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QUASIREGULAR MAPPINGS AND d-THINNESS

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0. Introduction

In the study of boundary behavior of solutions of the classical Dirichlet problem thinness is an important notion. Recently notion of p-thinness (or p-thickness) is introduced in nonlinear potential theory and studied deeply. For $p=2$, p-thinness (resp. p-thickness) coinsides with thinness (resp. thickness) with respect to the classical potential theory. In this note we are especially concerned with d-thinness (or d-thickness) on the d dimensional Euclidean space $\boldsymbol{R}^d$ $(d \geq 2)$. The purpose of this note is to consider whether d-thinness (or d-thickness) is quasiregularly invariant or not. We obtain

Theorem 0.1. *Let G be a subdomain of $\boldsymbol{R}^d$ $(d \geq 2)$, E a subset of G, ξ a point of $G \backslash E$, and f a quasiregular mapping from G into $\boldsymbol{R}^d$. If E is d-thick at ξ, then $f(E)$ is d-thick at $f(\xi)$.*

The following theorem is an immediate conclusion of our main Theorem 0.1.

Theorem 0.2 (O. Martio and J. Sarvas [11]). *Let G be a subdomain of $\boldsymbol{R}^d$ $(d \geq 2)$, E a subset of G, ξ a point of $G \backslash E$ and f a quasiconformal mapping from G into $\boldsymbol{R}^d$. If E is d-thin (resp. d-thick) at ξ, then $f(E)$ is d-thin (resp. d-thick) at $f(\xi)$.*

Theorem 0.1 is obtained by the results of nonlinear potential theory. For $d=2$, H. Shiga [15] also obtains Theorem 0.2 by a different method from [11].

This note is organized as follows. In §1 we give preliminaries and discuss whether d-thickness (or d-thinness) is invariant by quasiregular mappings (see Theorem 1.1). In §2 and §3 we are concerned with applications of Theorem 0.2. In §2, comparison between thinness and minimal thinness, and Theorem 0.2 give us the quasiconformal invariance of minimal thinness under a condition (see Theorem 2.1). In §3 we prove that the harmonic dimension of Heins' covering surface is quasiconformally invariant under a condition (see Theorem 3.1).

1. Preliminaries and Proof of Theorem 0.1

1.1. First we give a mapping $\mathscr{A}: \boldsymbol{R}^d \times \boldsymbol{R}^d \to \boldsymbol{R}^d (d \geq 2)$ which satisfies the following assumptions for some constants $0 < \alpha \leq \beta < \infty$:

$$\begin{aligned} &\text{the function } x \mapsto \mathscr{A}(x,\xi) \text{ is measurable for all } \xi \in \boldsymbol{R}^d, \text{ and} \\ &\text{the function } \xi \mapsto \mathscr{A}(x,\xi) \text{ is continuous for a.e. } x \in \boldsymbol{R}^d; \end{aligned} \tag{1}$$

for all $\xi \in \boldsymbol{R}^d$ and a.e. $x \in \boldsymbol{R}^d$

$$\mathscr{A}(x,\xi) \cdot \xi \geq \alpha |\xi|^d, \tag{2}$$

$$|\mathscr{A}(x,\xi)| \leq \beta |\xi|^{d-1}, \tag{3}$$

$$(\mathscr{A}(x,\xi) - \mathscr{A}(x,\zeta)) \cdot (\xi - \zeta) > 0, \tag{4}$$

whenever $\xi \neq \zeta$, and

$$\mathscr{A}(x,\lambda\xi) = \lambda |\lambda|^{d-2} \mathscr{A}(x,\xi), \tag{5}$$

for all $\lambda \in \boldsymbol{R}$, $\lambda \neq 0$.

Let G be a subdomain of $\boldsymbol{R}^d$. We denote by $W_{loc}^{1,d}(G)$ the set of all locally L^d-integrable functions on G whose gradients in distributional sense are locally L^d-integrable $\boldsymbol{R}^d$-valued functions on G. We consider the partial differential operator

$$Tu = -\operatorname{div} \mathscr{A}(x, \nabla u),$$

where $u \in W_{loc}^{1,d}(G)$. We can develop the potential theory associated with the operator T. This potential theory is called *nonlinear potential theory*. For basic properties and notions of nonlinear potential theory we refer to [4].

A continuous weak solution $u \in W_{loc}^{1,d}(G)$ to the equation

$$Tu = -\operatorname{div} \mathscr{A}(x, \nabla u) = 0 \tag{6}$$

is called *$\mathscr{A}$-harmonic* on G. We denote by $\mathscr{H}_{\mathscr{A}}(G)$ the set of all $\mathscr{A}$-harmonic functions on G.

A lower semicontinuous function $u: G \to (-\infty, \infty]$ is called *$\mathscr{A}$-superharmonic* on G if u is not identically infinite on G, and if for all relatively compact and open subset D of G, and all $h \in C(\bar{D}) \cap \mathscr{H}_{\mathscr{A}}(D)$, $h \leq u$ on ∂D implies $h \leq u$ on D. A function v on G is called *$\mathscr{A}$-subharmonic* on G if $-v$ is $\mathscr{A}$-superharmonic on G. We denote by $\mathscr{S}_{\mathscr{A}}(G)$ the set of all nonnegative $\mathscr{A}$-superharmonic functions on G.

Next we introduce a notion of balayage to $\mathscr{S}_{\mathscr{A}}(G)$ (cf. [1]). For a subset E of G and $u \in \mathscr{S}_{\mathscr{A}}(G)$, we define the *balayage* $\hat{R}_u^E(G, \mathscr{A})$ of u on E by the following:

$$\hat{R}_u^E(G,\mathscr{A})(z) = \liminf_{x \to z} \inf\{s(x) : s \in \mathscr{S}_{\mathscr{A}}(G),\ s \geq u \text{ on } E\}.$$

By balayage we can give a definition of $\mathscr{A}$-thinness.

DEFINITION 1.1 (cf. [2]). Let E be a subset of G and z a point of $G\backslash E$. E is called *$\mathscr{A}$-thin* at z if there exist open neighborhoods U and V of z with $U\subset V$ such that

$$\hat{R}_1^{E\cap U}(V,\mathscr{A})(z)<1.$$

Otherwise E is called *$\mathscr{A}$-thick* at z.

1.2. We denote by dm the d-dimensional Lebesgue measure.

DEFINITION 1.2 (cf. [4]). Let G be a subdomain of $\boldsymbol{R}^d\,(d\geq 2)$ and C a compact subset of G. The *d-capacity* of C is defined by

$$\mathrm{cap}_d(C,G)=\inf_u\int_C|\nabla u|^d dm,$$

where the infimum is taken over all nonnegative functions u which belong to $W_{loc}^{1,d}(G)$, $u\,|\,C\geq 1$, and have compact supports in G.

For an arbitrary Borel set its d-capacity is defined as usual. We give the definition of d-thinness.

DEFINITION 1.3 (cf. [4]). Let G be a subdomain of $\boldsymbol{R}^d\,(d\geq 2)$, E a subset of G, and z a point of $G\backslash E$. Then, we say that E is *d-thin* (resp. *d-thick*) at z, if

$$W_d(E,z)=\int_0^1\left(\frac{\mathrm{cap}_d(B(z,t)\cap E,\,B(z,2t))}{\mathrm{cap}_d(B(z,t),\,B(z,2t))}\right)^{\frac{1}{d-1}}\frac{dt}{t}<+\infty$$

$$(\text{resp. } W_d(E,z)=+\infty).$$

For $d=2$ d-thinness coincides with the classical thinness. P. Lindqvist and O. Martio [10] essentially proved that $\mathscr{A}$-thinness is equivalent to d-thinness.

Proposition 1.1 (cf. [10],[7]). *Let G be a subdomain of $\boldsymbol{R}^d\,(d\geq 2)$, E a subset of G, and z a point of $G\backslash E$. Then, E is d-thin at z if and only if E is $\mathscr{A}$-thin at z.*

1.3. We begin with recalling the definition of quasiregular mapping.

DEFINITION 1.4 (cf. [14]). Let G be a subdomain of $\boldsymbol{R}^d\,(d\geq 2)$. Then, a non-constant continuous mapping $f:G\to\boldsymbol{R}^d$ is called a *quasiregular mapping* if f

satisfies the following conditions:

(i) $f \in W_{loc}^{1,d}(G)$;

(ii) there exists $K, 1 \leq K < \infty$ such that

$$|f'(x)|^d \leq K J_f(x) \text{ a.e. on } G,$$

where $f'(x) = (\frac{\partial f_i}{\partial x_j})_{1 \leq i,j \leq d} (f = (f_1, f_2, \cdots, f_d) \in \boldsymbol{R}^d)$,

$$|f'(x)| = \max_{|h|=1} |f'(x)h|,$$

and $J_f(x)$ is the Jacobian determinant of f at x.

Furthermore, if f is injective on G, f is called a *quasiconformal mapping*.

We need the next two propositions.

Proposition 1.2 (cf. [4]). *Let $\mathscr{A}: \boldsymbol{R}^d \times \boldsymbol{R}^d \to \boldsymbol{R}^d (d \geq 2)$ be a mapping as in* 1.1, *G a subdomain of $\boldsymbol{R}^d$ and $f: G \to \boldsymbol{R}^d$ a quasiregular mapping. We define a mapping $f^{\#}\mathscr{A}: \boldsymbol{R}^d \to \boldsymbol{R}^d$ by the following*:

$$f^{\#}\mathscr{A}(x,\xi) = \begin{cases} J_f(x) f'(x)^{-1} \mathscr{A}(f(x), f'(x)^{-1*}\xi) & \text{if } J_f(x) \neq 0, \\ (f'(x)^{-1*} \text{ is the transpose of } f'(x)^{-1}) & \\ \xi |\xi|^{d-2} & \text{if } J_f(x) = 0,\ J_f(x) \text{ is undefined, or } x \in \boldsymbol{R}^d \setminus G. \end{cases}$$

Then, $f^{\#}\mathscr{A}$ is a mapping which satisfies the same conditions (1)–(5) *as those of $\mathscr{A}$ in* 1.1.

Proposition 1.3 (cf. [4]). *Let $\mathscr{A}$, $G, f, f^{\#}\mathscr{A}$ be as in* Proposition 1.2, *and s an element of $\mathscr{S}_{\mathscr{A}}(f(G))$. Then, $s \circ f$ is an element of $\mathscr{S}_{f^{\#}\mathscr{A}}(G)$.*

1.4. Proof of Theorem 0.1.

First we need the next lemma.

Lemma 1.1. *Let G be a subdomain of $\boldsymbol{R}^d (d \geq 2)$, f a quasiregular mapping from G into $\boldsymbol{R}^d$, $\tilde{E}$ a subset of $f(G)$, and $u \in \mathscr{S}_{\mathscr{A}}(f(G))$. Then,*

$$\hat{R}_u^{\tilde{E}}(f(G), \mathscr{A}) \circ f \geq \hat{R}_{u \circ f}^{f^{-1}(\tilde{E})}(G, f^{\#}\mathscr{A}).$$

In addition, if f a quasiconformal mapping from G into $\boldsymbol{R}^d$, then

$$\hat{R}_u^{\tilde{E}}(f(G), \mathscr{A}) \circ f = \hat{R}_{u \circ f}^{f^{-1}(\tilde{E})}(G, f^{\#}\mathscr{A}).$$

Proof. The former part follows from definition of balayage and Proposition 1.3. Suppose that f is a quasiconformal mapping from G into $\boldsymbol{R}^d$. We remark that, for all $\xi \in \boldsymbol{R}^d$ and a.e. $x \in \boldsymbol{R}^d$,

$$(f^{-1})^{\#}(f^{\#}\mathscr{A})(x,\xi) = \mathscr{A}(x,\xi).$$

By the former part of this theorem and the above remark we have

$$\begin{aligned}\hat{R}_u^{\tilde{E}}(f(G),\mathscr{A}) &\geq \hat{R}_{u\circ f}^{f^{-1}(\tilde{E})}(G, f^{\#}\mathscr{A})\circ f^{-1} \\ &= \hat{R}_{u\circ f}^{f^{-1}(\tilde{E})}(f^{-1}(f(G)), f^{\#}\mathscr{A})\circ f^{-1} \\ &\geq \hat{R}_{u\circ f\circ f^{-1}}^{f\circ f^{-1}(\tilde{E})}(f(f^{-1}(f(G))), (f^{-1})^{\#}(f^{\#}\mathscr{A}))\circ f\circ f^{-1} \\ &= \hat{R}_u^{\tilde{E}}(f(G),\mathscr{A}).\end{aligned}$$

Therefore we have the latter part.

Proof of Theorem 0.1. By Lemma 1.1, for all neighborhoods U, V of $f(\xi)$ with $U \subset V$,

$$\begin{aligned}\hat{R}_1^{f(E)\cap U}(V,\mathscr{A})\circ f(\xi) &\geq \hat{R}_1^{f^{-1}(f(E)\cap U)}(f^{-1}(V), f^{\#}\mathscr{A})(\xi) \\ &\geq \hat{R}_1^{E\cap f^{-1}(U)}(f^{-1}(V), f^{\#}\mathscr{A})(\xi) = 1.\end{aligned}$$

By Proposition 1.1, we obtain the desired result.

1.5. In this subsection we are concerned with quasiregular mappings from subdomains of $\boldsymbol{R}^2$ into $\boldsymbol{R}^2$.

Theorem 1.1. *Let G be a subdomain of $\boldsymbol{R}^2$, E a subset of G, ξ a point of $G \setminus E$, and f a quasiregular mapping from G into $\boldsymbol{R}^2$. Then, it holds that*

(i) *if E is thin at ξ with $\bar{E} \cap f^{-1}(f(\xi)) = \{\xi\}$, and there exists a neighborhood W of ξ with $W \subset G$ such that $\overline{f(E \setminus W)} \cap \{f(\xi)\} = \emptyset$, then $f(E)$ is thin at $f(\xi)$;*

(ii) *if E is thick at ξ, then $f(E)$ is thick at $f(\xi)$.*

Proof. By Theorem 0.1 we obtain (ii) and hence, have only to prove (i). Suppose that E is thin at a point ξ of G with $\bar{E} \cap f^{-1}(f(\xi)) = \{\xi\}$, and there exists a neighborhood W of ξ with $W \subset G$ such that $\overline{f(E \setminus W)} \cap \{f(\xi)\} = \emptyset$. Thus we have only to prove that, for a neighborhood U' of ξ, $f(E \cap U')$ is thin at $f(\xi)$. In the following discussion, we identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$. It is well-known that quasiregular mappings in dimension two are written as compositions of analytic functions and quasiconformal mappings (cf. [9]) and hence, by Theorem 0.2 we may suppose that f is analytic on G. There exist a disc $B = B(0,r)$ with $B + \xi \subset G$, a conformal mapping $g: B \to g(B)$ and an integer n such that $f(z)=[g(z-\xi)]^n+f(\xi)$ on $B+\xi$. By Theorem 0.2, we may suppose that $\xi=0$ and $f(z)=z^n$. It is easily seen that, for any $K \subset B$ $f^{-1}(f(K))=\cup_{k=0}^{n-1} e^{\frac{2k\pi}{n}i} K$. By Theorem 0.2 each $e^{\frac{2k\pi}{n}i} E$ is thin at 0 and hence, $f^{-1}(f(E \cap B))$ is thin at 0. On the other hand, we have the following equality, for open neighborhoods U, V of 0 in B with $U \subset V$,

$$(*) \qquad \hat{R}_1^{f(E \cap U)}(V, \xi) \circ f = \hat{R}_1^{f^{-1}(f(E \cap U))}(f^{-1}(V), f^{\#}\xi),$$

and hence, by Definition 1.1 we find that $f(E \cap U)$ is thin at $f(0)$. We must prove (*). By Lemma 1.1

$$\hat{R}_1^{f(E \cap U)}(V, \xi) \circ f \geq \hat{R}_1^{f^{-1}(f(E \cap U))}(f^{-1}(V), f^{\#}\xi).$$

To prove the converse inequality, we take an element $s \in \mathscr{S}_{f^{\#}\xi}(f^{-1}(V))$ such that $s \geq 1$ on $f^{-1}(f(E \cap U))$. We set

$$\tilde{s}(z) := \min\{s(e^{\frac{2k\pi}{n}i} z),\ k=0, \cdots, n-1\},$$

for every $z \in f^{-1}(V)$. We remark that $\tilde{s}(z)=\tilde{s}(e^{\frac{2k\pi}{n}i} z)$ on $f^{-1}(V)$ and $f^{-1}(f(E \cap U)) = \cup_{k=0}^{n-1} e^{\frac{2k\pi}{n}i} (E \cap U)$. We set

$$s^{\#}(z) := \tilde{s}(\sqrt[n]{z}).$$

We find that $s^{\#} \in \mathscr{S}_{\xi}(V)$ and $s^{\#} \geq 1$ on $f(E \cap U)$. Therefore we have the desired result.

2. Minimal thinness and quasiconformal mappings

Throughout this section we consider a half plane H of $\boldsymbol{R}^2$. Let $P_\zeta(z)$ ($z \in H$, $\zeta \in \partial H$) be the Poisson kernel with pole at ζ.

Definition 2.1. (cf. [2]) Let E be a subset of H and ζ a point of ∂H. Then, we say that E is *minimally thin* at ζ if

$$\hat{R}_{P_\zeta}^{E}(H, \xi)(z) \neq P_\zeta(z)$$

on H.

J. Lelong-Ferrand proved that thinness is equivalent to minimal thinness under a condition.

Proposition 2.1 (J. Lelong-Ferrand [8]). *Let E be a subset of H and ζ a point of ∂H. Suppose that E belongs to a Stolz domain at vertex ζ. Then, E is thin at ζ if and only if E is minimally thin at ζ.*

The next theorem gives us the quasiconformal invariance of minimal thinness under the same condition as that of Proposition 2.1.

Theorem 2.1. *Let H be a half plane of $\boldsymbol{R}^2$, E a subset of H and f a quasiconformal mapping from H onto H. Suppose that E is a subset of a Stolz domain at vertex ζ. If E is minimally thin at a point ζ of ∂H, then $f(E)$ is minimally thin at $f(\zeta)$.*

Proof. First we recall that f is extended as a quasiconformal mapping on $\boldsymbol{R}^2$. Suppose that E is minimally thin at ζ and belongs to a Stolz domain at vertex ζ. By Proposition 2.1 E is thin at ζ in $\boldsymbol{R}^2$. By Theorem 0.2 we find that $f(E)$ is thin at $f(\zeta)$ in $\boldsymbol{R}^2$. On the other hand, it is well-known that thinness means minimal thinness (cf. [6]). Therefore we obtain the desired result.

3. Quasiconformal invariance of Harmonic dimension of Heins' covering surfaces

Let F be an open Riemann surface of null boundary which has a single ideal boundary component in the sense of Kerékjártó-Stoïlow (cf. [3]). A relatively noncompact subregion Ω of F is said to be an *end* of F if the relative boundary $\partial\Omega$ consists of finitely many analytic Jordan curves (cf. Heins [5]). We denote by $\mathscr{P}(\Omega)$ the class of all nonnegative harmonic functions on Ω with vanishing boundary values on $\partial\Omega$. The *harmonic dimension* of Ω, $\dim\mathscr{P}(\Omega)$ in notation, is defined as the minimum number of elements of $\mathscr{P}(\Omega)$ generating $\mathscr{P}(\Omega)$ provided that such a finite set exists, otherwise as ∞. It is well-known that $\dim\mathscr{P}(\Omega)$ does not depend on a choice of end of F: $\dim\mathscr{P}(\Omega)=\dim\mathscr{P}(\Omega')$ for any pair (Ω,Ω') of ends of F (cf. [5]). In terms of the Martin compactification $\dim\mathscr{P}(\Omega)$ coincides with the number of minimal points over the ideal boundary (cf. Constantinesc and Cornea [3]).

In this section we are especially concerned with ends W which are subregions of p-sheeted unlimited covering surfaces of $\{0<|z|\leq\infty\}$. For these W it is known that $1\leq\dim\mathscr{P}(W)\leq p$ (cf. [5]).

Consider two positive sequences $\{a_n\}$ and $\{b_n\}$ satisfying $b_{n+1}<a_n<b_n<1$ and $\lim_{n\to\infty}a_n=0$. Set $G=\{0<|z|<1\}\backslash I$ where $I=\cup_{n=1}^{\infty}I_n$ and $I_n=[a_n,b_n]$. We take $p(>1)$ copies $G_1,\cdots,G_p$ of G. Joining the upper edge of I_n on G_j and the lower edge of I_n on $G_{j+1}(j \bmod p)$ for every n, we obtain a p-sheeted covering surface $W=W_p^I$ of $\{0<|z|<1\}$ which is naturally considered as an end of a p-sheeted covering surface of $\{0<|z|\leq\infty\}$. Such a covering surface W is referred to as the *Heins' covering surface.* We recall the characterization of harmonic dimension of Heins' covering surface.

Proposition 3.1 ([12] and [13]). *For every integer $p(>1)$, it holds that*

(i) $\dim \mathscr{P}(W)=p$ *if and only if I is thin at* $z=0$;

(ii) $\dim \mathscr{P}(W)=1$ *if and only if I is not thin at* $z=0$.

The next theorem gives us the quasiconformal invariance of harmonic dimension of Heins' covering surface under a condition.

Theorem 3.1. *Let* $W=W_p^I$ *(resp.* $W'=W_p^{I'}$*) be Heins' covering surfaces of* $D=\{0<|z|<1\}$ *with the covering map* $\pi: W \to D$ *(resp.* $\pi': W' \to D$*) which is constructed by* G_j $(j=1,\cdots,p)$ *(resp.* G'_j $(j=1,\cdots,p)$*) as above. Suppose that there exists a quasiconformal mapping f from W onto W' such that, for* $z,z' \in W$ *with* $\pi(z)=\pi(z') \in I$, $\pi'(f(z))=\pi'(f(z')) \in I'$. *Then,* $\dim \mathscr{P}(W)=\dim \mathscr{P}(W')$.

Proof. Suppose that there exists a quasiconformal mapping f from W onto W' such that, for $z,z' \in W$ with $\pi(z)=\pi(z') \in I$, $\pi'(f(z))=\pi'(f(z')) \in I'$. Then, we find that $f(W \backslash \pi^{-1}(I))=W' \backslash \pi'^{-1}(I')$. Thus, for every j $(j=1,\cdots,p)$, there exists an integer $n(j)$ $(n(j)=1,\cdots,p)$ such that $f(G_j)=G'_{n(j)}$ because each G_j $(j=1,\cdots,p)$ is connected and $f(\pi^{-1}(I))=\pi'^{-1}(I')$. Hence, we can consider each $f|_{G_j}$ $(j=1,\cdots,p)$ as a quasiconformal mapping g_j from $\boldsymbol{R}^2$ onto $\boldsymbol{R}^2$ such that $g_j(I)=I'$. By Theorem 0.2 I is thin (resp. thick) at $z=0$ if and only if I' is thin (resp. thick) at $z=0$. Therefore, by Proposition 3.1, we have the desired result.

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