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TOTALLY GEODESIC HYPERSURFACES OF NATURALLY REDUCTIVE HOMOGENEOUS SPACES

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1. Introduction

Totally geodesic submanifolds of Riemannian symmetric spaces have been well investigated and it has been shown that they have beautiful and fruitful properties. In particular, due to the (M_+, M_-)-theory by B.Y. Chen and T. Nagano [1] this subject has made great progress. Naturally reductive homogeneous spaces are known as a natural generalization of Riemannian symmetric spaces. K. Tojo [6] investigated totally geodesic submanifolds of naturally reductive homogeneous spaces and obtained a necessary and sufficient condition of their existence. We will recall his result in section 3. Moreover he implicitly made the following conjecture.

Conjecture. *If a simply connected irreducible naturally reductive homogeneous space M admits a totally geodesic hypersurface, then M has constant sectional curvature.*

The conjecture is regarded as a generalization of the result which was shown in the case of Riemannian symmetric spaces by B.Y. Chen and T. Nagano [1]. K. Tojo gave an affirmative answer to the conjecture in the case that $\dim M=3$, 4 and 5 [6] and in the case that M is a normal homogeneous space [7]. We shall prove that the conjecture above is true.

Main Theorem. *If a simply connected irreducible* (*as a Riemannian manifold*) *naturally reductive homogeneous space M admits a totally geodesic hypersurface, then M has constant sectional curvature.*

We shall discuss the irreducibility of naturally reductive homogeneous spaces in Section 2 and prove the main theorem in Section 3.

2. Irreducibility of naturally reductive homogeneous spaces

We first recall basic definitions and properties of naturally reductive homogeneous spaces, following J.E. D'Atri and W. Ziller [2] and S. Kobayashi and K. Nomizu [3]. See also O. Kowalski and L. Vanhecke [4], [5]. Let (M,g) be a homogeneous Riemannian manifold. Let K be a connected Lie group of isometries which acts transitively and almost effectively on M and let H be the isotropy subgroup at a point $o \in M$. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{h}$ the subalgebra corresponding to H. Let $\mathfrak{m}$ be an $Ad(H)$-invariant subspace which is complementary to $\mathfrak{h}$ in $\mathfrak{k}$. We denote by $x_{\mathfrak{h}}$ and $x_{\mathfrak{m}}$ the $\mathfrak{h}$-component and the $\mathfrak{m}$-component of $x \in \mathfrak{k}$, respectively. As usual we identify $\mathfrak{m}$ with the tangent space T_oM at o and denote by $\langle , \rangle$ the inner product on $\mathfrak{m}$ induced from the metric g_o on T_oM.

DEFINITION 2.1. A homogeneous Riemannian manifold (M,g) is said to be a *naturally reductive homogeneous space* if there exist K and $\mathfrak{m}$ as above such that

$$\langle [x,y]_{\mathfrak{m}}, z \rangle + \langle y, [x,z]_{\mathfrak{m}} \rangle = 0 \quad \text{for any } x,y,z \in \mathfrak{m}. \tag{2.1}$$

From now on we assume that (M,g) is a naturally reductive homogeneous space. Then by a theorem of Kostant we may assume that $\mathfrak{k} = \mathfrak{m} + [\mathfrak{m},\mathfrak{m}]$. Let $\Lambda_{\mathfrak{m}}: \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ be a linear mapping which corresponds to the Riemannian connection ∇ (see [3] Chapter X), where $\mathfrak{so}(\mathfrak{m})$ denotes the Lie algebra consisting of skew symmetric endomorphisms of $(\mathfrak{m}, \langle , \rangle)$. Then $\Lambda_{\mathfrak{m}}$ is given by

$$\Lambda_{\mathfrak{m}}(x)(y) = \frac{1}{2}[x,y]_{\mathfrak{m}} \quad \text{for } x,y \in \mathfrak{m} \tag{2.2}$$

(cf. Theorem 3.3 p.201 in [3]),

DEFINITION 2.2. A subspace V of $\mathfrak{m}$ is said to be $\Lambda_{\mathfrak{m}}$-*invariant* if it satisfies $\Lambda_{\mathfrak{m}}(x)(V) \subset V$ for any $x \in \mathfrak{m}$. Moreover a $\Lambda_{\mathfrak{m}}$-invariant subspace V is $\Lambda_{\mathfrak{m}}$-*irreducible* if V has only trivial $\Lambda_{\mathfrak{m}}$-invariant subspaces.

We set $\mathfrak{m}_0 = \{v \in \mathfrak{m} \mid \Lambda_{\mathfrak{m}}(x)(v) = 0 \text{ for any } x \in \mathfrak{m}\}$. Then we evidently have the following orthogonal decomposition into $\Lambda_{\mathfrak{m}}$-invariant subspaces:

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r, \tag{2.3}$$

where for each i $(1 \leq i \leq r)$ $\mathfrak{m}_i$ is $\Lambda_{\mathfrak{m}}$-irreducible and $\Lambda_{\mathfrak{m}}(x)|_{\mathfrak{m}_i} \neq 0$ for some $x \in \mathfrak{m}$.

Theorem 2.3. *Let $M = K/H$ be a naturally reductive homogeneous space with $Ad(H)$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$. We assume that $\mathfrak{k} = \mathfrak{m} + [\mathfrak{m},\mathfrak{m}]$. Let*

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$$

be the decomposition of $\mathfrak{m}$ *which satisfies* (2.3). *If we set*

$$\mathfrak{k}_i = \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i] \quad (i = 0, 1, \cdots, r)$$

$$\mathfrak{h}_i=\mathfrak{k}_i\cap\mathfrak{h} \qquad (i=0,1,\cdots,r),$$

then we have $\mathfrak{k}=\mathfrak{k}_0\oplus\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_r$ *and* $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_r$ *as direct sums of Lie algebras.*

Proof. We first show the following identity.

Lemma 2.4. *Let $M=K/H$ be a homogeneous space with $Ad(H)$-invariant decomposition $\mathfrak{k}=\mathfrak{h}\oplus\mathfrak{m}$. Then the following holds:*

$$[[x,y]_{\mathfrak{m}},z]_{\mathfrak{h}}+[[y,z]_{\mathfrak{m}},x]_{\mathfrak{h}}+[[z,x]_{\mathfrak{m}},y]_{\mathfrak{h}}=0$$

for $x,y,z\in\mathfrak{m}$.

Proof of Lemma 2.4. By the Jacobi's identity, we have

$$\begin{aligned}0&=[[x,y],z]+[[y,z],x]+[[z,x],y]\\&=[[x,y]_{\mathfrak{h}},z]+[[y,z]_{\mathfrak{h}},x]+[[z,x]_{\mathfrak{h}},y]\\&\quad+[[x,y]_{\mathfrak{m}},z]+[[y,z]_{\mathfrak{m}},x]+[[z,x]_{\mathfrak{m}},y]\end{aligned}$$

for $x,y,z\in\mathfrak{m}$.

Comparing the $\mathfrak{h}$-components of both sides, we obtain the identity in Lemma 2.4. □

By (2.2) and (2.3), we have $[\mathfrak{m},\mathfrak{m}_i]_{\mathfrak{m}}\subset\mathfrak{m}_i$. In particular,

(2.4) $$[\mathfrak{m}_i,\mathfrak{m}_j]_{\mathfrak{m}}=0 \qquad \text{for } i\neq j,$$

(2.5) $$[\mathfrak{m}_i,\mathfrak{m}_i]_{\mathfrak{m}}=\mathfrak{m}_i \qquad \text{for } i\geq 1.$$

Lemma 2.5. *The following relations hold:*

(1) $[\mathfrak{m}_i,\mathfrak{m}_j]=0$ *for $i\neq j$.*
(2) $[[\mathfrak{m}_i,\mathfrak{m}_i],\mathfrak{m}_j]=0$ *for $i\neq j$.*
(3) $[[\mathfrak{m}_i,\mathfrak{m}_i]_{\mathfrak{h}},\mathfrak{m}_i]\subset\mathfrak{m}_i$.
(4) $[[\mathfrak{m}_i,\mathfrak{m}_i],\mathfrak{m}_i]\subset\mathfrak{m}_i+[\mathfrak{m}_i,\mathfrak{m}_i]$.

Proof of Lemma 2.5. (1) It is sufficient to prove that $[\mathfrak{m}_i,\mathfrak{m}_j]_{\mathfrak{h}}=0$ for $i\neq j$. We may assume that $i\geq 1$. By Lemma 2.4, we have for $x,y\in\mathfrak{m}_i$ and $z\in\mathfrak{m}_j$,

$$[[x,y]_{\mathfrak{m}},z]_{\mathfrak{h}}=-[[y,z]_{\mathfrak{m}},x]_{\mathfrak{h}}-[[z,x]_{\mathfrak{m}},y]_{\mathfrak{h}}=0.$$

Since $[\mathfrak{m}_i,\mathfrak{m}_i]_{\mathfrak{m}}=\mathfrak{m}_i$ for $i\geq 1$, we have $[\mathfrak{m}_i,\mathfrak{m}_j]_{\mathfrak{h}}=0$.

(2) From the Jacobi's identity and (1), it follows that for $x,y\in\mathfrak{m}_i$, $z\in\mathfrak{m}_j$

$$[[x,y],z]=-[[y,z],x]-[[z,x],y]=0.$$

(3) By (1) and (2), we obtain $[[x,y]_{\mathfrak{h}},z]=0$ for $x,y\in\mathfrak{m}_i$, $z\in\mathfrak{m}_j$ $(i\neq j)$. Therefore for $x,y,v\in\mathfrak{m}_i$, $z\in\mathfrak{m}_j$ $(i\neq j)$

$$\langle[[x,y]_{\mathfrak{h}},v],z\rangle=-\langle v,[[x,y]_{\mathfrak{h}},z]\rangle=0,$$

that is, $[[x,y]_{\mathfrak{h}},\mathfrak{m}_i]\subset\mathfrak{m}_i$.
(4) By (3) and (2.5), we obtain (4). □

Proof of Theorem 2.3. We first prove that each $\mathfrak{k}_i$ is an ideal of $\mathfrak{k}$. In fact applying the relations in Lemma 2.5, we obtain the following:

$$\begin{aligned}
[\mathfrak{m},\mathfrak{m}_i] &\subset [\mathfrak{m}_i,\mathfrak{m}_i],\\
[\mathfrak{m},[\mathfrak{m}_i,\mathfrak{m}_i]] &\subset [\mathfrak{m}_i,[\mathfrak{m}_i,\mathfrak{m}_i]]\subset\mathfrak{m}_i+[\mathfrak{m}_i,\mathfrak{m}_i],\\
[[\mathfrak{m},\mathfrak{m}],\mathfrak{m}_i] &\subset [\sum_{j=0}^{r}[[\mathfrak{m}_j,\mathfrak{m}_j],\mathfrak{m}_i],\\
&\subset [[\mathfrak{m}_i,\mathfrak{m}_i],\mathfrak{m}_i]\subset\mathfrak{m}_i+[\mathfrak{m}_i,\mathfrak{m}_i],\\
[[\mathfrak{m},\mathfrak{m}],[\mathfrak{m}_i,\mathfrak{m}_i]] &\subset [[[\mathfrak{m},\mathfrak{m}],\mathfrak{m}_i],\mathfrak{m}_i]\\
&\subset [\mathfrak{m}_i+[\mathfrak{m}_i,\mathfrak{m}_i],\mathfrak{m}_i]\subset\mathfrak{m}_i+[\mathfrak{m}_i,\mathfrak{m}_i].
\end{aligned}$$

Since $[\mathfrak{m}_i,\mathfrak{m}_i]_{\mathfrak{m}}\subset\mathfrak{m}_i$ $((i=0,1,\cdots,r)$, we have $\mathfrak{h}_i=[\mathfrak{m}_i,\mathfrak{m}_i]_{\mathfrak{h}}$ and hence $\mathfrak{k}_i=\mathfrak{m}_i\oplus\mathfrak{h}_i$ (direct sum). Finally we shall show that $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_r$ as a direct sum of vector spaces. Let x be a vector of $(\mathfrak{h}_0+\cdots+\mathfrak{h}_i)\cap\mathfrak{h}_{i+1}$. Since $x\in\mathfrak{h}_{i+1}$ by (1) and (2), it follows $[x,v]=0$ for any $v\in\mathfrak{m}_0+\cdots+\mathfrak{m}_i+\mathfrak{m}_{i+2}+\cdots+\mathfrak{m}_r$. On the other hand since $x\in\mathfrak{h}_0+\cdots+\mathfrak{h}_i$ again by (1) and (2), it follows $[x,v]=0$ for any $v\in\mathfrak{m}_{i+1}$. These imply $[x,v]=0$ for any $v\in\mathfrak{m}$. Since K acts almost effectively on M, we have $x=0$. Hence $(\mathfrak{h}_0+\cdots+\mathfrak{h}_i)\cap\mathfrak{h}_{i+1}=0$. Since $[\mathfrak{m},\mathfrak{m}]_{\mathfrak{h}}=\mathfrak{h}$, we have $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_r$. Noticing that $\mathfrak{k}_i$ are ideals of $\mathfrak{k}$, we have $\mathfrak{k}=\mathfrak{k}_0\oplus\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_r$ and $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_r$ as direct sums of Lie algebras. □

Corollary 2.6. *Let $M=K/H$ be a simply connected irreducible (as a Riemannian manifold) naturally reductive homogeneous space. If $\Lambda_{\mathfrak{m}}\neq 0$, $\mathfrak{m}$ is $\Lambda_{\mathfrak{m}}$-irreducible.*

Proof. Let $\mathfrak{m}=\mathfrak{m}_0\oplus\mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_r$ be the decomposition of $\mathfrak{m}$ which satisfies (2.3). By Theorem 2.3, we see that each $\mathfrak{m}_i$ is an invariant subspace by the holonomy algebra of the Riemannian connection (cf. see [3] Chapter X §4). Therefore the above decomposition has the only one factor. Since $\Lambda_{\mathfrak{m}}\neq 0$, $\mathfrak{m}\neq\mathfrak{m}_0$ and thus $\mathfrak{m}$ is $\Lambda_{\mathfrak{m}}$-irreducible. □

3. Proof of the Main Theorem

We first recall a theorem of K. Tojo ([6]). Let $M=K/H$ be a naturally reductive homogeneous space with $Ad(H)$-invariant decomposition $\mathfrak{k}=\mathfrak{h}\oplus\mathfrak{m}$. According to [6], we put $\varphi_x=\Lambda_{\mathfrak{m}}(x)$ for simplicity. Since φ_x is a skew symmetric endomorphism on $(\mathfrak{m},\langle\,,\rangle)$, e^{φ_x} is defined as a linear isometry on $(\mathfrak{m},\langle\,,\rangle)$. Then K. Tojo showed the following (Theorem 3.2 in [6]).

Theorem 3.1. *Let V be a subspace of $\mathfrak{m}$ (which is canonically identified with T_0M). Then there exists a totally geodesic submanifold of M through o whose tangent space at o is V if and only if the following holds*:

$$R(e^{\varphi_x}(V),e^{\varphi_x}(V))e^{\varphi_x}(V)\subset e^{\varphi_x}(V)\quad \textit{for any } x\in V,$$

where R denotes the Riemannian curvature tensor of M.

The above theorem is considered as a generalization of the Lie triple system in Riemannian symmetric spaces due to E. Cartan.

Now we shall prove Main Theorem. Let M be as in Main Theorem. If $\Lambda_{\mathfrak{m}}=0$, then M is a simply connected irreducible Riemannian symmetric space. In this case, our theorem has been proved by B.Y. Chen and T. Nagano [1]. Therefore we assume that $\Lambda_{\mathfrak{m}}\neq 0$. By Corollary 2.6, it follows that $\mathfrak{m}$ is $\Lambda_{\mathfrak{m}}$-irreducible. Let S be a totally geodesic hypersurface of M. Since M is a homogeneous Riemannian manifold, we may assume that S is through o. Let V be a hyperplane (i.e., a subspace with codimension 1) of $\mathfrak{m}$ which is a tangent space of S at o. We denote by ξ the unit vector of $\mathfrak{m}$ which is orthogonal to V. We set

$$V_1=\{\varphi_\xi x\mid x\in\mathfrak{m}\}=\{\varphi_\xi x\mid x\in V\}.$$

Then V_1 is a subspace of V. In fact for any $x\in\mathfrak{m}$, $\langle\varphi_\xi x,\xi\rangle=-\langle x,\varphi_\xi\xi\rangle=0$. Since $\mathfrak{m}$ is $\Lambda_{\mathfrak{m}}$-irreducible, $V_1\neq 0$. We set $O_1=\boldsymbol{R}\xi\oplus V_1$.

Lemma 3.2. *The following equations hold*:

(1) $\langle R(x,y)z,\xi\rangle=0$.

(2) $\langle\varphi_\xi x,y\rangle\langle R(z,\xi)\xi,w\rangle-\langle\varphi_\xi x,z\rangle\langle R(y,\xi)\xi,w\rangle=\langle R(y,z)w,\varphi_\xi x\rangle$

for $x,y,z\in V,w\in\mathfrak{m}$.

Proof of Lemma 3.2. Applying Theorem 3.1, we obtain

$$\langle R(e^{t\varphi_x}y,e^{t\varphi_x}z)e^{t\varphi_x}w,e^{t\varphi_x}\xi\rangle=0 \tag{3.1}$$

for $x,y,z,w\in V,t\in\boldsymbol{R}$.

Putting $t=0$ in (3.1), we obtain (1). Differentiating (3.1) with respect to t at $t=0$,

$$\begin{aligned}\langle R(\varphi_x y,z)w,\xi\rangle+\langle R(y,\varphi_x z)w,\xi\rangle\\ +\langle R(y,z)\varphi_x w,\xi\rangle+\langle R(y,z)w,\varphi_x\xi\rangle=0.\end{aligned}\tag{3.2}$$

We put $\varphi_x y = \langle \varphi_x y, \xi \rangle \xi + v$, where $v \in V$. Then by the equation (1) in this lemma

$$\begin{aligned}\langle R(\varphi_x y, z)w, \xi \rangle &= \langle \varphi_x y, \xi \rangle \langle R(\xi, z)w, \xi \rangle + \langle R(v, z)w, \xi \rangle \\ &= \langle \varphi_\xi x, y \rangle \langle R(z, \xi)\xi, w \rangle.\end{aligned}$$

Similarly we have

$$\langle R(y, \varphi_x z)w, \xi \rangle = -\langle \varphi_\xi x, z \rangle \langle R(y, \xi)\xi, w \rangle$$

$$\langle R(y, z)\varphi_x w, \xi \rangle = \langle \varphi_\xi x, w \rangle \langle R(y, z)\xi, \xi \rangle = 0.$$

Substituting them in (3.2), we obtain (2) for $w \in V$. If $w = \xi$, the both sides of (2) are equal to 0. Therefore the equation (2) holds for all $w \in \mathfrak{m}$. □

By Lemma 3.2 (2), it follows that

$$\langle v, y \rangle \langle R(z, \xi)\xi, w \rangle - \langle v, z \rangle \langle R(y, \xi)\xi, w \rangle = -\langle R(y, z)v, w \rangle \tag{3.3}$$

for $v \in V_1$, $y, z \in V$, $w \in \mathfrak{m}$.
For $x \in \mathfrak{m}$, we define a symmetric endomorphism $R_x : \mathfrak{m} \to \mathfrak{m}$ by $R_x y = R(y, x)x$.

Lemma 3.3. *There exists a constant c such that $R_\xi x = cx$ for any $x \in V_1$.*

Proof of Lemma 3.3. Let x be an arbitrary non-zero vector of V_1 and y be a vector of V which is orthogonal to x. Putting $v = z = w = x$ in (3.3), we have $\langle R(x, \xi)\xi, y \rangle = 0$. On the other hand, clearly $\langle R(x, \xi)\xi, \xi \rangle = 0$. This implies that V_1 is a subspace of some eigenspace with respect to R_ξ. We may take its eigenvalue as the constant c. □

Lemma 3.4. *For any $v \in O_1$, the following relations hold:*
(1) $R(y, z)v = 0$ *for any* $y, z \in v^\perp$,
(2) $R_v x = c\{\langle v, v \rangle x - \langle x, v \rangle v\}$ *for* $x \in O_1$,
(3) $R_v x = \langle v, v \rangle R_\xi x$ *for* $x \in O_1^\perp$,
where $v^\perp$ and $O_1^\perp$ denote the orthogonal complements in $\mathfrak{m}$ of v and O_1, respectively and the constant c in (2) *is given in Lemma* 3.3.

Proof of Lemma 3.4. We consider the following three cases for $v \in O_1$:
Case 1. $v = \xi$;
Case 2. v is a unit vector of V_1. In this case we denote e by such a v;
Case 3. v is an arbitrary unit vector of O_1.

Case 1. By Lemma 3.2 (1), $R(y, z)\xi = 0$ for any $y, z \in V$. By Lemma 3.3

$$R_\xi x = c\{x - \langle x, \xi \rangle \xi\} \quad \text{for } x \in O_1.$$

Therefore (1), (2), and (3) in Lemma 3.4 hold for this case.

Case 2. Let y, z be vectors of $e^{\perp}\cap V$. Putting $v=e$ in (3.3), we have $R(y,z)e=0$. Moreover it holds that $R(y,\xi)e=0$. In fact, for $w\in V$,

$$\langle R(y,\xi)e,w\rangle=\langle R(e,w)y,\xi\rangle=0$$

and

$$\langle R(y,\xi)e,\xi\rangle=-\langle R(e,\xi)\xi,y\rangle=-\langle R_{\xi}e,y\rangle=-c\langle e,y\rangle=0.$$

From these, we see that (1) holds. Applying (3.3) for $v=z=e$ and $y\in e^{\perp}\cap V$, we obtain $R_e y=R_{\xi}y$. Hence (2) and (3) hold.

Case 3. It is easily seen that the following relations hold:

$$R(y,e)\xi=-c\langle y,\xi\rangle e$$
$$R(y,\xi)e=-c\langle y,e\rangle\xi$$

for a unit vector $e\in V_1$ and any $y\in\mathfrak{m}$.
We put $v=\cos\theta e+\sin\theta\xi$ for some unit vector $e\in V_1$ and some $\theta\in\boldsymbol{R}$. For $y,z\in e^{\perp}\cap V$, we have

$$R(y,z)v=\cos\theta R(y,z)e+\sin\theta R(y,z)\xi=0,$$

$$\begin{aligned}&R(y,-\sin\theta e+\cos\theta\xi)v\\&\quad=-\sin\theta\cos\theta R(y,e)e-\sin^2\theta R(y,e)\xi\\&\qquad+\cos^2\theta R(y,\xi)e+\sin\theta\cos\theta R(y,\xi)\xi\\&\quad=\sin\theta\cos\theta\{R_{\xi}y-R_e y\}=0.\end{aligned}$$

Hence in this case (1) holds.

For $x\in\mathfrak{m}$, we have

$$\begin{aligned}R_v x&=\cos^2\theta R_e x+\sin^2\theta R_{\xi}x+\sin\theta\cos\theta\{R(x,e)\xi+R(x,\xi)e\}\\&=\cos^2\theta R_e x+\sin^2\theta R_{\xi}x-c\sin\theta\cos\theta\{\langle x,\xi\rangle e+\langle x,e\rangle\xi\}.\end{aligned}\tag{3.4}$$

For $x\in O_1$, (3.4) implies

$$\begin{aligned}R_v x&=c\cos^2\theta\{x-\langle x,e\rangle e\}+c\sin^2\theta\{x-\langle x,\xi\rangle\xi\}\\&\quad-c\sin\theta\cos\theta\{\langle x,\xi\rangle e+\langle x,e\rangle\xi\}\\&=c\{x-\langle x,v\rangle v\}.\end{aligned}$$

For $x\in O_1^{\perp}$, (3.4) implies $R_v x=R_{\xi}x$. □

Lemma 3.5. *The following identity holds:*

$$\underset{x,y,z}{\mathfrak{S}} \{\varphi_x(R(y,z)w) - R(\varphi_x y,z)w - R(y,\varphi_x z)w - R(y,z)\varphi_x w\} = 0$$

for $x,y,z,w \in \mathfrak{m}$.
Here the symbol $\mathfrak{S}$ *denotes the cyclic sum with respect to the indicated variables.*

Proof of Lemma 3.5. It is known that the covariant derivative ∇R of R is given as follows

$$\begin{aligned}(\nabla_x R)(y,z)w &= (\varphi_x \cdot R)(y,z)w \\ &= \varphi_x(R(y,z)w) - R(\varphi_x y,z)w - R(y,\varphi_x z)w - R(y,z)\varphi_x w.\end{aligned}$$

By this and Bianchi's 2nd identity of ∇R, we have the identity in this lemma. □

We consider the symmetric endomorphism $R_\xi : \mathfrak{m} \to \mathfrak{m}$. Evidently we have $R_\xi(V) \subset V$. Then V is decomposed into the eigenspaces of R_ξ:

$$V = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_l,$$

where each $\mathfrak{p}_i$ $(i=1,\cdots,l)$ is the eigenspace of R_ξ with eigenvalue λ_i. Here we set $\lambda_1 = c$, where the constant c has been given in Lemma 3.3. By Lemma 3.3, it follows that $V_1 \subset \mathfrak{p}_1$.

Lemma 3.6. (1) *For* $x,y \in \mathfrak{p}_1$, $\varphi_x y \in \boldsymbol{R}\xi \oplus \mathfrak{p}_1$.
(2) *For* $x \in \mathfrak{p}_i$, $y \in \mathfrak{p}_j$ $(j \neq 1)$, $\varphi_x y$ *is contained in the eigenspace of* R_ξ *with eigenvalue* $\frac{\lambda_i + \lambda_j}{2}$.

Proof of Lemma 3.6. By Lemma 3.5, we have for $x \in \mathfrak{p}_i$, $y \in \mathfrak{p}_j$

$$\begin{aligned}0 &= \varphi_\xi(R(x,y)\xi) - R(\varphi_\xi x,y)\xi - R(x,\varphi_\xi y)\xi - R(x,y)\varphi_\xi \xi \\ &\quad + \varphi_x(R(y,\xi)\xi) - R(\varphi_x y,\xi)\xi - R(y,\varphi_x \xi)\xi - R(y,\xi)\varphi_x \xi \\ &\quad + \varphi_y(R(\xi,x)\xi) - R(\varphi_y \xi,x)\xi - R(\xi,\varphi_y x)\xi - R(\xi,x)\varphi_y \xi \\ &= \lambda_j \varphi_x y - 2R_\xi(\varphi_x y) - R(y,\xi)\varphi_x \xi - \lambda_i \varphi_y x + R(x,\xi)\varphi_y \xi \\ &= (\lambda_i + \lambda_j)\varphi_x y - 2R_\xi(\varphi_x y) + 2c\langle \varphi_x \xi, y\rangle \xi.\end{aligned}$$

Hence

$$2R_\xi(\varphi_x y) = (\lambda_i + \lambda_j)\varphi_x y + 2c\langle \varphi_x \xi, y\rangle \xi. \tag{3.5}$$

If $i=j=1$, then (3.5) implies $R_\xi(\varphi_x y) = c\{\varphi_x y - \langle \varphi_x y, \xi\rangle \xi\}$. Therefore (1) in this lemma holds. If $j \neq 1$, (3.5) implies $R_\xi(\varphi_x y) = \frac{\lambda_i + \lambda_j}{2} \varphi_x y$. Therefore (2) in this lemma holds. □

Lemma 3.7. *If $x \in \mathfrak{p}_i$, $y \in \mathfrak{p}_j$ $(i \neq j)$, then we have $\varphi_x y = 0$.*

Proof of Lemma 3.7. We assume that $j \neq 1$ and that $\varphi_x y \neq 0$. We set $\varphi_x y = z$. Then by Lemma 3.6 (2), z is an eigenvector of R_ξ with eigenvalue $\frac{\lambda_i + \lambda_j}{2}$. Since $0 \neq \langle \varphi_x y, z \rangle = -\langle y, \varphi_x z \rangle$, y and $\varphi_x z$ are eigenvectors of R_ξ with same eigenvalue. Therefore we have $\lambda_j = \frac{1}{2}(\frac{\lambda_i + \lambda_j}{2} + \lambda_i)$ and hence $\lambda_i = \lambda_j$, that is, $i = j$. It is contrary to our assumption $i \neq j$. Therefore we have $\varphi_x y = 0$. □

Since $V_1 \subset \mathfrak{p}_1$, together with Lemmas 3.6 and 3.7, we see that $\boldsymbol{R}\xi \oplus \mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_l$ are $\Lambda_{\mathfrak{m}}$-invariant subspaces. By $\Lambda_{\mathfrak{m}}$-irreducibility, we have $\mathfrak{m} = \boldsymbol{R}\xi \oplus \mathfrak{p}_1$. By this and Lemma 3.4, it holds that

$$R(v,x)y = c\{\langle x,y \rangle v - \langle v,y \rangle x\} \quad \text{for } v \in O_1 = \boldsymbol{R}\xi \oplus V_1 \quad \text{and } x,y \in \mathfrak{m}.$$

We define a tensor R_0 of type (1,3) by

$$R_0(u,v)w = \langle v,w \rangle u - \langle u,w \rangle v$$

and define a subspace $\mathfrak{n}$ of $\mathfrak{m}$ by

$$\mathfrak{n} = \{x \in \mathfrak{m} \mid i(x)(R - cR_0) = 0\}.$$

The preceding result means that $O_1 \subset \mathfrak{n}$. Now we note that the curvature tensor R is given as follows (cf [3] p.202):

$$\begin{aligned} R(x,y)z &= -[[x,y]_{\mathfrak{h}}, z] \\ &\quad + \frac{1}{4}[x,[y,z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{4}[y,[x,z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{2}[[x,y]_{\mathfrak{m}}, z]_{\mathfrak{m}} \\ &= -[[x,y]_{\mathfrak{h}}, z] + \varphi_x \varphi_y z - \varphi_y \varphi_x z - \varphi_{(\varphi_x y - \varphi_y x)} z \end{aligned}$$

for $x,y,z \in \mathfrak{m}$.
Since R and R_0 are invariant by the action of $\mathfrak{h}$, the subspace $\mathfrak{n}$ is invariant by the action of $\mathfrak{h}$. In particular we see that $[[y,z]_{\mathfrak{h}}, v] \in \mathfrak{n}$ for $v \in \mathfrak{n}$ and $y,z \in \mathfrak{m}$.

We first assume that $c \neq 0$. For an arbitrary vector $x \in V$, we have

$$R(x,\xi)\xi = -[[x,\xi]_{\mathfrak{h}}, \xi] - \varphi_{\varphi_x \xi}\xi.$$

Hence ξ and $\varphi_{\varphi_x \xi}\xi$ are contained in $\mathfrak{n}$. By the preceding remark, it follows that $[[x,\xi]_{\mathfrak{h}}, \xi] \in \mathfrak{n}$. Hence $R(x,\xi)\xi \in \mathfrak{n}$. On the other hand, since $V = \mathfrak{p}_1$, $R(x,\xi)\xi = cx$. Since $c \neq 0$, we have $x \in \mathfrak{n}$. Therefore we see that $\mathfrak{n} = \mathfrak{m}$, that is, R has constant sectional curvature c.

We secondly assume that $c = 0$. We define subspaces V_i $(i = 0,1,2,\cdots)$ inductively as follows. Set $V_0=\boldsymbol{R}\xi$. We define V_{i+1} by a subspace linearly spanned by $\varphi_x z$ for $x\in\mathfrak{m}$, $z\in V_i$. We remark that V_1 coincides with the subspace defined at the beginning in this section.

Lemma 3.8. *For each i, $V_i\subset\mathfrak{n}=\{x\in\mathfrak{m}\,|\,i(x)R=0\}$.*

Proof of Lemma 3.8. We shall prove our assertion by the induction with respect to i. It is already shown that $V_0\subset\mathfrak{n}$ and $V_1\subset\mathfrak{n}$. Suppose that our assertion holds for $0,1,\cdots,i$ $(i\geq 1)$. Then we shall prove that $V_{i+1}\subset\mathfrak{n}$, that is, $\varphi_x z\in\mathfrak{n}$ for $x\in\mathfrak{m}$, $z\in V_i$. We consider the following three cases.
Case 1. $x\in V_j$, $0\leq j\leq i-1$;
Case 2. $x\in V_i$;
Case 3. $x\in(V_0+V_1+\cdots+V_i)^\perp$.

Case 1. Since $\varphi_x z=-\varphi_z x\in V_{j+1}$ and $j+1\leq i$, $\varphi_x z\in\mathfrak{n}$.
Case 2. By Lemma 3.5, we have for $u,v\in\mathfrak{m}$

$$\begin{aligned}0&=\varphi_x(R(z,u)v)-R(\varphi_x z,u)v-R(z,\varphi_x u)v-R(z,u)\varphi_x v\\&\quad+\varphi_z(R(u,x)v)-R(\varphi_z u,x)v-R(u,\varphi_z x)v-R(u,x)\varphi_z v\\&\quad+\varphi_u(R(x,z)v)-R(\varphi_u x,z)v-R(x,\varphi_u z)v-R(x,z)\varphi_u v\\&=-2R(\varphi_x z,u)v.\end{aligned}$$

Therefore we have $\varphi_x z\in\mathfrak{n}$.

Case 3. It is sufficient to prove our assertion when $z=\varphi_u v$ for $u\in\mathfrak{m}$, $v\in V_{i-1}$. We first remark that $\varphi_x v=0$. In fact, for any $w\in\mathfrak{m}$, $\langle\varphi_x v,w\rangle=-\langle\varphi_w v,x\rangle$ and since $\varphi_w v\in V_i$ and $x\in(V_0+V_1+\cdots+V_i)^\perp$, we have $\langle\varphi_x v,w\rangle=0$. It follows that

$$\begin{aligned}R(x,u)v&=-[[x,u]_{\mathfrak{h}},v]+\varphi_x\varphi_u v-\varphi_u\varphi_x v-2\varphi_{\varphi_x u}v\\&=-[[x,u]_{\mathfrak{h}},v]+\varphi_x z-2\varphi_{\varphi_x u}v.\end{aligned}$$

On other hand, $R(x,u)v=-R(u,v)x-R(v,x)u=0$ by the assumption of induction. Then we have $\varphi_x z=[[x,u]_{\mathfrak{h}},v]+2\varphi_{\varphi_x u}v$. Since the right hand side is contained in $\mathfrak{n}$, so is $\varphi_x z$. $\square$

We set $O_i=V_0+V_1+\cdots+V_i$. Evidently we have $O_0\subseteq O_1\subseteq\cdots\subseteq O_i\subseteq O_{i+1}\subseteq\cdots$. Therefore there exists an integer i such that $O_i=O_{i+1}$. Then O_i is an invariant subspace with respect to $\Lambda_{\mathfrak{m}}$. Since $O_i\neq 0$, we have $O_i=\mathfrak{m}$. By Lemma 3.8, it follows that $\mathfrak{n}=\mathfrak{m}$, that is, the curvature tensor R vanishes. Thus our theorem has been completely proved.

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