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HANDLEBODY DECOMPOSITIONS OF 4-MANIFOLDS AND TORUS FIBRATIONS

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1. Introduction

The purpose of this paper is to prove the following

Theorem A. *Let M be a closed smooth 4-manifold which has a handlebody decomposition of this type:*

$$M=H^0\cup aH^2\cup bH^3\cup H^4$$

where $b\leq 1$. Then M admits a torus fibration over the 2-sphere $f\colon M\to S$. The projection f is smooth except at a point $\in M$, and f has only one singular fiber.

In this theorem, a and b are the numbers of 2 and 3 handles, respectively. The type of the singular fiber in the above fibration is not necessarily 'good' in the sense of [7].

Theorem A was first announced in 1982 in [6]. (See also [7], [8].) The main reason for the long delay in publishing the proof is, of course, the author's laziness. But a reason was partly because the author was not fully convinced of the usefulness of the result; the variety of the singular fibers appearing in the construction seemed quite uncontrolable. As a matter of fact, such wide variety was a key to the proof of the existence theorem. Recently, the author received an enquiry from Daniel Ruberman about the proof. In trying to answer him, the author found a new example of a smooth torus fibration of S^4 over S^2 applying the general construction in this paper to S^4. Also, he found that, if $H_2(M;\boldsymbol{Z})\neq\{0\}$, we can arrange so that the general fiber is not homologous to zero in M (Theorem B in Section 3). He hopes that these improvements might justify this late publication of the proof. The author thanks D. Ruberman, whose enquiry gave him an opportunity to publish this paper.

2. Multiple fibered links

We begin by recalling 'multiple fibered links' from [6], [8]. Let L^3 be an oriented closed 3-manifold.

DEFINITION. A smooth map $g: L^3 \to C$ is called a *multiple fibered link* in L^3 if it satisfies the following:

(i) $g^{-1}(0) \neq \phi$;

(ii) the map $\varphi(x) = g(x)/|g(x)|: L^3 - g^{-1}(0) \to S^1$ is a submersion with connected fibers;

(iii) around each point x_0 ($\in g^{-1}(0)$), there exist local coordinates u_1, u_2, u_3 of L^3 satisfying

$$g(x) = (u_2(x) + \sqrt{-1}u_3(x))^m$$

for all x near x_0, m being a positive integer called the *multiplicity* at x_0.

The first coordinate u_1 is along the component of the link $g^{-1}(0)$. The multiplicity is constant along a component of $g^{-1}(0)$, which is the *multiplicity* of the component. Sometimes $g^{-1}(0)$ itself is called a *multiple fibered link* for simplicity. A fiber of $\varphi: L^3 - g^{-1}(0) \to S^1$ is a punctured surface, whose genus is the *genus* of the multiple fibered link.

Lemma 1. *Let $\hat{\beta}$ be a closed braid in a* 3*-sphere Σ^3. Suppose Dehn surgery along the components $K_0^*, K_1^*, \cdots, K_a^*$ of $\hat{\beta}$ with integer coefficients $n_0, n_1, \cdots, n_a$ gives a* 3*-manifold $L^3 = \chi_{\Sigma^3}(K_0^*, \cdots, K_a^*; n_0, \cdots, n_a)$. Then the meridians $K_0, K_1, \cdots, K_a$ of $K_0^*, K_1^*, \cdots, K_a^*$ and the circle C_0 (in Fig.* 1) *become a multiple fibered link of genus* 0 *in L^3. A typical fiber is the punctured disk shaded in the diagram. Also circles belonging to $\{C_1, \cdots, C_r\}$ (in Fig.* 1) *are isotopic in L^3 to a component K_i of the mutiple fibered link if and only if they link the component K_i^* of $\hat{\beta}$ in Σ^3.*

We chose the notation $K_0^*, K_1^*, \cdots, K_a^*$ for the components of $\hat{\beta}$, and Σ^3 for the 3-sphere, so that they are concordant with the notation in Section 3.

Proof of Lemma 1. $\Sigma^3 - \operatorname{int} N(C_0 \cup K_0^* \cup \cdots \cup \cdots K_a^*)$ is fibered over the circle S^1 with fiber the shaded punctured disk in Figure 1. For a component K_i^* of $\hat{\beta}$, let $\varphi_i: \partial D^2 \times S^1 \to \partial N(K_i^*)$ be the diffeomorphism used for the Dehn surgery. We may assume

$$\begin{cases} \varphi_i(\partial D^2 \times \{pt\}) = n_i\mu_i + \lambda_i \\ \varphi_i(\{pt\} \times S^1) = -\mu_i. \end{cases}$$

Here λ_i is a preferred longitude of K_i^*, and μ_i is a meridian. Then on $\partial D^2 \times S^1$ the preimages $\varphi_i^{-1}(\lambda_i)$, $\varphi_i^{-1}(\mu_i)$ look as shown in Figure 2.

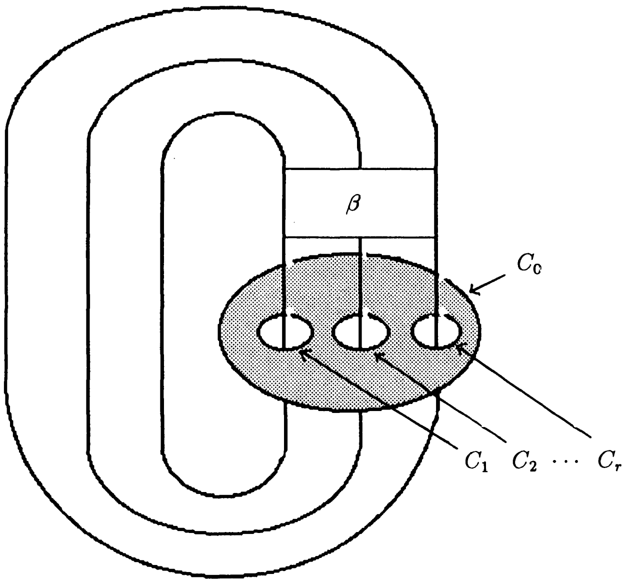
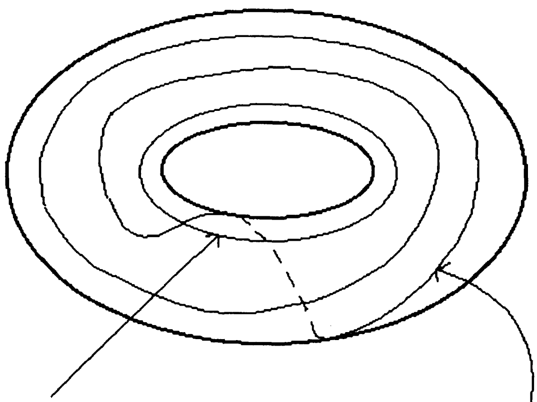


Fig. 1.



$\varphi_i^{-1}(\mu_i) = -\lambda'$, $\qquad\qquad \varphi_i^{-1}(\lambda_i) = \mu' + n_i\lambda'$,

where $\lambda' = [\{\text{pt}\} \times S^1]$, and $\mu' = [\partial D^2 \times \{\text{pt}\}]$.

Fig. 2.

Thus if $C_1^{(i)}, C_2^{(i)}, \cdots, C_{m_i}^{(i)}$ are the boundary circles of the punctured disk (different from C_0) which link the component K_i^*, then $\varphi_i^{-1}(C_1^{(i)}), \cdots, \varphi_i^{-1}(C_{m_i}^{(i)})$ look as shown in Figure 3:

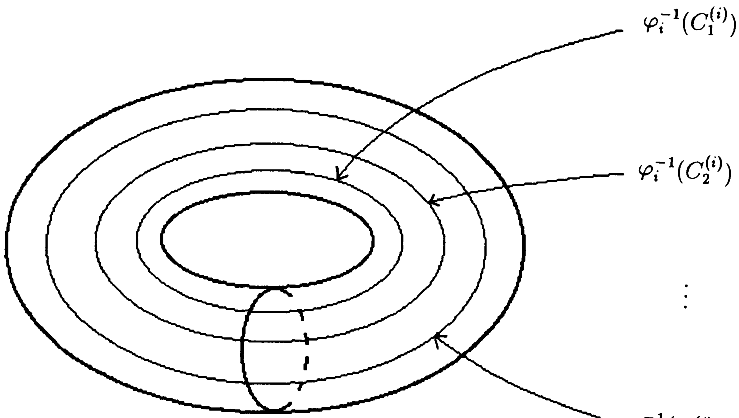


Fig. 3.

A tubular neighborhood of a component of the fibered link is obtained by identifying the 2-disks at the right and left ends of the cylinder in Figure 4 in an untwisted manner. (In this figure, the multiplicity of the component is 3.) This tubular neighborhood is identified with the solid torus of Figure 3.

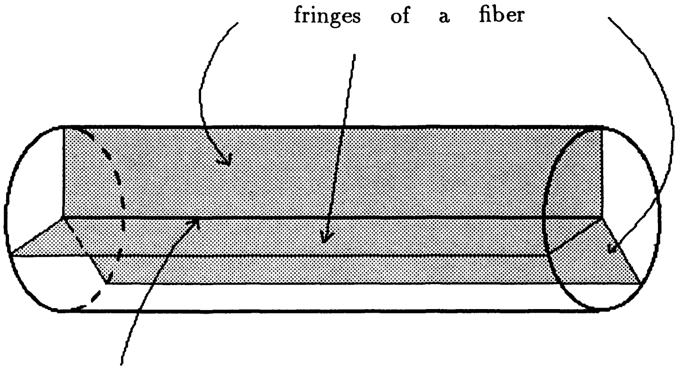


Fig. 4

Now it would be clear that in L^3 the circles $C_1^{(i)},\cdots,C_{m_i}^{(i)}$ are isotopic to the central circle of the above tubular neighborhood, which is a component K_i of the asserted multiple fibered link in L^3.

The argument is similar for the component C_0.

Finally it is an easy task to construct a projection $g: L^3 \to \boldsymbol{C}$ of the multiple fibered link. $\square$

REMARK. The multiplicity m_i of the component K_i is equal to the number of the circles $C_1^{(i)},\cdots,C_{m_i}^{(i)}$, which is equal to the linking number $|\mathrm{link}_{\Sigma^3}(K_i^*, C_0)|$, $(i=1,\cdots,a)$.

3. Proof of Theorem A

By the assumption M has a handlebody decomposition $M=H^0\cup aH^2\cup bH^3\cup H^4$ with $b\leq 1$. If $b=0$, we introduce an extra pair of 2, 3-handles, and we may assume $b=1$. Then $\partial(H^0\cup aH^2)=S^1\times S^2$. Suppose an unknot K_0^* of Figure 5 with framing 0 is a surgery description of $S^1\times S^2$:

$$S^1\times S^2=\chi_{\Sigma^3}(K_0^*,0),$$

where Σ^3 is a 3-sphere in which K_0^* is drawn.

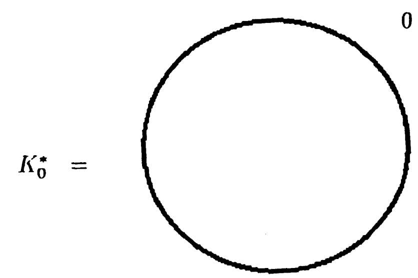


Fig. 5.

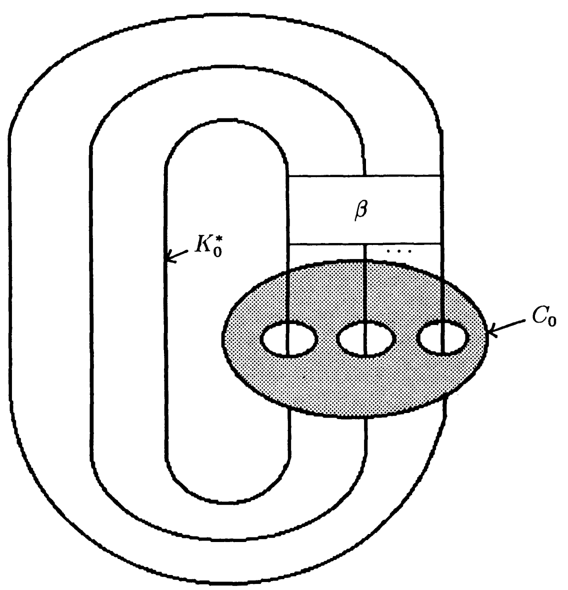


Fig. 6.

Let $H_1^*, H_2^*, \cdots, H_a^*$ be the 2-handles dual to the 2-handles $H_1^2, H_2^2, \cdots, H_a^2$ in the decomposition of M. Let $K_1^*, K_2^*, \cdots, K_a^*$ be the attaching circles (in $S^1 \times S^2 = \chi_{\Sigma^3}(K_0^*, 0)$) of $H_1^*, H_2^*, \cdots, H_a^*$ with certain framings $n_1, n_2, \cdots, n_a$. Then the framed link in Σ^3

$$\mathscr{L} = \{(K_0^*, 0), (K_1^*, n_1), \cdots, (K_a^*, n_a)\}$$

gives a surgery description of $S^3 = \partial H^0$. By Alexander's theorem [1], [2, Thm.2.1] we can put $\mathscr{L}$ into the form of a closed braid $\hat{\beta}$. Applying Lemma 1, we use $\hat{\beta}$ and draw a surgery description of a multiple fibered link $\boldsymbol{L}$ (in $S^3 = \partial H^0$) of genus zero as shown in Figure 6, where the curve K_0^* is arranged to link C_0 geometrically once.

Let $K_1, K_2, \cdots, K_a$ be the attaching circles (in $\partial H^0 = S^3 = \chi_{\Sigma^3}(\mathscr{L})$) of the 2-handles $H_1^2, H_2^2, \cdots, H_a^2$. Since H_i^2 is dual to H_i^*, we may assume that K_i is a meridian of K_i^* for each i and that the attaching framings of these handles $H_1^2, \cdots, H_a^2$ are all zero (we are speaking of the framing in the diagram of Figure 6, in other words, on the 'canvas' Σ^3).

We denote a meridian of K_0^* by K_0.

By Lemma 1, the multiple fibered link $\boldsymbol{L}$ is the union

$$C_0 \cup K_0 \cup K_1 \cup \cdots \cup K_a.$$

Now the torus fibration $f: M \to S^2$ will be constructed in 3 steps:

(1) construction of a map $G: H^0 \to D^2$,
(2) extension of G to a map $F: H^0 \cup aH^2 \to D^2$, and
(3) extension of F to the desired torus fibration $f: M \to S^2$.

Step 1. Let $\boldsymbol{L}$ be the multiple fibered link constructed above, $g: S^3 \to \boldsymbol{C}$ the projeciton of $\boldsymbol{L}$. Let us recall the definition of the 'cone extension' of g. (See [6], [8].) In this definition, we identify S^3 with the unit sphere of $\boldsymbol{R}^4$.

DEFINITION. A function $\tilde{g}: \boldsymbol{R}^4 \to \boldsymbol{C}$ is called the *d-th cone extension* of $g: S^3 \to \boldsymbol{C}$ if $\tilde{g}$ satisfies the following:

$$\tilde{g}(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\|^d g\left(\dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) & \boldsymbol{x} \neq \boldsymbol{0} \\ 0 & \boldsymbol{x} = \boldsymbol{0} \end{cases}$$

The integer $d(\geq 1)$ is usually equal to 1, in which case $\tilde{g}$ is simply called *the cone extension* of g. (See Figure 7.)

Topological properties of $\tilde{g}$ are independent of d. $\tilde{g}$ is smooth except at the origin $\boldsymbol{0} \in \boldsymbol{R}^4$. (Fig. 7.)

REMARK. The cone extension construction of a multiple fibered link is intended to mimic topologically the local fibering structure of a holomorphic function. See Milnor [9].

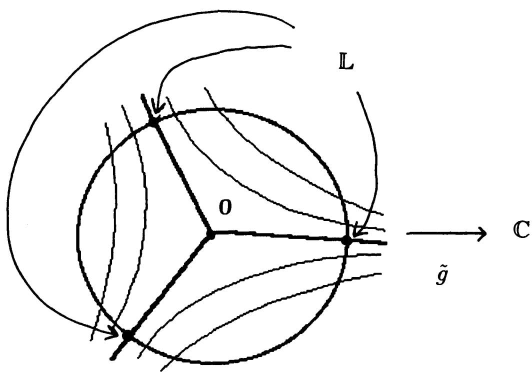


Fig. 7.

Let D^4 be the unit 4-disk of $\boldsymbol{R}^4$. Put $S_\varepsilon^1=\{z\in\boldsymbol{C}\,|\,|z|=\varepsilon\}$ and $D_\varepsilon^2=\{z\in\boldsymbol{C}\,|\,|z|\leq\varepsilon\}$.

Lemma 2. *For a sufficiently small* $\varepsilon>0$, *the fiber bundle* $\tilde{g}\,|\,(D^4\cap\tilde{g}^{-1}(S_\varepsilon^1))$: $D^4\cap\tilde{g}^{-1}(S_\varepsilon^1)\rightarrow S_\varepsilon^1$ *is smoothly isomorphic to the bundle* $\varphi=g/|g|:(S^3-\operatorname{int}T)\rightarrow S^1$ *in the complement of a tubular neighborhood* T *of* $\boldsymbol{L}$.

Proof. The proof is similar to Theorem 5.11 or Lemma 11.3 in [9], but here easier, because there exists a standard vector field

$$\boldsymbol{v}(\boldsymbol{x})=\sum_{k=1}^{4}x_k\frac{\partial}{\partial x_k},\qquad \boldsymbol{x}=(x_1,x_2,x_3,x_4)\in\boldsymbol{R}^4$$

that is transverse to both $\tilde{g}^{-1}(S_\varepsilon^1)$ and S^3. $\square$

Let Δ_ε^4 denote $D^4\cap\tilde{g}^{-1}(D_\varepsilon^2)$. Δ_ε^4 is a smooth manifold with corners. By 'straightening corners', we can find a homeomorphism $h:H^0\rightarrow\Delta_\varepsilon^4$ of the 0-handle H^0 of M to Δ_ε^4 such that

(i) h is the identity on the tubular neighborhood T of $\boldsymbol{L}$ in S^3, and
(ii) h is a diffeomorphism except at ∂T. (See Figure 8)

We construct the map $G:H^0\rightarrow D_\varepsilon^2$ by setting

$$G=(\tilde{g}\,|\,\Delta_\varepsilon^4)\circ h$$

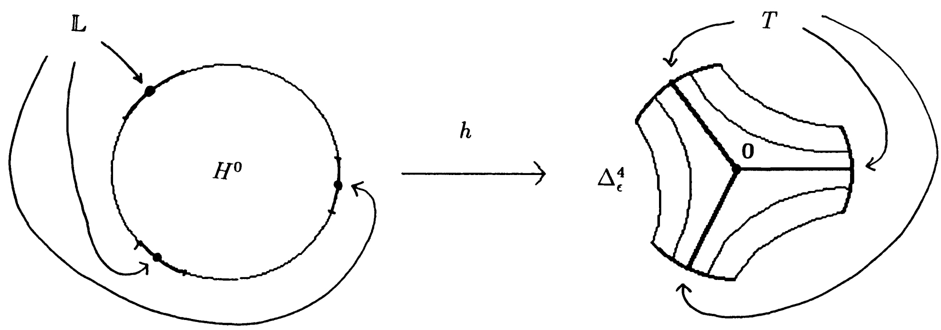


Fig. 8.

G has the following properties: G becomes a fiber bundle over $D^2_\varepsilon \backslash \{0\}$, and has a singular fiber over 0. By Lemma 2, the general fiber $G^{-1}(z_0)$, $z_0 \in D^2_\varepsilon \backslash \{0\}$, is diffeomorphic to the shaded punctured disk in Figure 1. Also the monodromy of the restricted bundle $G|G^{-1}(S^1_\varepsilon): G^{-1}(S^1_\varepsilon) \to S^1_\varepsilon$ is isomorphic to the bundle associated with the multiple fibered link $\boldsymbol{L}$.

The singular fiber $G^{-1}(0)$ is a cone $\mathbf{0} * \boldsymbol{L}$ over $\boldsymbol{L}$ from the origin $\mathbf{0}$. (Cf. Fig. 8)

Another property of G is that $G|T = \tilde{g}|T (= g|T)$, which follows from Property (i) of h.

Step 2. Note that the tubular neighborhood T of L in $\partial H^0 (= S^3)$ is a union $T = T(C_0) \cup T(K_0) \cup \cdots \cup T(K_a)$ of tubular neighborhoods of the components $C_0, K_0, \cdots, K_a$. From the 'fringe structure' shown in Figure 4 and the definition of a multiple fibered link (in particular, condition (iii)), it follows that each tubular neihborhood has a natural product structure in the following sense : In the case of $T(K_i)$ $(i = 0, 1, \cdots, a)$, it is diffeomorphic to $S^1 \times D^2_{\delta_i}$:

$$T(K_i) \cong S^1 \times D^2_{\delta_i} \tag{3.1}$$

and under this identification, $G|T(K_i): T(K_i) \to D^2_\varepsilon$ is written as

$$G(z_1, z_2) = z_2^{m_i}, \qquad (z_1, z_2) \in S^1 \times D^2_{\delta_i} \tag{3.2}$$

where $\delta_i = (\varepsilon)^{1/m_i} > 0$, m_i being the multiplicity of K_i. In the case of $T(C_0)$, the situation is similar and simpler; $T(C_0) \cong S^1 \times D^2_\varepsilon$ and $G(z_1, z_2) = z_2$ for each $(z_1, z_2) \in S^1 \times D^2_\varepsilon$.

Recall that the 2-handles $H^2_1, H^2_2, \cdots, H^2_a$ of M are attached along $K_1, K_2, \cdots, K_a$ $(\subset \partial H^0 = S^3 = \chi_{\Sigma^3}(\mathscr{L}))$. We may assume

$$H^2_i \cap \partial H^0 = T(K_i).$$

We saw at the begining of Section 3 that the attaching framing of H^2_i to $\partial H^0 = \chi_{\Sigma^3}(\mathscr{L})$

is zero framing on Σ^3. From the proof of Lemma 1, the natural product structure (3.1) of $T(K_i)$ is zero framingt on Σ^3, too. Thus the attaching framing of H_i^2 coincides with the natural framing of $T(K_i)$. In the other words, the product structure $H_i^2 \cong D^2 \times D^2$ as a handle coincides on $T(K_i)$ with the natural product structure (3.1) up to the scaling of the second facter $D^2 \cong D^2_{\delta_i}$. Consequently we can extend $G: H^0 \to D^2_\varepsilon$ to a map $F: H^0 \cup aH^2 \to D^2_\varepsilon$ by defining it on $H_i^2 \cong D^2 \times D^2_{\delta_i}$ $(i=1,2,\cdots,a)$ as

$$F(z_1,z_2)=z_2^{m_i}, \qquad (z_1,z_2)\in D^2 \times D^2_{\delta_i} \tag{3.3}$$

where m_i is the multiplicity of K_i (Cf. (3.2).)

This completes the construction of the map F.

Before proceeding to Step 3, we study the fibering structure of $F: H^0 \cup aH^2 \to D^2_\varepsilon$. We will henceforth identify H^0 with Δ^4_ε through the homeomorphism $h: H^0 \to \Delta_\varepsilon$. Then we may consider $H^0 \cup aH^2$ to be a smooth manifold with corners along $\partial T(C_0) \cup \partial T(K_0)$.

As we saw in Step 1, the general fiber $G^{-1}(z_0)$ of $G: H^0 \to D^2_\varepsilon$ is diffeomorphic to the shaded punctured disk in Figure 1. This punctured disk has $2+\Sigma_{i=1}^a m_i$ boundary components: one isotopic to C_0, one isotopic to K_0, and m_i isotopic to K_i, $i=1,\cdots,a$.

The m_i boundary components of $G^{-1}(z_0)$ which are isotopic to K_i are situated in $T(K_i) \cong S^1 \times D^2_{\delta_i}$ as parallel copies of K_i, more precisely, as

$$S^1 \times \{m_i\text{-th roots of } z_0\} \quad (\subset S^1 \times D^2_{\delta_i}).$$

When the handle H_i^2 is attached to H^0 along $T(K_i)$, these m_i circles are capped off by m_i disks parallel to the core of H_i^2:

$$D^2 \times \{m_i\text{-th roots of } z_0\} \quad (\subset D^2 \times D^2_{\delta_i}),$$

and these m_i disks coincide with $F^{-1}(z_0) \cap H_i^2$. Thus the general fiber $F^{-1}(z_0)$, $z_0 \neq 0$, is obtained from the shaded punctured disk in Figure 1 by capping off all the boundary components except C_0 and K_0. Hence $F^{-1}(z_0)$ is an annulus with boundary components parallel to $C_0 \cup K_0$. (Cf. Fig. 9.)

The restriction $F|F^{-1}(S^1_\varepsilon): F^{-1}(S^1_\varepsilon) \to S^1_\varepsilon$ is an annulus bundle over a circle. The monodromy of this bundle relative to $\partial T(C_0) \cup \partial T(K_0)$ is trivial, because attaching the 2-handles $H_1^2, H_2^2, \cdots, H_a^2$ along $K_1, K_2, \cdots, K_a$ (with 0-framings on Σ^3) has the effect, on the boundary, of neglecting $K_1^*, K_2^*, \cdots, K_a^*$, which leaves the surgery description (in the sense of Lemma 1) of a fibered link $C_0 \cup K_0$ in $S^1 \times S^2$ whose fibering structure (in the complement of an open tubular neighborhood of $C_0 \cup K_0$) is trivial and isomorphic to $F|F^{-1}(S^1_\varepsilon): F^{-1}(S^1_\varepsilon) \to S^1_\varepsilon$. (See Figure 10.)

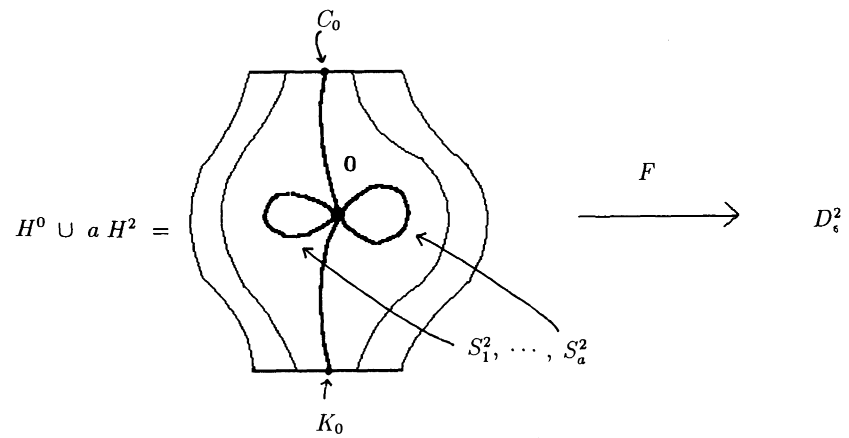


Fig. 9.

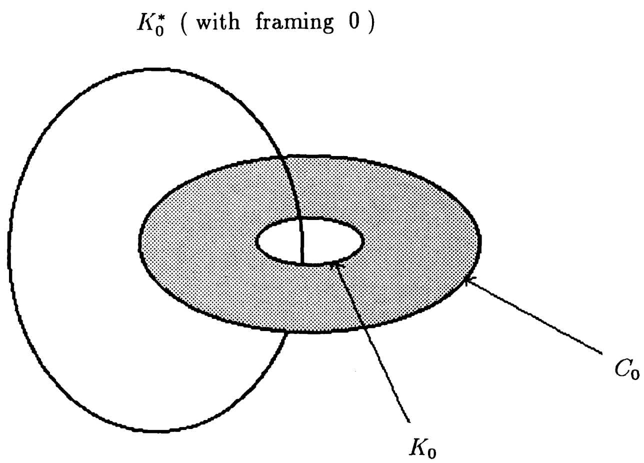


Fig. 10.

So far we have been considering general fibers. Now we describe the singular fiber of $F: H^0 \cup aH^2 \to D^2_\varepsilon$.

The singular fiber $F^{-1}(0)$ is obtained by attaching the cores of the 2-handles $H^2_1, \cdots, H^2_a$ to $G^{-1}(0)$ along $K_1, \cdots, K_a$. Thus $F^{-1}(0)$ is a bouquet of 2-spheres $S^2_1, \cdots, S^1_a$ (with 'equators' $K_1, \cdots, K_a$) and 2-disks $\mathbf{0} * C_0 \cup \mathbf{0} * K_0$. (See Figure 9.)

As $z_0 \to 0$, the general fiber $F^{-1}(z_0)$ covers the sphere S^2_i m_i times $(i = 1, \cdots, a)$ and covers $\mathbf{0} * C_0$ and $\mathbf{0} * K_0$ once.

Step 3. Let $\Phi: S^3 \to S^2$ be the Hopf fibering. Regarding S^2 as $\boldsymbol{C} \cup \{\infty\}$, we take an ε-disk $D^2_\varepsilon = \{z \in \boldsymbol{C} \mid |z| \leq \varepsilon\}$ in S^2. Let p_0 be a point $\Phi^{-1}(0)$. We take a closed neighborhood N of p_0 in S^3 which is identified with $D^2_\varepsilon \times [-1, 1]$ via an appropriate diffeomorphism such that

(i) p_0 is identified with (0,0) in $D^2_\varepsilon \times [-1,1]$, and

(ii) $\Phi | N : N \to D^2_\varepsilon$ is identified with the projection $D^2_\varepsilon \times [-1,1] \to D^2_\varepsilon : (z,t) \mapsto z$.

Consider the projection

$$\Phi' : (S^3 - \operatorname{int} N) \times S^1 \to S^2$$

defined by $\Phi' = (\Phi | (S^3 - \operatorname{int} N)) \circ P$, where $P : (S^3 - \operatorname{int} N) \times S^1 \to (S^3 - \operatorname{int} N)$ is the projection to the first factor.

It is easy to see that the restriction of Φ' to the boundary $\partial(S^3 - \operatorname{int} N) \times S^1$,

$$\Phi' | \partial(S^3 - \operatorname{int} N) \times S^1 : \partial(S^3 - \operatorname{int} N) \times S^1 \to D^2_\varepsilon,$$

has the same trivial fibering structure as the restriction of F (constructed in Step 2) to the boundary $\partial(H^0 \cup aH^2)$,

$$F | \partial(H^0 \cup aH^2) : \partial(H^0 \cup aH^2) \to D^2_\varepsilon.$$

Thus by gluing $H^0 \cup aH^2$ and $(S^3 - \operatorname{int} N) \times S^1$ along their boundaries, we obtain a smooth closed 4-manifold $M' = (H^0 \cup aH^2) \cup (S^3 - \operatorname{int} N) \times S^1$ and a map $f : M' \to S^2$.

By straightening corners, we may assume that $H^0 \cup aH^2$ and $(S^3 - \operatorname{int} N) \times S^1$ are smooth 4-manifolds. The latter manifold is diffeomorphic to $D^3 \times S^1$. By [5] and [10], the diffeomorphism type of the resulting closed 4-manifold $(H^0 \cup aH^2) \cup (S^3 - \operatorname{int} N) \times S^1$ is indepenednt of the gluing diffeomorphism. Thus M' is diffeomorphic to our original $M (= H^0 \cup aH^2 \cup H^3 \cup H^4)$, and we obtain a map $f : M \to S^2$.

From the fibering structure of $F : H^0 \cup aH^2 \to D^2_\varepsilon$ and $\Phi' : (S^3 - \operatorname{int} N) \times S^1 \to S^2$, it is easy to see that the map $f : M \to S^2$ is a torus fibration of M. f is smooth except at the center $\mathbf{0}$ of H^0.

The fiber $f^{-1}(0)$ over 0 ($\in \mathbf{C} \subset S^2$) is a single singular fiber of f. It is a bouquet of 2-spheres $S^2_1, \cdots, S^2_a$ and a 'croissant' X. Note that $\Phi^{-1}(0) \cap (S^3 - N)$ is an interval, and X is obtained by attaching the annulus $(\Phi^{-1}(0) \cap (S^3 - N)) \times S^1$ to the bouquet of disks $\mathbf{0} * C_0 \cup \mathbf{0} * K_0$. Hence X is an immersed 2-sphere with a single transverse self-intersection point at $\mathbf{0}$.

This completes Step 3 and thus the proof of Theorem A. $\square$

PROBLEM. Find a condition on a closed 4-manifold M under which M admits a torus fibration $f : M \to S^2$ which is smooth everywhere.

Let M be the same 4-manifold as in Theorem A. The following theorem answers Ruberman's question.

Theorem B. *Suppose $H_2(M;\mathbf{Z})\neq\{0\}$. Then there exists a torus fibration $f:M\to S^2$ such that a general fiber is not homologous to 0 in M.*

Proof. Let $f:M\to S^2$ be the torus fibration constructed in the proof of Theorem A. $H_2(M)$ is generated by the homology classes $[S_1^2],[S_2^2],\cdots,[S_a^2]$ of the 'irreducible components' $S_1^2,S_2^2,\cdots,S_a^2$ of the singular fiber $f^{-1}(0)$. This is because these 2-spheres belong to the relative homology classes $[H_1^2],[H_2^2],\cdots,[H_a^2]$ of the 2-handles in $H_2(H^0\cup aH^2,H^0)$, and the natural homomorphism

$$H_2(H^0\cup aH^2,H^0)\to H_2(M,H^0)\cong H_2(M)$$

is onto.

By the observation at the end of Step 2 (of the proof of Theorem A), a general fiber $f^{-1}(z_0)$ covers S_i^2 m_i times $(i=1,\cdots,a)$ and covers X once. Thus in $H_2(M)$, we have

$$[f^{-1}(z_0)]=\sum_{i=1}^{a}m_i[S_i^2]+[X], \tag{3.4}$$

if S_i^2's and X are appropriately oriented.

By Remark at the end of Section 2, the multiplicity m_i is equal to the linking number $|\mathrm{link}_{\Sigma^3}(K_i^*,C_0)|$. On the other hand, we can arbitrarily increase this linking number by introducing extra linking between K_i^* and C_0 when applying the Alexander theorem as we did at the begining of Section 3. (See Figure 11.)

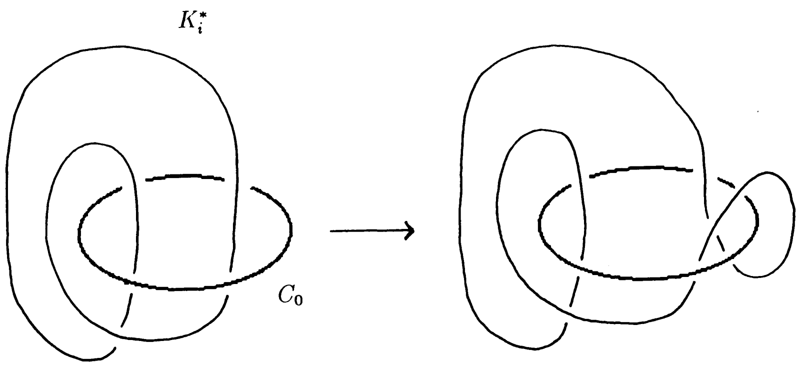


Fig. 11

Therefore, there exists a large positive number n_0 such that, given any integers $m_1,m_2,\cdots,m_a$ greater than n_0 $(i=1,\cdots,a)$, we can construct a torus fibration $f:M\to S^2$ in which (3.4) holds with the given coefficients $m_1,m_2,\cdots,m_a$. Choosing $m_1,m_2,\cdots,m_a$ appropriately, we can accomplish $[f^{-1}(z_0)]\neq 0$ in $H_2(M)$.

This completes the proof of Theorem B. □

4. An example

In [6], [7], [8], we showed that the 4-sphere S^4 has torus fibrations over S^2 with one singular fiber of type *Twin* or with two singular fibers of types I_1^+, I_1^-. A singular fiber of type *Twin* consists of two 2-spheres cutting each other in two points. A singular fiber of type I_1^+ (or I_1^-) is an immersed 2-sphere with one transverse self-intersection point of positive (or negative) sign.

Applying our construction in this paper to S^4, we obtain a new example of a torus fibration of S^4.

Let us start with the following handlebody decomposition of S^4:

$$S^4 = H^0 \cup H_1^2 \cup H_1^3 \cup H^4,$$

where H_1^2 and H_1^3 are a cancelling pair. The boundary $\partial(H^0 \cup H_1^2)$ is diffeomorphic to $S^1 \times S^2$ and has the surgery description $\chi_{\Sigma^3}(K_0^*,0)$. Let H_1^* be the dual 2-handle to H_1^2. The attaching circle K_1^* of H_1^* may be assumed to have attaching framing 0, and $(K_1^*,0)$ together with $(K_0^*,0)$ gives the surgery description of $S^3 = \partial H^0 = \chi_{\Sigma^3}(K_0^*, K_1^*; 0,0)$. See Figure 12

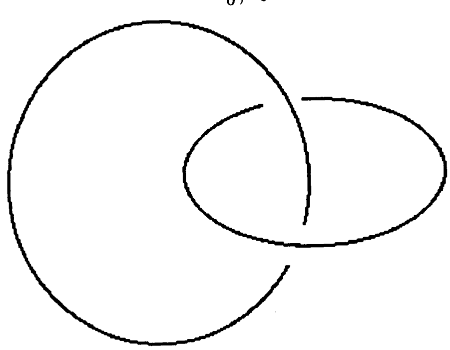


Fig. 12.

By Lemma 1, Figure 13 gives a surgery description of a multiple fibered link $\boldsymbol{L} = C_0 \cup K_0 \cup K_1$ in $S^3 = \chi_{\Sigma^3}(K_0^*, K_1^*; 0,0)$.

Let us see what $\boldsymbol{L}$ looks like actually in S^3. We apply Kirby's calculus [4] for framed links; Blow up a point on K_0^*, then we have Figure 14(a). Figure 14(b) is essentially the same as (a). We proceed as indicated in Fig.14.

We obtain a fibered link of Figure 14(f), which is nothing but the fibered link denoted by B_1^3 in Kanenobu's list [3, p.32].

As we observed at the end of Step 3 (of the proof of Theorem A), the singular fiber of the resulting $f: S^4 \to S^2$ is obtained by attaching an annulus along $K_0 \cup C_0$, and a disk along K_1, to the bouquet of disks $\mathbf{0} * C_0 \cup \mathbf{0} * K_0 \cup \mathbf{0} * K_1$, $\mathbf{0}$ being the

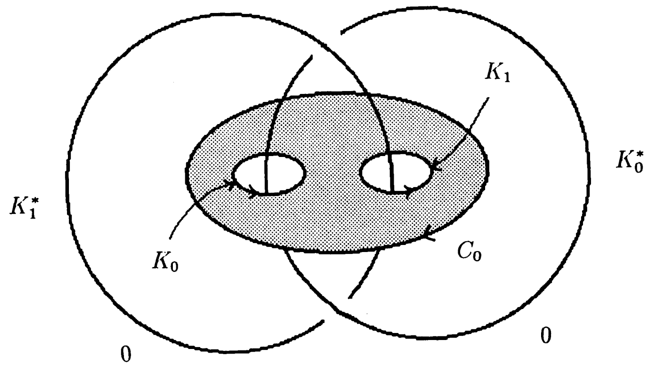


Fig. 13.

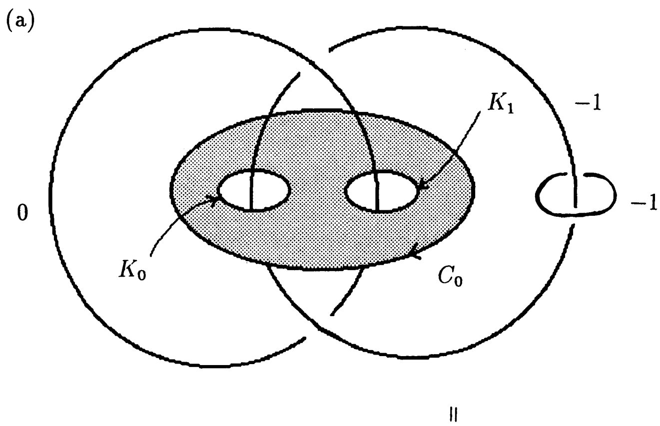


Fig. 14 (a)

center of H^0. Thus it is homeomorphic to a one-point union of an embedded 2-sphere and an immersed 2-sphere. The latter has a transverse self-intersection point at $\mathbf{0}$ with negative sign. (Cf. Fig. 15(a))

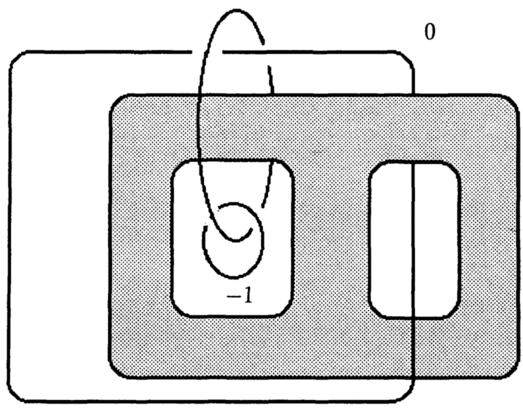
If we choose the opposite orientation of S^4 in every stage, then we will obtain a torus fibration with a singular fiber of the same type but in which the immersed sphere has a positive self-intersection point (Fig. 15(b)).

For the singular fiber of Fig.15 (a), the projection $f: S^4 \to S^2$ can be locally written around the intersection point $\mathbf{0}$ as

$$f(z_1,z_2)=z_1z_2\overline{(z_1+z_2)}$$

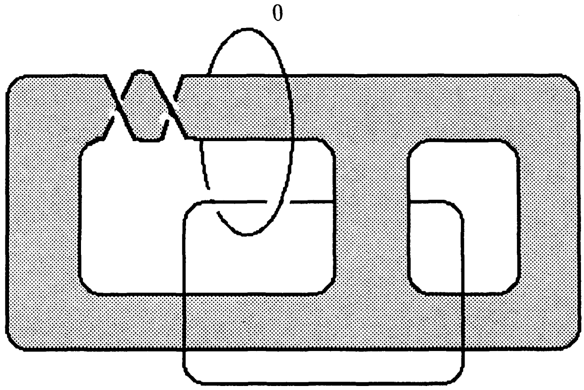
with certain complex coordinates (z_1,z_2) satisfying $\mathbf{0}=(0,0)$. Consequently $f: S^4 \to S^2$ may be taken to be smooth at the intersection point, thus smooth

(b)



blow down

(c)



blow down

Fig. 14 (b)(c)

everywhere. The same thing can be said for the singular fiber of Fig. 15(b).

We remark that once the results have been obtained, the singular fibers of Fugures 15 (a) and (b) are seen to be the results of blowing up and down starting from a *Twin* singular fiber. (See Figure 16, where attached integers are self-intersection numbers.)

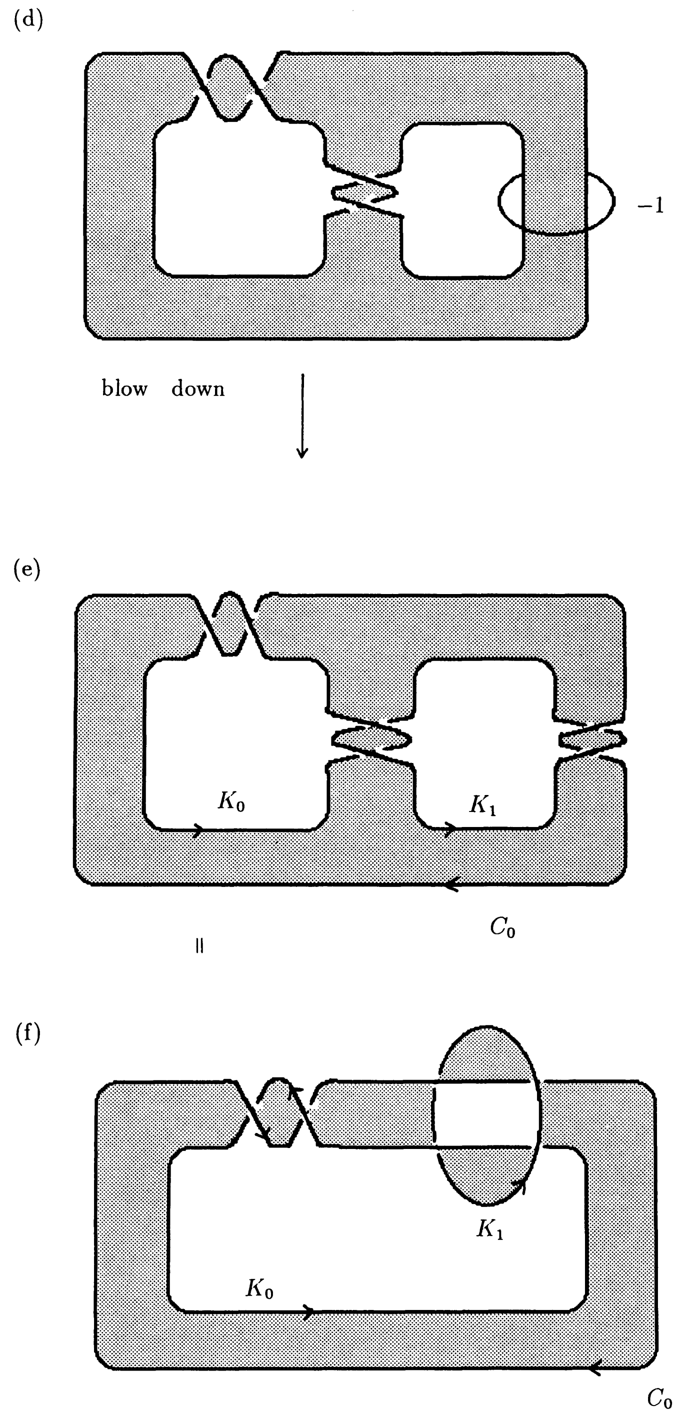


Fig. 14 (d)(e)(f)

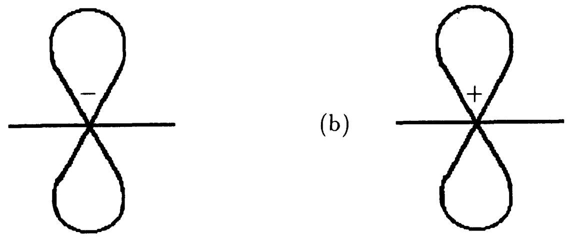


Fig. 15.

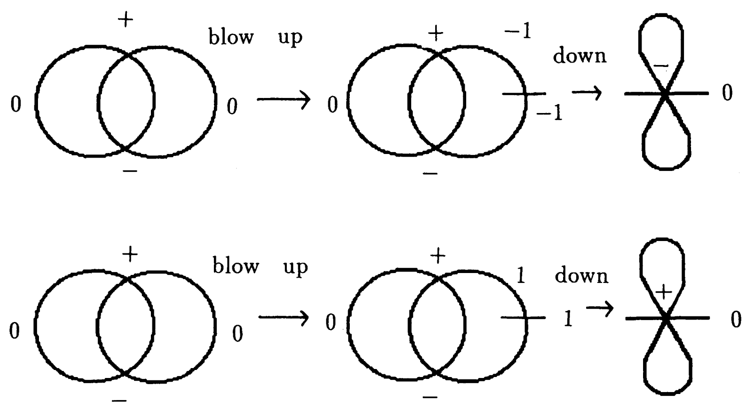


Fig. 16.

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