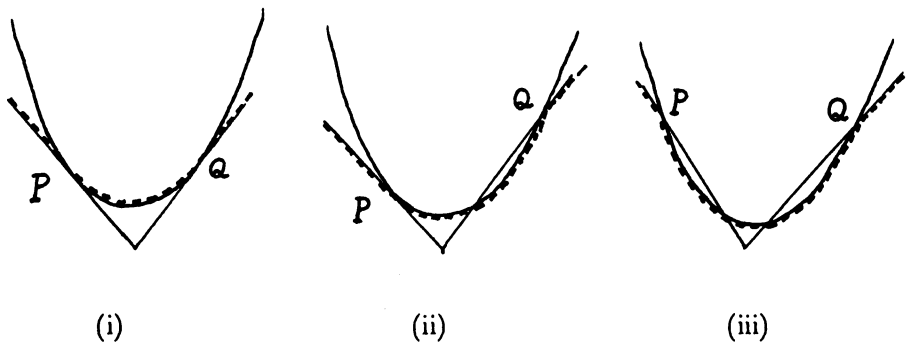


Figure 2.2

Note that in each case $\tilde{c}$ has one maximal vertex (resp. one minimal vertex) if $\theta>0$ (resp. $\theta<0$).

Proposition 2.3 deals with an operation of a curve with a corner of type (A) of Figure 2.3. Similar results are obtained also for operations of type (B) and type (C). For type (B), we have the new curve $\tilde{c}$ with 3, 2 and 1 vertices under the assumptions of (i), (ii) and (iii) of Proposition 2.3, respectively. For type (C), we have $\tilde{c}$ with 3,4 and 5 vertices, respectively.

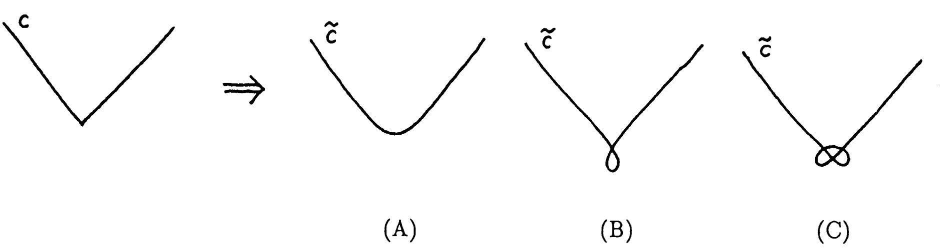


Figure 2.3

In order to demonstrate how to use proposition 2.3 in applications, we give a proof of a result of Jackson.

Lemma 2.4([3]). *A positive* (resp. *negative*) *shell has a maximal* (resp. *minimal*) *vertex.*

Proof. A shell is considered a simple closed curve with a corner. For the sake of simplicity, we suppose the shell is positive. Using Proposition 2.3, we round out the corner to get a simple closed regular curve. This process increases the number of maximal vertices by one. From the classical 4-vertex theorem, the new curve has at least 2 maximal vertices.

Therefore the positive shell has a maximal vertex. □

For example we have, from this lemma, at least 6 vertices on any curve geotopic to the curve in Figure 2.4.

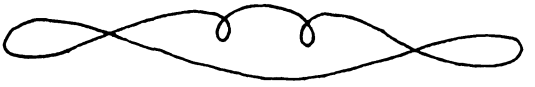


Figure 2.4

Theorem 2.5. *Let I be a compact interval or S^1. Then any regular curve $c: I \to \boldsymbol{R}^2$ can be deformed to a smooth regular curve $\tilde{c}: I \to \boldsymbol{R}^2$ with the following properties*:

(i) *$\tilde{c}$ is a normal curve and C^1-close to c.*

(ii) *The number of vertices of $\tilde{c}$ is finite and less than or equal to the number of vertices of c.*

(iii) *There is no vertex at any intersection point.*

(iv) *At any vertex of $\tilde{c}$, the second derivative of curvature does not vanish.*

Proof. In case c has infinitely many vertices, we split the curve into finitely many small segments, then replace the segments by vertex-free arcs to get a piecewise smooth curve, and then smooth out corners following Proposition 2.3. In this way we obtain a smooth curve with finitely many vertices. So we may assume c has only finitely many vertices. In view of Corollary 2.2, we may also assume c has only honest vertices.

We now look at a vertex of c. We modify the curve near the vertex in the following way (Figure 2.5): First make a curve with a corner such that near the corner the curvature is monotone in each side of the corner, then apply Proposition 2.3 (i) to the corner, and we have another smooth curve.

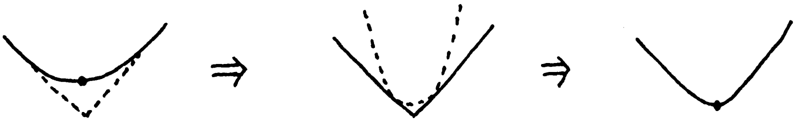


Figure 2.5

Note that by this operation we can make the new curve have the same number of vertices as the old one. As a result we can move the position of a vertex freely in a neiborhood of the original position of the vertex.

By moving vertices in this manner, we can make the curve have no self-intersections near the vertices. Now we modify part of the curve which is the complement of a neighborhood of vertices, using the standard transversality arguments (cf. [8]). In this part of the curve, the derivative of the curvature is nowhere zero, and we can make this property unchanged by the modification. Now it is easy to see that the obtained curve has the desired properties. □

REMARK. This modification preserves various topological properties of a curve. For example, if c bounds an immersed surface, then $\tilde{c}$ also does.

3. Shells

For a regular curve $c:[a,b]\to \boldsymbol{R}^2$, we denote by C_t the circle of curvature at t. The complement of C_t in $\boldsymbol{R}^2$ consists of two connected open regions. D_t denotes the one of the two regions that lies in the left side of C_t with respect the orientation of C_t induced from c. Thus, if the curvature $\kappa(t)>0$ (resp. $\kappa(t)\leq 0$), then D_t is a bounded (resp. unbounded) region. Using this notation, Kneser's lemma ([5]) is stated as follows: if the curvature is strictly increasing (resp. decreasing) in an interval $[t_1,t_2]$, then $\bar{D}_{t_2}$ $(=D_{t_2}\cup C_{t_2})\subset D_{t_1}$(resp. $\bar{D}_{t_1}\subset D_{t_2}$).

In this section, we discuss the number of vertices when we are given a cluster of shells. The following lemma will be core of our argument.

Lemma 3.1. *Let $c:[a,b]\to \boldsymbol{R}^2$ be a positive shell at p, S its image in $\boldsymbol{R}^2$.*
(i) *If c has only one vertex (necessarily maximal), then $S\backslash\{p\}\subset D_a\cap D_b$.*
(ii) *If c has exactly two vertices, maximal at $t_1\in(a,b)$ and minimal at $t_2\in(a,b)$, then $S\backslash\{p\}\subset D_a$ or $S\backslash\{p\}\subset D_b$ according as $t_1<t_2$ or $t_2<t_1$.*
(iii) *If c has three vertices in all, two of which are maximal and the other is minimal, then either $S\backslash\{p\}\subset D_a$ or $S\backslash\{p\}\subset D_b$ holds.*

Proof. (i): $S\backslash\{p\}\subset D_a$ iff $a=s_a$, where $s_a=\inf\{s\in(a,b)|c(t)\in D_a \text{ if } t\in(s,b)\}$. We will show there is a minimal vertex in (a,s_a) if $a<s_a$. Note that the curvature κ of c is increasing at a. So, if $S\not\subset\bar{D}_a$, then we can extend c to $\tilde{c}:[\tilde{a},b]\to\boldsymbol{R}^2$ such that $\tilde{c}|_{[a,b]}=c$, $\tilde{c}|_{[\tilde{a},a]}$ is a curvature increasing curve, $\tilde{c}|_{(\tilde{a},a)}$ does not intersect with S, and that $\tilde{c}(\tilde{a})\in S$. Then we find a negative shell at $\tilde{c}(\tilde{a})$ and a minimal vertex in (a,s_a). In case $S\backslash\{p\}\subset\bar{D}_a$, c is tangent to C_a at s_a and we have a minimal vertex in (a,s_a) ([3; Lemma 4.1]).

Likewise $S\backslash\{p\}\subset D_b$ if $s_b=b$, where $s_b=\sup\{s\in(a,b)|c(t)\in D_b \text{ if } t\in(a,s)\}$, and we have a minimal vertex in (s_b,b) if $s_b<b$.

(ii): For $t\in(a,b)$ and $k\in\boldsymbol{R}$, we denote by $C_t(k)$ the circle of curvature k which is tangent to c in the same direction at $c(t)$. In particular, $C_t(\kappa(t))=C_t$. Let $D_t(k)$ denote the left side open region of $C_t(k)$. Note that $S\cap D(k)=\phi$ for sufficiently large k. We put $k_t^*=\inf\{k\in\boldsymbol{R};\ S\cap D_t(k)=\phi\}$, $C_t^*=C_t(k_t^*)$ and $D_t^*=D_t(k_t^*)$. Obviously, $k^*\geq\kappa(t)$ and $S\cap D_t^*=\phi$. Also we have $p\notin C_t^*$ because S is a positive shell. If c has no maximal vertex at t, we can find a $t^*\in(a,b)$ such that $t\neq t^*$ and $c(t^*)\in C_t^*$. For this t^*, we have $D_t^*\subset D_{t^*}$. Hence we have a maximal vertex between t and t^* (cf. [3; Lemma 4.1]).

Now suppose $t_1 < t_2$. If $t \in (a, t_1]$ then $c(t) \in \bar{D}_t \subset D_a$. If $t \in (t_1, b)$ then we have, from the above argument, $t^* \in (a, t_1)$, and thus $c(t) \subset \bar{D}_t^* \subset \bar{D}_{t^*} \subset D_a$. Hence $S \backslash \{p\} \subset D_a$. The proof for the case $t_2 < t_1$ is completely similar.

(iii): In the same way as in (ii), we see $S \backslash \{p\} \subset D_a \cup D_b$. Hence $s_a < s_b$, where s_a and s_b are as defined in (i). The argument in the proof of (i) is valid in this case, and we have two minimal vertices in $(a, s_a) \cup (s_b, b)$ if $S \backslash \{p\} \not\subset D_a$ and $S \backslash \{p\} \not\subset D_b$. □

Corresponding assertion for a negative shell is as follows.

Lemma 3.2. *Let $c: [a, b] \to \boldsymbol{R}^2$ be a negative shell, S its image in $\boldsymbol{R}^2$ and $p = c(a) = c(b)$.*
(i) *If c has only one vertex (necessarily minimal), then $S \cap (\bar{D}_a \cup \bar{D}_b) = \{p\}$.*
(ii) *If c has exactly two vertices, minimal at $t_1 \in (a, b)$ and maximal at $t_2 \in (a, b)$, then $S \cap \bar{D}_a = \{p\}$ or $S \cap \bar{D}_b = \{p\}$ according as $t_1 < t_2$ or $t_2 < t_1$.*
(iii) *If c has three vertices in all, two of which are minimal and the other is maximal then either $S \cap \bar{D}_a = \{p\}$ or $S \cap \bar{D}_b = \{p\}$ holds.*

The following form will be useful in applications.

Corollary 3.3. *Let $c: [a, b] \to \boldsymbol{R}^2$ be a curve such that $c|_{[a_1, b_1]}$, $[a_1, b_1] \subset [a, b]$, is a positive* (resp. *negative*) *shell with at most one minimal* (resp. *maximal*) *vertex and that $c(a) \in S$ and $c(b) \in S$, where $S = c([a_1, b_1])$.*
(i) *If the shell has only one vertex, then there is a vetex in each of (a, a_1) and (b_1, b).*
(ii) *If the shell has exactly two vertices, maximal at t_1 and minimal at t_2, then there is a vertex in (a, a_1) or (b_1, b), according as $t_1 < t_2$ or $t_1 > t_2$* (resp. *$t_1 > t_2$ or $t_1 < t_2$*).
(iii) *If the shell has three vetices in all, then there is a vertex in $(a, a_1) \cup (b_1, b)$.*

Proposition 3.4. *Let $c: I \to \boldsymbol{R}^2$ be a curve. If there are two subintervals I_1, $I_2 \subset I$ such that $c|_{I_1}$ and $c|_{I_2}$ are positive* (resp. *negative*) *shells, then c has at least two maximal* (resp. *minimal*) *vertices in $\mathring{I}_1 \cup \mathring{I}_2$.*

Proof. We consider the case of positive shells. If $(a_1, b_1) \cap (a_2, b_2) = \phi$, where $(a_i, b_i) = \mathring{I}_i$, then the conclusion is obvious. So we assume $a_1 < a_2 < b_1 < b_2$. Note that $c|_{(a_1, a_2)}$ has an intersection with $c|_{(b_1, b_2)}$. Let $a_0 \in (a_1, a_2)$ and $b_0 \in (b_1, b_2)$ be such that $c(a_0) = c(b_0)$ and b_0 is the time closest to b_1 among such intersection times. Then $c|_{[a_0, b_0]}$ turns out a negative shell. Then it is easy to find two maximal vertices by applying Corollary 3.3 to the negative shell. □

One may expect three maximal (resp. minimal) vertices if there are three positive (resp. nagative) shells. This is not true, however, as Figure 3.1 shows. We can see many positive and nagative shells and it is easy to modify this curve to obtain a 4-vertex curve with the same crossing pattern.

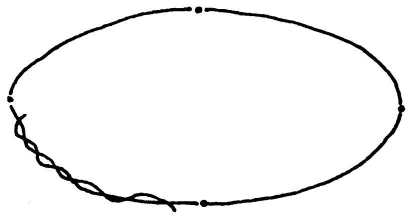


Figure 3.1

For a closed curve, we have the following.

Theorem 3.5. *If a closed planar curve contains three positive shells or three negative shells, then it has at least six vertices.*

Proof. We adopt the following notation : For $a, b \in S^1$, we denote by $[a, b]$ the closed connected arc in S^1 from a to b in the positive direction. We use $(a, b), (a, b]$, etc., similarly.

Let $c: S^1 \to \boldsymbol{R}^2$ denote the curve. We consider the case when c has 3 positive shells $c|_{I_1}$, $c|_{I_2}$ and $c|_{I_3}$, $I_i = [a_i, b_i] \subset S^1$. Suppose that c has at most 4 vertices, that is, at most 2 maximal (minimal) vertices. Then it follows from Propositon 3.4 that there are 2 maximal vertices in $(J_1 \cap J_2) \cup (J_2 \cap J_3) \cup (J_3 \cap J_1)$, where $J_i = \mathring{I}_i$. Therefore, one of J_i's, say J_1, has 2 maximal vertices. Then, we see that $J_1 \cap J_2 \neq \phi$, $J_1 \cap J_3 \neq \phi$, and we have 2 negative shells contained in $c|_J$, *where* $J = J_1 \cup J_2 \cup J_3$. Hence we have, from Propostion 3.4, two minimal vertices in J. Thus, J has altogether 4 vertices.

Obviously, $c|_{S^1 \setminus J_1}$ contains a shell $c|_{I_0}$, $I_0 = [a_0, b_0] \subset [b_1, a_1]$. This shell is negative, and $I_0 \cap J \neq \phi$, because J_1 (resp. J) already has 2 (resp. 4) maximal vertices. Hence we have one vertex in I_0 and three vertices in J_1. The curvature is decreasing (resp. increasing) on $[b_1, a_0]$ (resp. $[b_0, a_1]$). So, $D_{b_1} \subset D_{a_0}$ and $D_{b_0} \supset D_{a_1}$. Hence, from Lemma 3.1(iii) and Lemma 3.2(i), we have $c(J_1) \cap c(I_0) \subset (D_{b_0} \cup D_{a_0}) \cap c(I_0) = \phi$. Therefore, $I_0 \cap I_2 = \phi$ and $I_0 \cap I_3 = \phi$, because the nodes of the shells $c|_{I_2}$ and $c|_{I_3}$ are in $c(J_1)$. Thus, $I_0 \cap J = \phi$. Contradiction. □

The last paragraph of the above proof shows the following. If a closed planar curve with four vertices contains a positive shell with two maximal vertices and one minimal vertex, then it contains another shell which has no intersection with the positive shell except at its node.

A typical example for Theorem 3.5 is given in Figure 3.2, where we can find 3 positive shells or 3 negative shells. This theorem is also best possible on this line. Figure 3.3 indecates a construction of a 6-vertex closed curve with many shells.

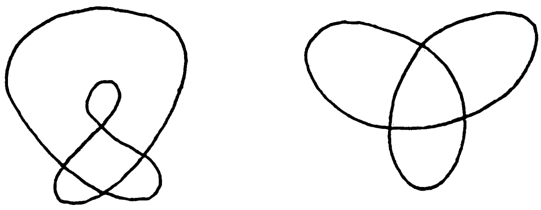


Figure 3.2

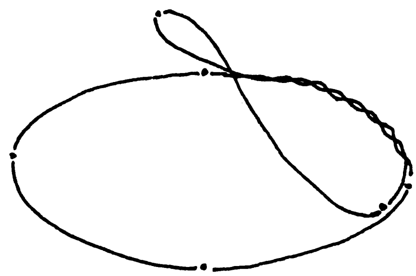


Figure 3.3

If a closed curve has a crossing at which there are two shells (e.g., 3-crossing curve in Figure 3.2), then the curve has odd number of crossings. If moreover the curve has more than one crossings, we easily find three (or more) shells of the same sign. Thus applying Theorem 3.5, we have

Corollary 3.6. *If a closed planar curve with more than one crossings has a pair of shells with a common node, it has at least six vertices.*

Here is another application to curves bounding immersed disks. It is known that *if a closed curve bounding an immersed surface has exactly four vertices, the curve can bound only an immersed disk* ([7], [1]). A key step in Umehara's proof [7] is to show that *a* 4-*vertex curve bounding an immersed surface is semisimple, that is, it can be divided into two simple curve segments.* Using Theorem 3.5, we have a refinement of this results.

Corollary 3.7. *A* 4-*vertex curve bounding an immersed surface does not contain a positive shell.*

Proof. Suppose the curve has a positive shell. Under our assumtion, it has already shown (cf. the proof of [7; Proposition 2.8]) that the curve has a portion shown in Figure 3.4. In this figure, we see two negative shells. We look at the part from P to Q that is indeterminate in the figure. If there is no self-intersection in this part between P and Q, we have third negative shell at P. Otherwise, we have another shell between P and Q. If the shell is positive, we have a negative shell inside it ([7; Theorem 2.1]). In any case, we have third negative shell, and 6 vertices on the whole curve, which gives a contradiction. □

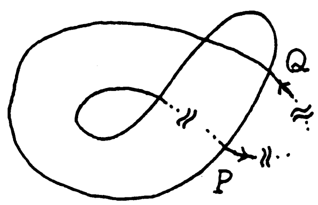


Figure 3.4

4. Scrolls of different kinds

As mentioned in Introduciton, whether the curvature of a vertex-free curve is increasing or decreasing is independent of the choice of orientation of the curve, and we have two different classes of scrolls. In this section, we study how two scrolls of different kinds can intertwine, and determine geotopy types of 1-vertex curves and 2-vertex closed curves completely. Throughout this section, we assume the following:

(I) c_+ and c_- denote compact vertex-free curves with increasing and decreasing curvature respectively.

(II) All intersections of c_+ and c_- except at ends are transversal. If c_+ and c_- have an intersection at their ends, then either it is transversal or c_+ and c_- are smoothly joined there.

(III) Since c_+ and c_- are themselves simple curves, any intersection of c_+ and c_- is a double point. In particular, any intersection point possibly except at ends is a crossing. We say a crossing is *positive* if c_+ crosses c_- from left side of c_- to right side of c_-. Otherwise the crossing point is said to be *negative*. (i.e., this convention is the standard one with respect to c_- (cf. [8]).)

DEFINITION. Fix orientations of $c_\pm$. Let $a \in \boldsymbol{R}^2$ be a positive crossing. We define $a^* \in \boldsymbol{R}^2$ to be the intersection point in the past part of c_+ from a that one encounters for the first time as one moves forward along c_- from a (see Figure 4.1).

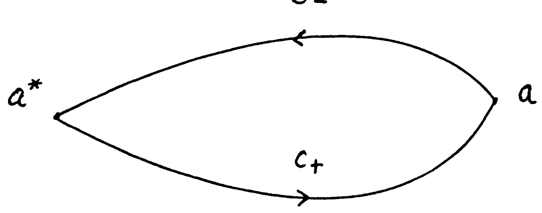


Figure 4.1

There may not exist a^* for a. A convenient sufficient condition for existence of a^* for a given a is that c_+ and c_- are joined smoothlty at the terminal point of c_- and the starting point of c_+. We remark that a^* depends on orientations of curves. So, for a crossing a, we can have different kinds of a^* by changing orientations.

Lemma 4.1. *Let a and b be positive crossings.*
(i) *a^* is a negative crossing.*
(ii) *a^* is the intersection point in the future part of c_- from a that one encounters for the first time as one moves backward along c_+ from a.*
(iii) *$a=b$ if $a^*=b^*$.*

Proof. (i) a^* is transversal intersection point, because otherwise a^* is the terminal point of c_- and we have a positive shell at a with one vertex which is minimal. Suppose a^* is a positive crossing. Then we have curve segments as shown in Figure 4.2 (i). Smooth out the corner at a^* using Proposition 2.3 (i), and we will have a positive shell with unique minimal vertex. Contradiciton.
(ii) Otherwise we have curve segments as shown in Figure 4.2(ii). Smooth out the corner at a using Propositon 2.3 (i), and we will have a curve forbidden by Corollary 3.3 (i).
(iii) Otherwise we have curve segments as shown in Figure 4.2 (iii). Smooth out the corner at $a^*=b^*$ using Propositon 2.3 (iii), and we will have a curve forbidden by Proposition 3.4. □

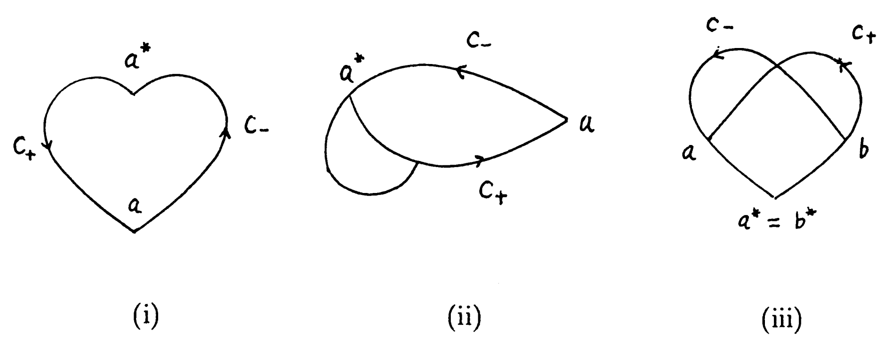


Figure 4.2

Corollary 4.2. *Let x and y be neighboring negative on c_- in this order. Then x is in the future part of c_+ from y.*

Proof. Reverse the orientation of c_- and apply Lemma 4.1 (i). □

Corollary 4.3 ([1]). *For a* 1*-vertex curve with a minimal* (resp. *maximal*) *vertex, the number of negative crossings is greater* (resp. *less*) *than or equal to the number* *of positive crossings.*

Proof. We may assume the curve has a minimal vertex, by reversing the orientation if necessary. Then the curve consists of two parts c_- and c_+, and it follows from Lemma 4.1 (iii) that $*$ defines an injective mapping from the set of positive crossings to the set of negative crossings. □

This is the key lemma of Cairns, McIntyre and Özdemir [1], which was used to prove their 6-vertex theorem.

To describe how c_- intersects c_+, we use Gauss word, which is simply a sequence of intersection points labeled by letters. For example, if we write "$\cdots a_1 a_2 \cdots$", this means there are 2 intersections a_1 and a_2 in this order on an oriented curve, and there is no intersection on the curve between a_1 and a_2. We will use the following notation: Let a_i be positive crossings and x a negative crossing. We write

$$\begin{aligned} &[a_1,a_2,\cdots,a_n] \quad \text{for} \quad a_1 a_2 \cdots a_n a_n^* \cdots a_2^* a_1^*,\\ &[a_1^*,a_2^*,\cdots,a_n^*] \quad \text{for} \quad a_1^* a_2^* \cdots a_n^* a_n \cdots a_2 a_1,\\ &[a_1,a_2,\cdots,a_n;x] \quad \text{for} \quad a_1 a_2 \cdots a_n x a_n^* \cdots a_2^* a_1^*,\\ \text{and} \quad &[a_1^*,a_2^*,\cdots,a_n^*;x] \quad \text{for} \quad a_1^* a_2^* \cdots a_n^* x a_n \cdots a_2 a_1. \end{aligned}$$

Proposition 4.4. *Let a be a positive crossing. Then the crossings between a and a^* are as follows*:
(i) *On c_- they are given by a word of the form $[a,a_1,a_2,\cdots,a_n]$ or $[a,a_1,a_2,\cdots,a_n;x]$. Moreover the crossings in this word appear on c_+ in the following order.*
$a^* \cdots a \cdots a_1^* \cdots a_1 \cdots a_n^* \cdots a_n \cdots (x)$.
(ii) *On c_+ they are given by a word of the form $[a^*,b_1^*,b_2^*,\cdots,b_n^*]$ or $[a^*,b_1^*,b_2^*,\cdots,b_n^*;x]$. Morever the crossings in this word appear on c_- in the following order.* $(x)\cdots b_n \cdots b_n^* \cdots b_1 \cdots b_1^* \cdots a \cdots a^*$.

Proof. Since the proofs are similar, we show the case (i). First note that the segment in c_- between a and a^* does not intersect the past part of c_+ from a^*, because of the definition of $*$.

We will go forward along c_+ from a, and read crossings which lie in c_- between a and a^*. There is no problem for the first crossing, say x. If the second crossing, say y, is as in Figure 4.3 (i), then we have $y^*=a^*$, which goes against Lemma 4.1 (iii). Hence, y must be on c_- between a and x (Figure 4.3 (ii)). If the third crossing, say z, is as in Figure 4.3 (iii), then we have $\tilde{x}=\tilde{z}=a$, which goes against Lemma 4.1 (iii), where $\tilde{}$ is the $*$-correspondence with respect to the orientation of c_+ and the opposite orientation of c_-. Hence, z must be on c_- between y and x (Figure 4.3 (iv)).

Now, put $a_1=y$, and we have $x=a_1^*$. Repeating this argument, we have the assertion. □

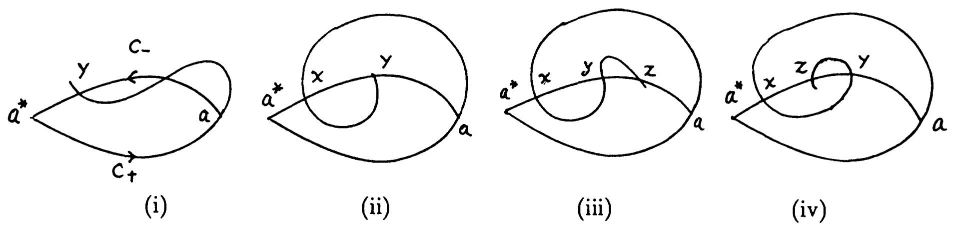


Figure 4.3

So far our proof is essentially an application of §3. In order to describe intersections of two vertex-free curves of different kinds completely, we need one more geometric lemma which is not covered by results in §3.

Lemma 4.5. *Suppose a positive crossing a has a*. Then the first crossing that one encounters as one moves forward along* c_+ (resp. *backward along* c_-) *from a is in the future part of* c_- (resp. *the past part of* c_+) *from a.*

Proof. Draw two circles of curvature at a (Figure 4.4). Then it is clear. □

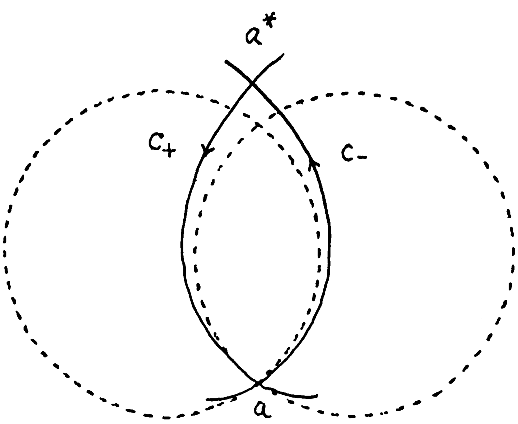


Figure 4.4

This lemma is of independent interest. For example, we see, from this lemma, 2 vertices on any curve geotopic to the curve (i) or (ii) of Figure 4.5, and 4 vertices for (iii) and (iv).

Figure 4.5

Lemma 4.6. *If the first crossing that one encounters as one moves backward along c_-* (resp. *forward along c_+*) *from a positive crossing a is negative, then it is between a^* and a on c_+* (resp. *between a and a^* on c_-*).

Proof. Let x be the negative crossing. Note that x is in the past part of c_+ from a (resp. in the future part of c_- from a^*), because of Lemma 4.5. Reverse the orientation of c_+, and we have $\tilde{x}=a$ (resp. $\tilde{x}=a^*$), where $\tilde{}$ is the $*$-correspondence for this orientation. Then the result follows from Lemma 4.1 (ii). □

Proposition 4.7. *Let $a_1a_2,\cdots,b_1,b_2,\cdots$ be positive crossings and $y,x_1,x_2,\cdots$ be negative crossings.*
(i) *Suppose the intersection sequence of c_- has $b_k\cdots b_1x_1\cdots x_l[a_1,\cdots,a_m]$, $m\geq 1$ as a part. Then, $k\leq l\leq m$, $b_i^*=x_i$ for $i=1,\cdots,k$, and a_m is the last crossing of c_+*
(ii) *Suppose the intersection sequence of c_- has $b_k\cdots b_1x_1\cdots x_l[a_1,\cdots,a_m;y]$, $m\geq 1$ as a part. Then, one of the following holds.*
(a) *$k\leq l$, $l\leq m$, $b_i^*=x_i$ for $i=1,\cdots,k$, and y is the last crossing of c_+.*
(b) *$k\leq l$, $l>m$, $b_i^*=x_i$ for $i=1,\cdots,k$. and b_1 is the last crossing of c_+.*
(c) *$k<l$, $l>m+1$, and $b_i^*=x_{i+1}$ for $i=1,\cdots,k$.*
Moreover, in each case, the order of crossings on c_+ is as shown in Figure 4.6.

Proof. Since the proof is now elementary, we only show the case (i). The point is to chase intersections on c_- backward starting from a_1^*. The arrangement of crossings on c_- between a_1^* and a_1 is described by Proposition 4.4 (i). Then, we see, from Lemma 4.6, that x_l is between a_1^* and a_1 on c_+. Since x_i are negative crossings, their possible locations are then naturally determined, and also we have $k\leq m$. Proposition 4.4 (ii) then says positive crossing b must be between x_1 and a_l on c_+. Finally, the locations of b_i are easily determined, because they are positive crossings. Now the assertion is obvious. □

From these results, Corollary 4.2, Proposition 4.4 and Propositon 4.7, or basically Lemmas 4.1 and 4.5, we can describle intersections of two scrolls of different kinds completely. Here, we summarize the results in the case of 1-vertex curves and 2-vertex closed curves. In order to state them, we prepare some notation and terminology. For a word X, we denote by $|X|_-$ the number of negative crossings in X. A word X is said to be *of type T*, if $X=x_1x_2\cdots x_k$, $k\geq 0$, where x_i are negative crossings. A word X is said to be *of type D*, if $X=[a_1,\cdots,a_k]$, $k\geq 1$, where a_i are positive crossings. A word X is said to be *of type S*, if $X=[a_1,\cdots,a_k;y]$, $k\geq 1$, where a_i are positive crossings and y a negative crossing.

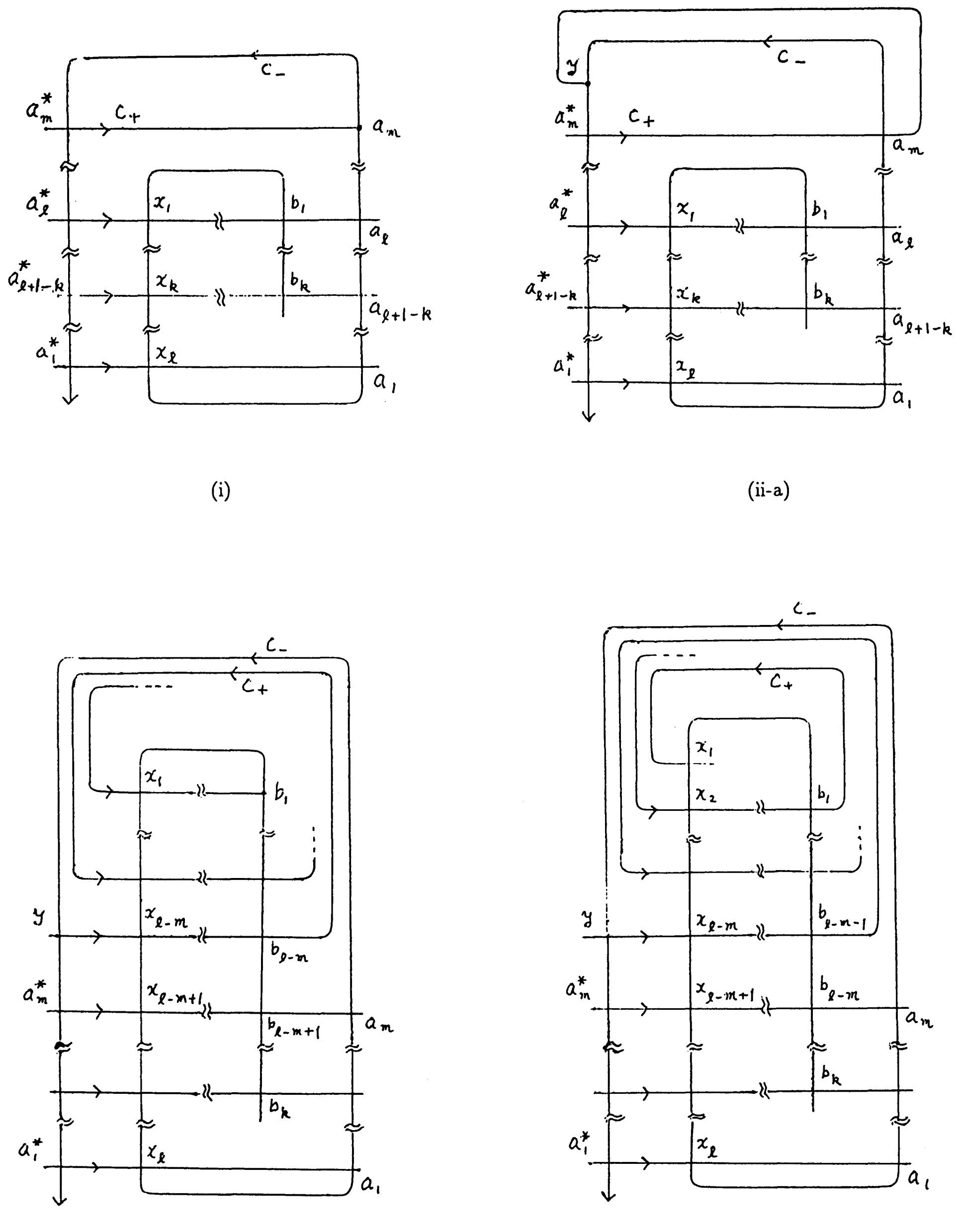


Figure 4.6

Theorem 4.8. *Let c be a 1-vertex normal curve with minimal vertex, and c_- and c_+ be the parts of c where the curvature is decreasing and increasing, respectively. Let W be the Gauss word of c, and $W_\pm$ the restriction of W to $c_\pm$.*

(i) *Then, W_- is of the form $X_1X_2\cdots X_k$, where any subword X_i is of type T, D or S.*
(ii) *If X_i is of type D, then X_j is not of type S for $j<i$. Moreover, if X_{i-1} is of type T, then $|X_{i-1}|_- \leq |X_i|_-$. Furthermore, if X_{i-2} is of type D, then $|X_{i-2}|_- + |X_{i-1}|_- \leq |X_i|_-$.*
(iii) *If X_i is of type S, X_{i-1} is of type T and X_{i-2} is of type S, then $|X_{i-2}|_- + |X_{i-1}|_- \geq |X_i|_-$.*
Conversely, if W_- is given abstractly with the above conditions (i), (ii) and (iii) satisfied, there exists unique (in the sense of geotopy) 1-vertex curve c with a minimal vertex such that W_- is the crossing sequence for c_-.

Proof. First remark that there is a^* for any positive crossing a. Then (i) follows immediately from Proposition 4.4 (i).

As in the proof of Proposition 4.7, we chase crossings on c_- backward starting from the minimal vertex. Suppose $X_k = x_1 \cdots x_l$, $l \geq 0$, is of type T, and $X_{k-1} = [a, \cdots]$ is of type D or S. Then we have Figure 4.7. It then follows from Proposition 4.4 (i) and Proposition 4.7 that the whole picture is obtained by making fragments in Figure 4.6 nested in Figure 4.7. Observe this procedure carefully, and we have (ii), (iii) and geotopical uniqueness.

Also it is not hard to see the existence of 1-vertex curve for a given W_-, from a schematic picture obtained in the manner above. □

For a 2-vertex closed curve, only the case (ii-c) of Propositon 4.7 is possible. Thus, the following is a corollary of the above Theorem.

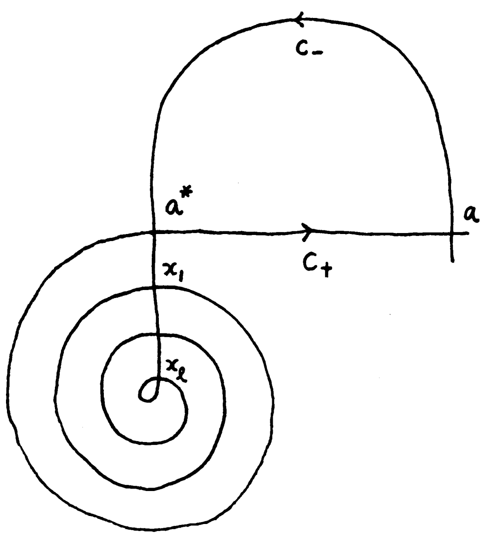


Figure 4.7

Theorem 4.9. *Let c be a 2-vertex closed normal curve, and c_- and c_+ be the parts of c where the curvature is decreasing and increasing, respectively. Let W be the Gauss word of c, $W_\pm$ the restriction of W to $c_\pm$. Then, W_- is either a word of type T or of the form $T_0S_1T_1S_2T_2\cdots S_kT_k$, where T_i and S_j are words of type T and S respectively, and $|T_0|_- > 0$, $|T_0|_- \geq |S_1|_-$, $|S_i|_- + |T_i|_- \geq |S_{i+1}|_-$. Conversely, if W_- is given abstractly with this condition satisfied, there exists unique* (*in the sense of geotopy*) *2-vertex closed curve c such that W_- is the crossing sequence for c_-.*

Theorem 4.8 as well as Theorem 4.9 says that W_+ is determined by W_-, and vice versa. Namely, the order of crossings on c_+ is determined by that of c_-. Here we clarify this combinatorially. Perhaps the best way is to explain it by an example. Suppose the crossing sequence of c_- is as follows:

$$W_- = [a_1][a_2,a_3]x_1[a_4,a_5;x_2]x_3x_4,$$

where a_i's (resp. x_j's) are positive (resp. negative) crossings. (See also Figure 4.8).

First step is to divide this sequence of letters into *atoms*. A word of type D or S forms an atom. A letter in a word of type T forms an atom. We collect them from the last, and we have

$$(x_4),\ (x_3),\ (a_4,a_5,x_2),\ (x_1),\ (a_2,a_3),\ (a_1).$$

Second step is to assemble these into a matrix by thinking of each atom as a row vector, following a rule that any atom is put to the leftmost place but places under an x_i to the bottom are avoided. Then we have

$$\begin{bmatrix} x_4 & & & & & \\ & x_3 & & & & \\ & & a_4 & a_5 & x_2 & \\ & & x_1 & & & \\ & & & a_2 & & a_3 \\ & & & a_1 & & \end{bmatrix}.$$

Third step is to take the transpose of the matrix, and then to take row vectors ignoring blanks. Then we have

$$(x_4),\ (x_3),\ (a_4,x_1),\ (a_5,a_2,a_1),\ (x_2),\ (a_3).$$

The last step is to add $*$ to the letters a_i, and to arrange these as follows to get W_+:

$$W_+ = x_4x_3[a_4^*;x_1][a_5^*,a_2^*,a_1^*]x_2[a_3^*],$$

which is exactly the crossing sequence of c_+.

The matrix which appeared in the course of this procedure is interesting in its own way. In terms of the matrix, the condition for 2-vertex closed curves can be stated as follows: the matrix is a square matrix and the bottom of the diagonal components is not blank.

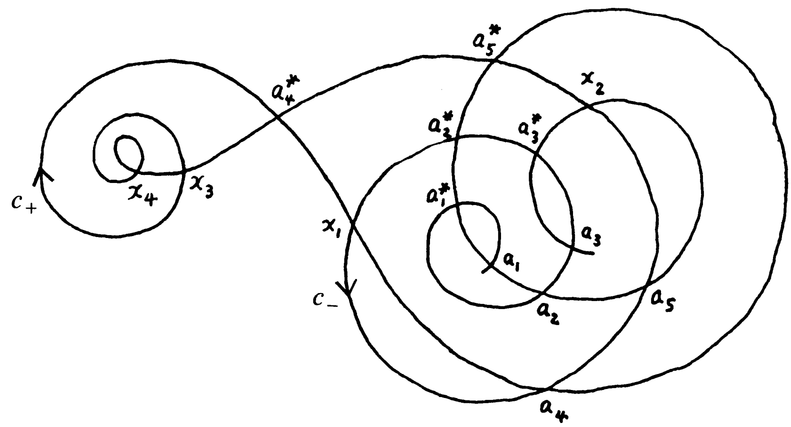


Figure 4.8

5. Scrolls of the same kind

Throughout this section, we assume the following:

(I) $c_\pm$ are compact vertex-free curves with increasing curvature.
(II) All intersections of c_+ and c_- are transversal. Hence they are crossings, i.e., transvers double points.
(III) The sign of crossing is taken with respect to c_- (cf. §4).

As in §4, for a positive crossing $a \in \boldsymbol{R}^2$, we define $a^* \in \boldsymbol{R}^2$ to be the crossing in the past part of a along c_+ that one encounters for the first time as one moves forward along c_- from a (see Figure 5.1). Note that there may not exist a^* for a given positive crossing a. Note also that the $*$-correspondence depends on orientations of the curves and on the choice which curve is thought of as c_-.

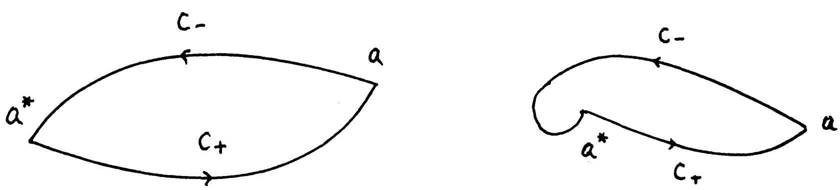


Figure 5.1

A crossing p is called an *ancestor* of a crossing q, if p is in the past of q along c_-, at the same time in the past of q along c_+. The following lemma is basic for deriving all the results in this section.

Lemma 5.1. *Suppose a positive crossing a has a^* and an ancestor p. Then*

a^* *is a negative crossing and it is situated between* p *and* a *on* c_+.

Proof. Draw the circles of curvature at a, or use technique employed in the proof of Lemma 4.1. □

Corollary 5.2.
(i) $a=b$ *if* $a^*=b^*$.
(ii) *The crossings on* c_- *between* a *and* a^* *are given by a word of the following form,* $aa_1a_2\cdots a_ka_k^*\cdots a_2^*a_1^*a^*$ *or* $aa_1a_2\cdots a_kxa_k^*\cdots a_2^*a_1^*a^*$, *where* a_i *are positive crossings,* a_i^* *and* x *are negative crossings.*
(iii) *Let* x *be the crossing succeeding* a^* *on* c_-. *Then* x *is in the past of* a *along* c_+.
(iv) *Suppose a positive crossing* a *has* a^* *and an ancestor. Let* x *be the crossing succeeding* a^* *on* c_-.

(a) *If* x *is a negative crossing, then* $x=y^*$ *for some positive crossing* y *in the past of* a *along* c_-.

(b) *If* x *is a positive crossing, then either* x *is on* c_+ *between* a^* *and* a *or* x *has no ancestor.*

Proof. (i): Easy. (ii): Similar to the proof of Proposition 4.4. (iii): Reverse the orientation of c_+ and apply the lemma. (iv-a): Use (iii).

(iv-b): Suppose x is not between a^* and a. Then, from (iii), x is in the past of a^* along c_+. Let p be the ancestor of a that is nearest to a along c_-. Then, from the lemma, p is in the past of a^* along c_+. By reversing the orientation of c_-, we see, from the lemma, that p is not in the past of x along c_+. Thus p is between x and a^* on c_+. Then it is obvious that x cannot have an ancestor. □

From now on, in addition to the general assumptions (I), (II) and (III), we assume that the starting points of $c_\pm$ are the same. Then, if $p_0,p_1,\cdots,p_n$ are the crossings on c_- taken successively, the crossing sequence $W_\pm$ of $c_\pm$ is given as follows: $W_-=p_0p_1p_2\cdots p_n$, $W_+=p_0p_{\sigma(1)}p_{\sigma(2)}\cdots p_{\sigma(n)}$, where $\sigma\in S_n$ is a permutation. Let $\varepsilon(i)$ denote the sign of the crossing p_i. Then $\varepsilon:\{0,1,2,\cdots,n\}\to\{-1,1\}$ and $\sigma\in S_n$ give complete data for the geotopy type of intersecting simple curves $c_\pm$. A crossing p_i is said to be *tame* if $\sigma(i)=i$ and $\sigma(i+1)=i+1$ (this last equality is required only when $i<n$). p_i is said to be *disordered* if it is not tame.

Lemma 5.3. *Let* p *be a tame crossing. Then the past* (resp. *future*) *part of* c_- *from* p *and the future* (resp. *past*) *part of* c_+ *from* p *do not intersect.*

Proof. Let q be the succeeding crossing. Without loss of generality, we may assume q is a positive crossing. Since p is an ancestor of q and there is no crossing between p and q, it follows from Lemma 5.1 that q^* does not exist. Hence the future part of c_- from p and the past part of c_+ from p do not intersect. This implies $A_- \supset A_+$, where $A_\pm$ denotes the set of crossings on the past part of $c_\pm$ from p. From our assumtion, $\sharp A_- = \sharp A_+ < \infty$. Hence $A_- = A_+$, which completes the proof. □

Suppose the curves are given as $c_\pm : I_\pm \to \boldsymbol{R}^2$. Then from the lemma, we have decompositions $I_\pm = I_1^\pm \cup \cdots \cup I_m^\pm$, where $I_i^\pm$ are closed intervals such that $\mathring{I}_i^\pm \cap \mathring{I}_j^\pm = \phi$ for $i \neq j$, $c_-|_{I_i^-}$ can intersect $c_+|_{I_j^+}$ only if $i = j$, and that $c_\pm|_{I_i^\pm}$ contains either only tame crossings or only disorderd crossings. This decompostion of $c_\pm$ reduces our problem to two special cases. Namely, we have only to treat the cases when all crossings are tame, and when all crossings are disorderd. The former case is easier. The result is

Proposition 5.4. *For any* $\varepsilon : \{0,1,2,\cdots,n\} \to \{-1,1\}$, *there are vertex-free curves* c_+ *and* c_- *of the same kind with only tame crossings* $p_0, p_1, \cdots, p_n$, *such that* $\varepsilon(i)$ *is the sign of crossing* p_i.

Proof. The curves are obtained by iterating the construction indicated by Figure 5.2. □

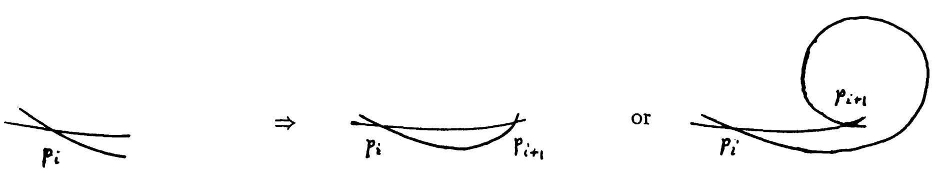


Figure 5.2

From this proposition and Proposition 2.3 (ii), we see that there are 4-vertex closed curves geotopic to the curves in Figure 4.5 (iii), (iv) of §4.

Somewhat different proof of the above proposition is as follows: For brevity's sake, we assume $\varepsilon(j) = (-1)^j$. Let $c_+ : [0,1] \to \boldsymbol{R}^2$ be a vertex-free curve with $\kappa' > 0$, where κ is the curvature of c_+. Put $c_\delta(t) = c_+(t) + \delta \sin(n\pi t) v(t)$, where v is unit normal vector field along c_+. Then, for a sufficiently small $\delta > 0$, $c_- = c_\delta$ is a desired vertex-free curve. This kind of construction is sometimes useful. Actually we have used it implicitly in Figures 3.1 and 3.3 of §3.

Now, we consider the case when all crossings are disordered. We use the notaion

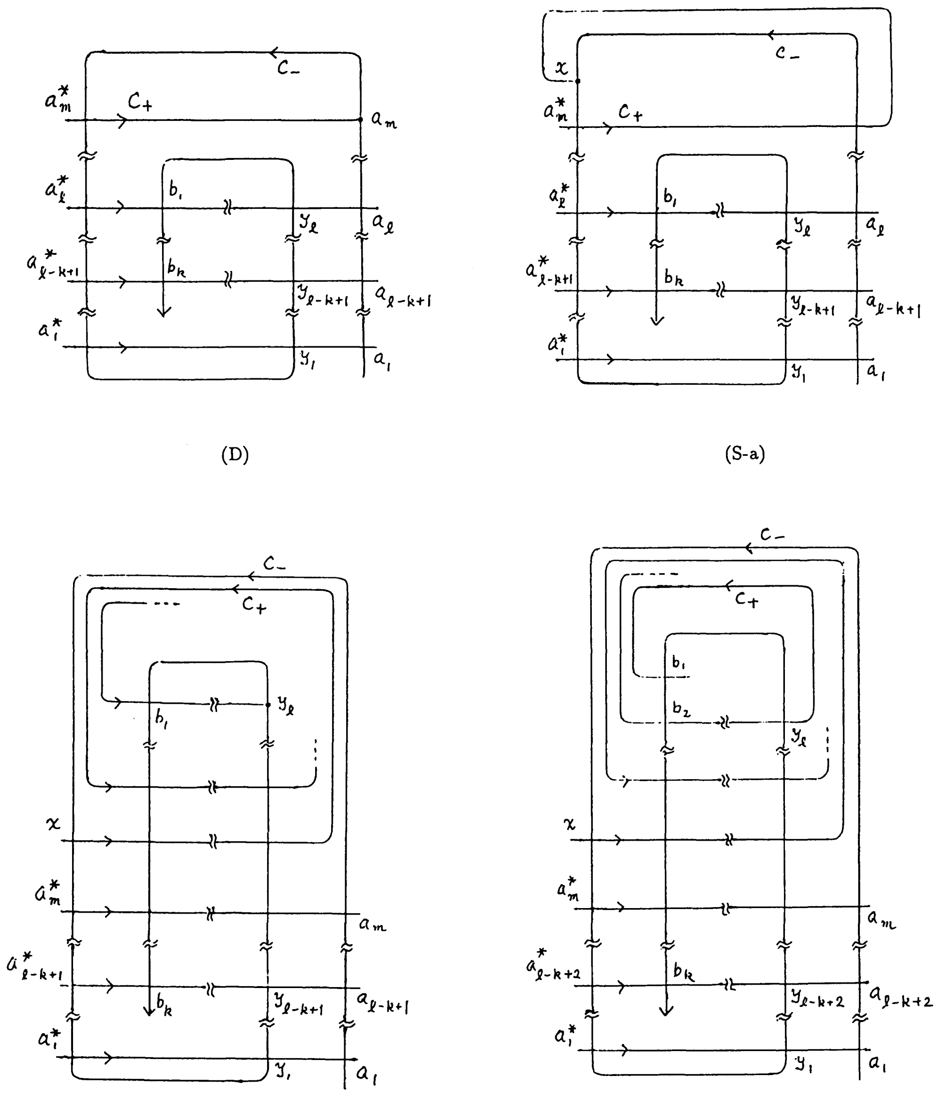
$$[a_1, a_2, \cdots, a_k] = a_1 a_2 \cdots a_n a_n^* \cdots a_2^* a_1^*,$$
$$[a_1, a_2, \cdots, a_k ; x] = a_1 a_2 \cdots a_n x a_n^* \cdots a_2^* a_1^*, \text{ etc.}$$

in the same way as in §4. Types of words are defined similarly. A word X is said to be *of type T*, if $X=y_1y_2\cdots y_k$, $k\geq 0$, where y_i are *positive* crossings. A word X is said to be *of type D* (resp. *of type S*), if X is of the form $[a_1,\cdots,a_k]$ (*resp.* $[a_1,\cdots,a_k;x]$), $k\geq 1$, where a_i's are positive crossings and x is a negative crossing. $|X|_+$ denotes the number of positive crossings in X.

Theorem 5.5. *Let $c_\pm$ be vertex-free curves with increasing curvature, and $W_\pm$ denote the crossing sequence of $c_\pm$. Suppose that both curves start from the same point $p\in \boldsymbol{R}^2$. If all intersections are disordered crossings, then the following* (i), (ii) *and* (iii) *hold.*
(i) *W_- is of the form $pX_1X_2\cdots X_k$, where X_1 is of type D or S and X_i are of type T, D or S.*
(ii) *If X_i is of type D, then X_j is not of type S for $j>i$. Moreover, if X_{i+1} is of type T, then $|X_i|_+\leq|X_{i+1}|_+$. Furthermore, if X_{i+2} is of type D, then $|X_i|_+\leq|X_{i+1}|_+ + |X_{i+2}|_+$.*
(iii) *If X_i is of type S, X_{i+1} is of type T and X_{i+2} is of type S, then $|X_i|_+\geq|X_{i+1}|_+ + |X_{i+2}|_+$.*
If both c_+ and c_- terminate at the same point q and all intersections except q are disordered crossings, then W_- is of the form $pS_1T_1S_2T_2\cdots S_kT_kq$, where T_i and S_j are words of type T and S repectively, and $|S_k|_+=|T_k|_+$, $|S_i|_-\leq|T_i|_+ + |S_{i+1}|_+$.
Covnersely, in any case, if W_- is given abstractly with the above conditions satisfied, there exist vertex-free curves $c_\pm$ with increasing curvature such that W_- is the crossing sequence for c_-, and the crossing sequence W_+ for c_+ is determied uniquely from W_-.

Proof. Using Lemma 5.1 and Corollary 5.2, we obtain the following in the same way as the proof of Proposition 4.7:
(D) If W_- contains a subword $[a_1,\cdots,a_m]y_1\cdots y_lb_1\cdots b_k$, where y_i (resp. b_j) are positive (resp. negative) crossings, then $m\geq l\geq k$, $b_i=y^*_{l-i+1}$, and a_m is the last crossing of c_+.
(S) If W_- contains a consecutive subword $[a_1,\cdots,a_m,x]y_1\cdots y_lb_1\cdots b_k$, where y_i (resp. b_j) are positive (resp. negative) crossings, then one of the following holds.
(a) $m\geq l\geq k$, $b_i=y^*_{l-i+1}$, and x is the last crossing of c_+.
(b) $k\leq l$, $l>m$, $b_i=y^*_{l-i+1}$, and y_l is the last crossing of c_+.
(c) $k\leq l+1$, $b_{i+1}=y^*_{l-i+1}$.
Moreover, in each case, the order of crossings on c_+ is as shown in Figure 5.3.

Now, we chase crossings on c_- starting from p. It follows from Lemma 5.1 and Corollary 5.2 (ii) that W_- begins with a subword of the form $p[a_1,\cdots,a_m(;x)]$, (see Figure 5.4). We see, from Corollary 5.2 (iv), that the next crossing is positive. So, we may assume W_- is of the form $p[a_1,\cdots,a_m(;x)]y_1\cdots y_lb_1\cdots b_k\cdots$, where y_i (resp. b_j) are positive (resp. negative) crossings. So, apply the above argument repeatedly, and we obtain geotopical picture of the curves. Then, our assertion naturally follows. □



(D) (S-a)

(S-b) (S-c)

Figure 5.3

There is a simple combinatorial way of reading crossings on c_+ from crossings on c_-. Suppose the crossing sequence of c_- is as follows:

$$W_- = p[a_1,a_2;x]y_1y_2[a_3,a_4][a_5]$$

where all crossings are disordered, and a_i's and y_j's (resp. x) are positive (resp.

negative) crossings. (See also Figure 5.5).

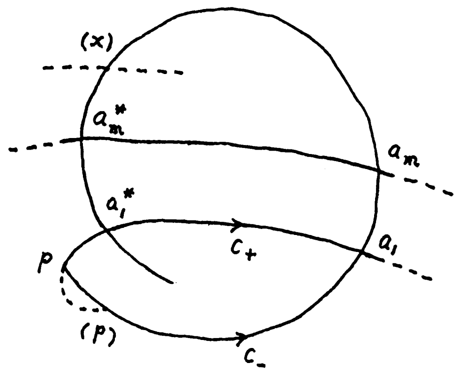


Figure 5.4

First step is to divide this sequence of letters into *atoms*. A word of type D or S forms an atom. A letter in a word of type T forms an atom. We collect them in order, and we have

$$(p),\ (a_1,a_2,x),\ (y_1),\ (y_2),\ (a_3,a_4),\ (a_5).$$

Second step is to assemble these into a matrix by thinking of each atom as a row vector, following a rule that any atom is put to the leftmost place but places under p, x and y_i to the bottom are avoided. Then we have

$$\begin{bmatrix} p & & & & & \\ & a_1 & a_2 & x & & \\ & y_1 & & & & \\ & & y_2 & & & \\ & & & & a_3 & a_4 \\ & & & & a_5 & \end{bmatrix}$$

Third step is to take the transpose of the matrix, and then to take row vectors ignoring blanks. Then we have

$$(p),\ (a_1,y_1),\ (a_2,y_2),\ (x),\ (a_3,a_5),\ (a_4).$$

The last step is to add $*$ to the letters a_i, and to arrange these as follows to get W_+:

$$W_+ = p[a_1^*;y_1][a_2^*;y_2]x[a_3^*,a_5^*][a_4^*],$$

which is exactly the crossing sequence of c_+.

Appendix: minimal number of vertices

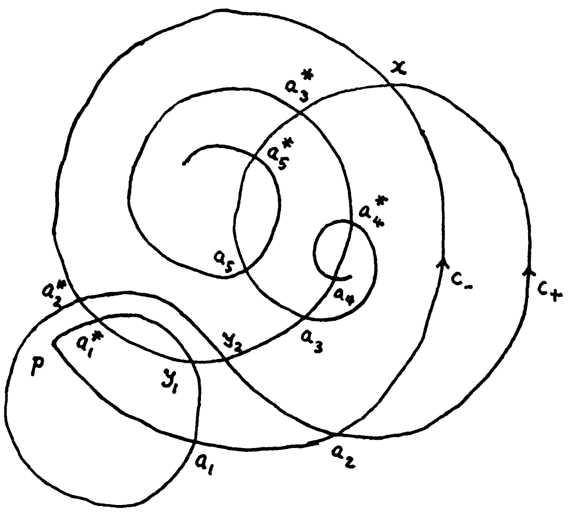


Figure 5.5

Let G be a geotopy class of closed normal curves in the plane. Define $\nu(G)$ as

$$\nu(G)=\min\{\text{the number of vertices on } \gamma \mid \gamma \in G\}.$$

$\nu(G)$ is a finite positive even integer. It is an interesting problem to determine $\nu(G)$. The classical 4-vertex theorem asserts that $\nu(G)=4$ for the geotopy class G of Jordan curves. Theorem 4.9 gives a criterion for G with $\nu(G)=2$. The results in this paper can be used to determine $\nu(G)$ for various geotopy classes. Here are some examples.

EXAMPLE **A.1.** *For the geotopy class G corresponding to the Gauss word* $ac^{-1}bd^{-1}cb^{-1}da^{-1}ee^{-1}$, $\nu(G)=6$.

Proof. It is an easy exercise to find a curve in G with 6 vertices, i.e., $\nu(G) \leq 6$. To show $\nu(G)\geq 6$, we devide γ, an arbitrary curve in G, into two parts as in Figure A.1. One portion has a maximal vertex because it contains a positive shell. The other has at least 2 maximal vertices, because otherwise we get, using Proposition 2.3, a curve geotopic to a curve in Figure 3.2 with only 2 maximal vertices, which contradicts Theorem 3.5. □

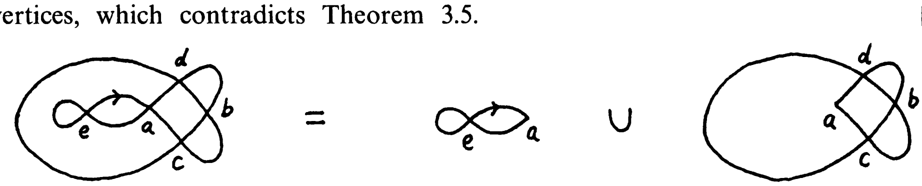


Figure A.1

EXAMPLE A.2. *For the geotopy class G corresponding to the Gauss word* $ab^{-1}bd^{-1}ca^{-1}dc^{-1}ee^{-1}$, $v(G)=6$.

Proof. There is a minimal vertex in the negative shell at b. We find two maximal vertices in the positive shells at c and e. Hence between them, we have second minimal vertex on the arc $\overline{ca^{-1}dc^{-1}ee^{-1}}$. If there is no other minimal vertex in the positive shell at c, we get third minimal vertex on $\overline{d^{-1}c}$ from Corollary 3.3 (ii). Therefore $v(G)\geq 6$. Again it is easy to see $v(G)\leq 6$. □

EXAMPLE A.3. *For the geotopy class G correponding to the Gauss word* $ad^{-1}be^{-1}ca^{-1}db^{-1}ec^{-1}$, $v(G)=8$.

For this geotopy class too, it is not hard to see $v(G)\leq 8$. The proof of $v(G)\geq 8$ needs efforts beyond this paper, and it will appear in a forthcoming paper [4].

In this way, we can determine $v(G)$ for all G with the number of crossings ≤ 5. The following are tables for $v(G)$. As above, we use Gauss words to express geotopy classes. For reference, we also add sketches of curves, which are not necessarily faithful to curvature of curves of least number of vertices.

Table 1

G	$v(G)$	G	$v(G)$
empty	4	$abb^{-1}a^{-1}$	2
aa^{-1}	2	$aa^{-1}bb^{-1}$	4

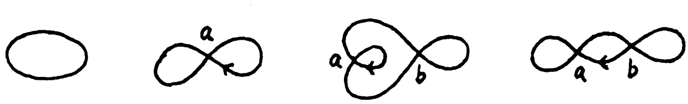


Figure A.2

Table 2

G	$v(G)$	G	$v(G)$
$abcc^{-1}b^{-1}a^{-1}$	2	$ac^{-1}ba^{-1}cb^{-1}$	6
$ab^{-1}bcc^{-1}a^{-1}$	4	$ab^{-1}ba^{-1}cc^{-1}$	4
$aa^{-1}bcc^{-1}b^{-1}$	4	$aa^{-1}bb^{-1}cc^{-1}$	6

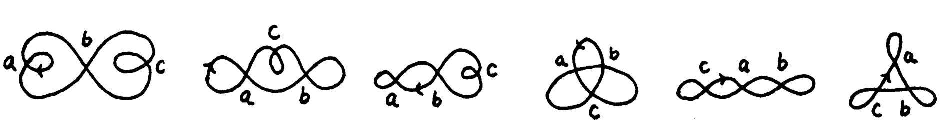


Figure A.3

Table 3

	G	$v(G)$		G	$v(G)$
(1)	$abcdd^{-1}c^{-1}b^{-1}a^{-1}$	2	(11)	$ab^{-1}bc^{-1}cdd^{-1}a^{-1}$	6
(2)	$abb^{-1}c^{-1}cdd^{-1}a^{-1}$	4	(12)	$ac^{-1}ba^{-1}cdd^{-1}b^{-1}$	4
(3)	$abd^{-1}a^{-1}cdb^{-1}c^{-1}$	4	(13)	$aa^{-1}bc^{-1}cdd^{-1}b^{-1}$	4
(4)	$abb^{-1}a^{-1}cdd^{-1}c^{-1}$	4	(14)	$ab^{-1}ba^{-1}cdd^{-1}c^{-1}$	4
(5)	$abb^{-1}cdd^{-1}c^{-1}a^{-1}$	4	(15)	$aa^{-1}bb^{-1}cdd^{-1}c^{-1}$	6
(6)	$ab^{-1}bcdd^{-1}c^{-1}a^{-1}$	4	(16)	$ac^{-1}bb^{-1}ca^{-1}dd^{-1}$	4
(7)	$aa^{-1}bcdd^{-1}c^{-1}b^{-1}$	4	(17)	$ac^{-1}ba^{-1}cb^{-1}dd^{-1}$	6
(8)	$ab^{-1}bcc^{-1}dd^{-1}a^{-1}$	6	(18)	$ab^{-1}ba^{-1}cc^{-1}dd^{-1}$	4
(9)	$aa^{-1}bcd^{-1}dc^{-1}b^{-1}$	4	(19)	$aa^{-1}bb^{-1}cc^{-1}dd^{-1}$	8
(10)	$aa^{-1}bcc^{-1}dd^{-1}b^{-1}$	6			

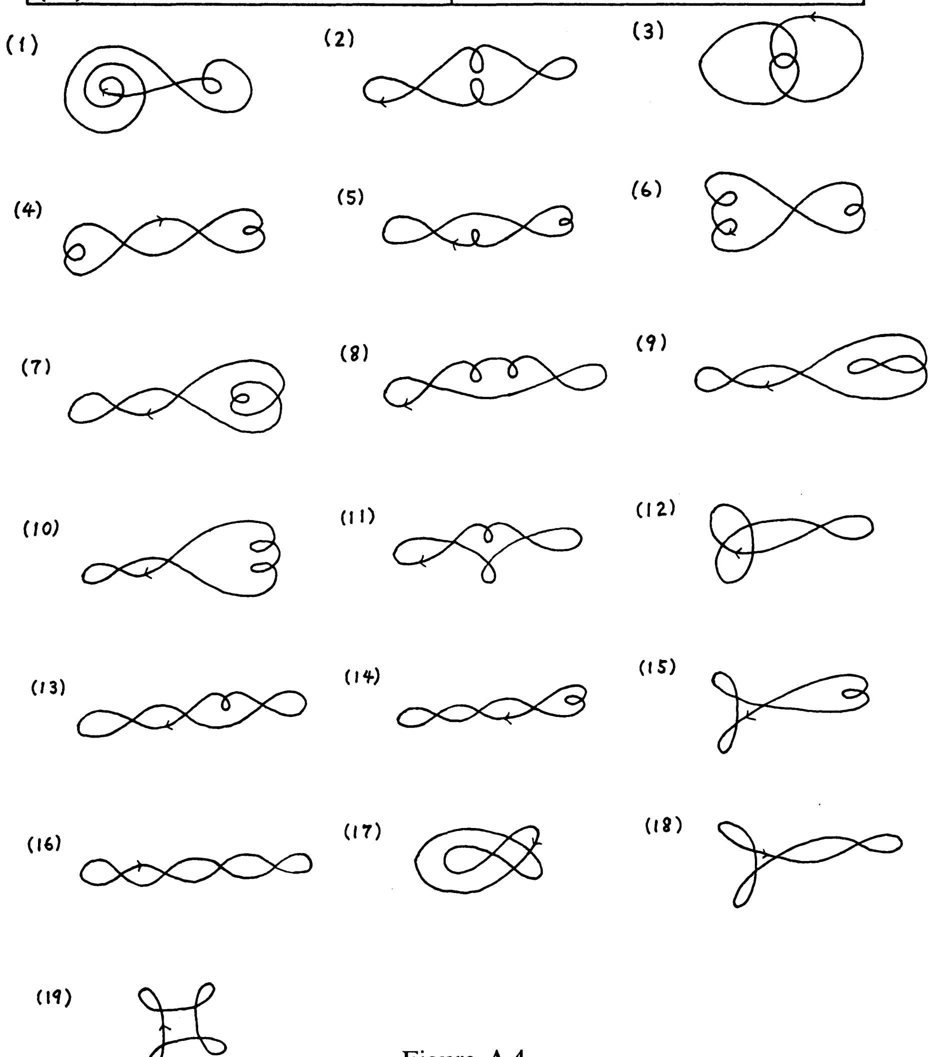
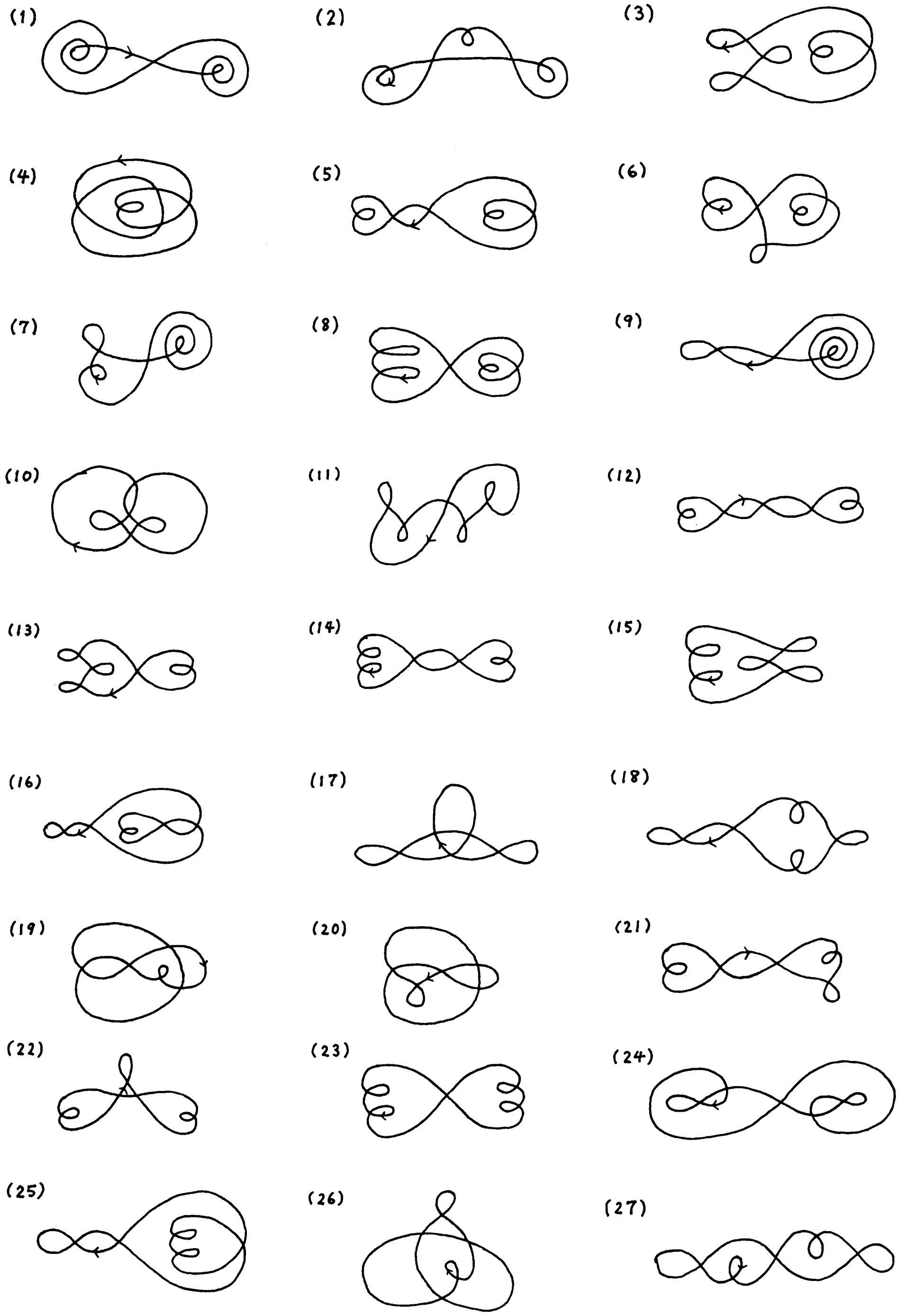


Figure A.4

Table 4

	G	$v(G)$		G	$v(G)$
(1)	$abcdee^{-1}d^{-1}c^{-1}b^{-1}a^{-1}$	2	(39)	$ab^{-1}bc^{-1}cde^{-1}ed^{-1}a^{-1}$	8
(2)	$abd^{-1}c^{-1}cdee^{-1}b^{-1}a^{-1}$	4	(40)	$ab^{-1}bc^{-1}cdd^{-1}ee^{-1}a^{-1}$	8
(3)	$abb^{-1}c^{-1}cdee^{-1}d^{-1}a^{-1}$	4	(41)	$ac^{-1}ba^{-1}cde^{-1}ed^{-1}b^{-1}$	6
(4)	$abd^{-1}a^{-1}cdee^{-1}b^{-1}c^{-1}$	4	(42)	$aa^{-1}bc^{-1}cde^{-1}ed^{-1}b^{-1}$	4
(5)	$abb^{-1}a^{-1}cdee^{-1}d^{-1}c^{-1}$	4	(43)	$ac^{-1}ba^{-1}cdd^{-1}ee^{-1}b^{-1}$	6
(6)	$abc^{-1}cdee^{-1}d^{-1}b^{-1}a^{-1}$	4	(44)	$aa^{-1}bc^{-1}cdd^{-1}ee^{-1}b^{-1}$	6
(7)	$abb^{-1}cdee^{-1}d^{-1}c^{-1}a^{-1}$	4	(45)	$ab^{-1}ba^{-1}cde^{-1}ed^{-1}c^{-1}$	4
(8)	$ab^{-1}bcdee^{-1}d^{-1}c^{-1}a^{-1}$	4	(46)	$aa^{-1}bb^{-1}cde^{-1}ed^{-1}c^{-1}$	6
(9)	$aa^{-1}bcdee^{-1}d^{-1}c^{-1}b^{-1}$	4	(47)	$ab^{-1}ba^{-1}cdd^{-1}ee^{-1}c^{-1}$	6
(10)	$ad^{-1}b^{-1}bca^{-1}dee^{-1}c^{-1}$	2	(48)	$aa^{-1}bb^{-1}cdd^{-1}ee^{-1}c^{-1}$	8
(11)	$aa^{-1}b^{-1}bcd^{-1}dee^{-1}c^{-1}$	6	(49)	$ac^{-1}bb^{-1}cd^{-1}dee^{-1}a^{-1}$	4
(12)	$ac^{-1}b^{-1}bca^{-1}dee^{-1}d^{-1}$	4	(50)	$ab^{-1}bc^{-1}cd^{-1}dee^{-1}a^{-1}$	8
(13)	$aa^{-1}b^{-1}bcc^{-1}dee^{-1}d^{-1}$	6	(51)	$aa^{-1}bd^{-1}cc^{-1}dee^{-1}b^{-1}$	6
(14)	$ab^{-1}bce^{-1}d^{-1}dec^{-1}a^{-1}$	6	(52)	$ac^{-1}ba^{-1}cd^{-1}dee^{-1}b^{-1}$	6
(15)	$ab^{-1}bcc^{-1}d^{-1}dee^{-1}a^{-1}$	6	(53)	$aa^{-1}bc^{-1}cd^{-1}dee^{-1}b^{-1}$	6
(16)	$aa^{-1}bce^{-1}d^{-1}dec^{-1}b^{-1}$	4	(54)	$ad^{-1}bb^{-1}ca^{-1}dee^{-1}c^{-1}$	4
(17)	$ad^{-1}bcc^{-1}a^{-1}dee^{-1}b^{-1}$	4	(55)	$ab^{-1}bd^{-1}ca^{-1}dee^{-1}c^{-1}$	4
(18)	$aa^{-1}bcc^{-1}d^{-1}dee^{-1}b^{-1}$	6	(56)	$aa^{-1}bd^{-1}cb^{-1}dee^{-1}c^{-1}$	6
(19)	$ab^{-1}bce^{-1}a^{-1}dec^{-1}d^{-1}$	4	(57)	$ab^{-1}ba^{-1}cd^{-1}dee^{-1}c^{-1}$	4
(20)	$aa^{-1}bce^{-1}b^{-1}dec^{-1}d^{-1}$	6	(58)	$aa^{-1}bb^{-1}cd^{-1}dee^{-1}c^{-1}$	6
(21)	$ab^{-1}bcc^{-1}a^{-1}dee^{-1}d^{-1}$	4	(59)	$ac^{-1}bb^{-1}ca^{-1}dee^{-1}d^{-1}$	4
(22)	$aa^{-1}bcc^{-1}b^{-1}dee^{-1}d^{-1}$	6	(60)	$ab^{-1}bc^{-1}ca^{-1}dee^{-1}d^{-1}$	4
(23)	$ab^{-1}bcdd^{-1}ee^{-1}c^{-1}a^{-1}$	6	(61)	$ac^{-1}ba^{-1}cb^{-1}dee^{-1}d^{-1}$	6
(24)	$aa^{-1}bcde^{-1}ed^{-1}c^{-1}b^{-1}$	4	(62)	$ab^{-1}ba^{-1}cc^{-1}dee^{-1}d^{-1}$	4
(25)	$aa^{-1}bcdd^{-1}ee^{-1}c^{-1}b^{-1}$	6	(63)	$aa^{-1}bb^{-1}cc^{-1}dee^{-1}d^{-1}$	8
(26)	$abd^{-1}cb^{-1}dee^{-1}c^{-1}a^{-1}$	4	(64)	$ad^{-1}be^{-1}ca^{-1}db^{-1}ec^{-1}$	8
(27)	$abb^{-1}cd^{-1}dee^{-1}c^{-1}a^{-1}$	4	(65)	$ac^{-1}be^{-1}ca^{-1}db^{-1}ed^{-1}$	6
(28)	$abc^{-1}cb^{-1}dee^{-1}d^{-1}a^{-1}$	4	(66)	$ad^{-1}bc^{-1}cb^{-1}da^{-1}ee^{-1}$	4
(29)	$abb^{-1}cc^{-1}dee^{-1}d^{-1}a^{-1}$	6	(67)	$ac^{-1}bd^{-1}cb^{-1}da^{-1}ee^{-1}$	6
(30)	$ab^{-1}bcd^{-1}dee^{-1}c^{-1}a^{-1}$	6	(68)	$ad^{-1}bc^{-1}ca^{-1}db^{-1}ee^{-1}$	4
(31)	$ab^{-1}bcc^{-1}dee^{-1}d^{-1}a^{-1}$	6	(69)	$ad^{-1}bb^{-1}ca^{-1}dc^{-1}ee^{-1}$	6
(32)	$aa^{-1}bcd^{-1}dee^{-1}c^{-1}b^{-1}$	4	(70)	$ab^{-1}bd^{-1}ca^{-1}dc^{-1}ee^{-1}$	6
(33)	$aa^{-1}bcc^{-1}dee^{-1}d^{-1}b^{-1}$	6	(71)	$ab^{-1}ba^{-1}cd^{-1}dc^{-1}ee^{-1}$	4
(34)	$ab^{-1}bc^{-1}cdee^{-1}d^{-1}a^{-1}$	6	(72)	$ac^{-1}bb^{-1}ca^{-1}dd^{-1}ee^{-1}$	6
(35)	$ac^{-1}ba^{-1}cdee^{-1}d^{-1}b^{-1}$	4	(73)	$ab^{-1}bc^{-1}ca^{-1}dd^{-1}ee^{-1}$	6
(36)	$aa^{-1}bc^{-1}cdee^{-1}d^{-1}b^{-1}$	4	(74)	$ac^{-1}ba^{-1}cb^{-1}dd^{-1}ee^{-1}$	8
(37)	$ab^{-1}ba^{-1}cdee^{-1}d^{-1}c^{-1}$	4	(75)	$ab^{-1}ba^{-1}cc^{-1}dd^{-1}ee^{-1}$	6
(38)	$aa^{-1}bb^{-1}cdee^{-1}d^{-1}c^{-1}$	6	(76)	$aa^{-1}bb^{-1}cc^{-1}dd^{-1}ee^{-1}$	10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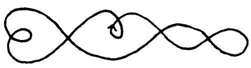
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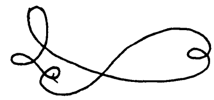
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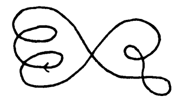
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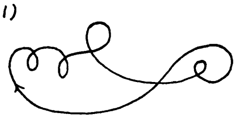
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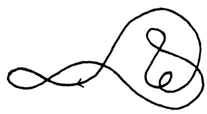
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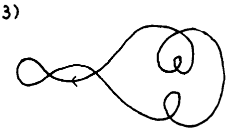
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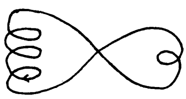
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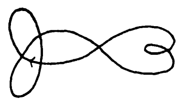
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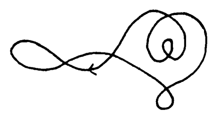
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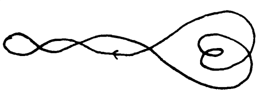
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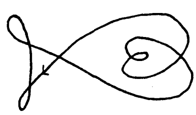
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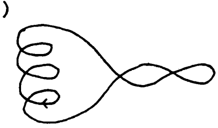
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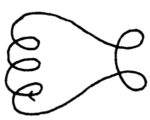
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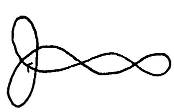
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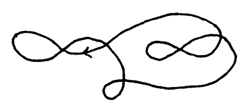
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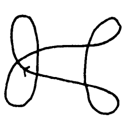
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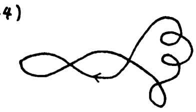
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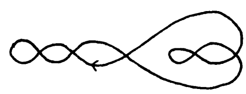
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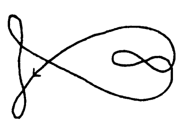
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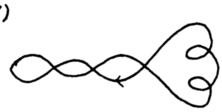
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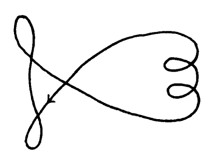
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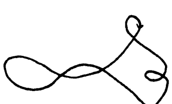
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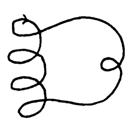
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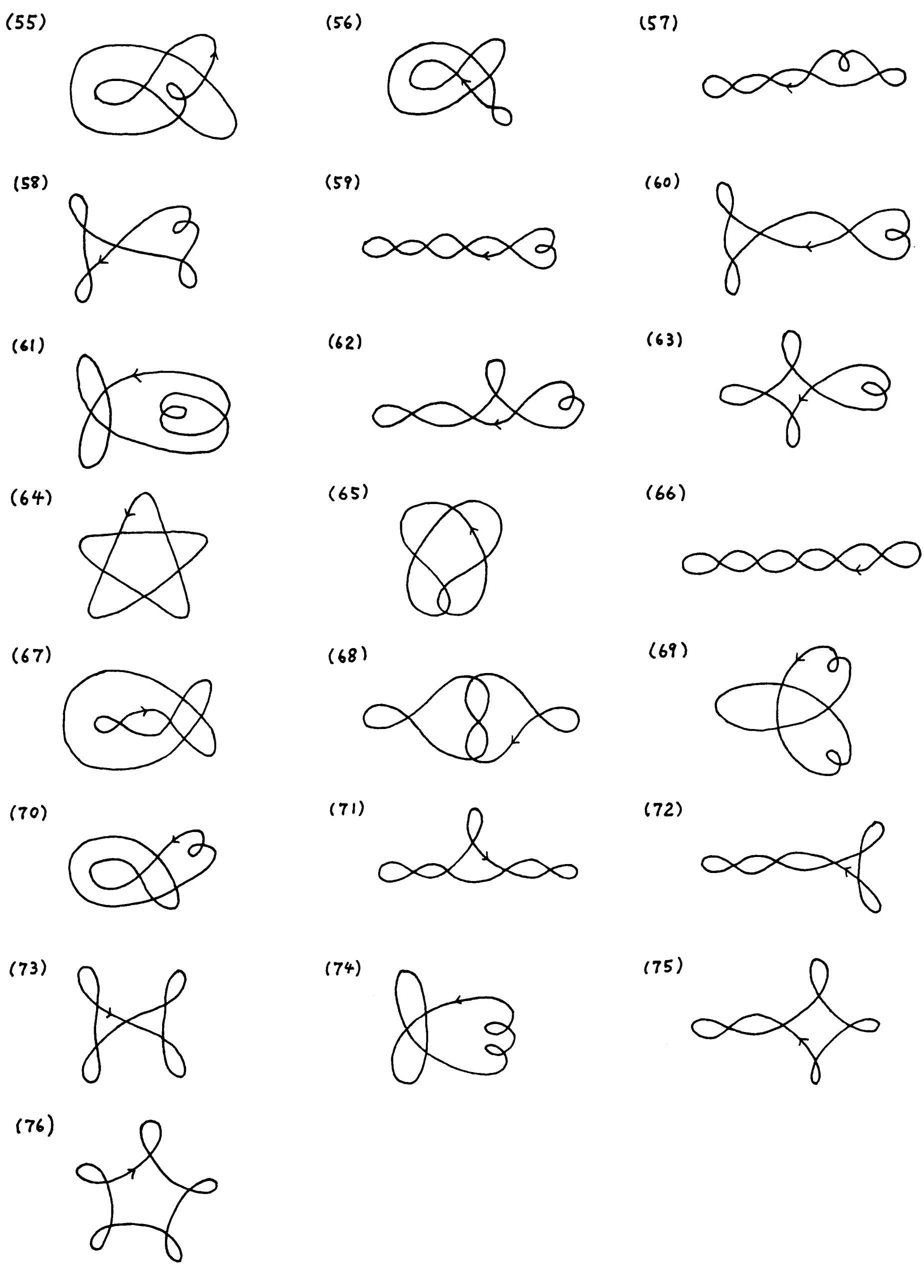

Figure A.5

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