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ON RATIONALITY OF LOGARITHMIC $\mathbb{Q}$-HOMOLOGY PLANES-I

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1. Introduction

Let V be a normal surface defined over $\mathbb{C}$. Following [3], we say V is *logarithmic* if all its singularities are of quotient type. It is called a $\mathbb{Q}$-*homology plane* if its reduced homology groups with rational coefficients all vanish. Let $\sum = \{p_1, \ldots, p_r\}$ denote the set of singularities of V. Then recall that the logarithmic Kodaira dimension of V is defined to be the logarithmic Kodaira dimension of $V \setminus \sum$. In a sequence of three articles beginning with this, we propose to probe the following questions:

Question A: Are all logarithmic $\mathbb{Q}$-homology planes rational?

All logarithmic $\mathbb{Q}$-homology planes with logarithmic Kodaira dimension ≤ 1 are known the be rational (see [3], [2], [7]). Therefore Question A immediately reduces to:

Question B: Are all logarithmic $\mathbb{Q}$-homology planes of logarithmic Kodaira dimension 2 rational?

It may be recalled that in [4], it is proved that all smooth $\boldsymbol{Z}$-homology planes are rational. Adopting the style therein, we can pose the following:

Question C: Let X be a smooth projective surface defined over $\mathbb{C}$. Suppose there is a reduced effective divisor Δ on X such that

i) the irreducible components of Δ generate the $Pic(X) \otimes \mathbb{Q}$;
ii) each connected component of Δ is simply connected;
iii) $\kappa(X, K + \Delta) = 2$.

Then is X a rational surface?

Observe that by blowing up points inside Δ, if necessary, we can assume that Δ is a normal crossing curve. By blowing down, if necessary, we can assume that

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it is minimal with respect to this property. This is what we are going to assume in the sequel without even bothering to mention it.

Given the logarithmic $\mathbb{Q}$-homology plane V as above, let $\psi : U \to V$ be the minimal resolution of singularities, X be a smooth minimal completion of U and $\tilde{D} = (X \setminus U) \cup \psi^{-1}(\sum)$. Below we recall some basic facts about logarithmic $\mathbb{Q}$-homology planes from [7].

Lemma 1.1. *For a logarithmic $\mathbb{Q}$-homology plane V, the following holds*:

(a) *V is an affine surface.*

(b) *A smooth projective completion X of U can be chosen such that $\tilde{D}$ has at worst ordinary double point singularities, and minimal with respect to this property.*

(c) *All the connected components of $\tilde{D}$ are simply connected and the irreducible components of $\tilde{D}$ generate Pic $X \otimes \mathbb{Q}$. Also, X is simply connected.*

(d) *The irregularity and the geometric genus of V (and hence that of X) vanish.*

(e) *The intersection form of $\tilde{D}$ has exactly one positive eigen value.*

Moreover, by the hypothesis in Question B, we have $\kappa(X, K+\tilde{D}) = 2$. Therefore, a positive answer to Question C will imply the same for Question B by taking $\Delta = \tilde{D}$. We adopt similar techniques as in [4], which heavily depends on an inequality of M. Miyaoka and the unimodularity of the adjacency matrix of the *divisor at infinity* of the $\boldsymbol{Z}$-homology plane. In the situation of $\mathbb{Q}$-homology planes and logarithmic $\mathbb{Q}$-homology planes, the divisor at infinity need not be unimodular. As if to compensate for this drawback, we now have a more generalized version of the Miyaoka-Yau type inequality proved by R. Kobayashi for open algebraic surfaces (see [5]). (This was brought to the notice of the authors by R. V. Gurjar for which they are grateful to him.) This inequality plays a crucial role in our study. Below, we breifly describe the outline of our strategy.

Assume that X is not rational. Then X is either a surface of general type or an elliptic surface. Starting with a reduced effective NC divisor D on X, we study the contractions $\Phi : (X, D) \to (X_c, D_c)$ where (X_c, D_c) denotes the *log-canonical model* for the pair (X, D). With the help of this study and Kobayashi's inequality, we derive an inequality (see (7) below), involving the *bark* of the divisor D, the Betti numbers of X and D and $(K.D)$ where K is the canonical divisor of X. The term $(K.D)$ is estimated in [4] by studying the contractions $\pi : X \to X''$, where X'' is the smooth *minimal model* for the function field of X. We reproduce this estimation with minor modifications for the sake of completeness of our treatment and for fixing up our notations. Using this estimation in the afore mentioned inequality, we get an important auxiliary inequality involving certain integral parameters of the surface X and the divisor D. Of course, we are often interested in the case when $D = \Delta$, but sometimes we shall use the inequalities for other divisors also. From this stage onwards, our study is divided into two parts-one, when X is a surface of general type and two, when X is an elliptic surface. A detailed study of the inequalities gives a bound on the number of irreducible components in D and the self-intersection number for the components appearing in D. Certain properties of these surfaces and *bark* computations impose restrictions on the admissible values for the parameters appearing in the auxiliary inequality. One then hopes that geometric configurations predicted by this study violates the auxiliary inequality, thus proving the rationality of X and hence rationality of V.

One can say that the case when Δ (and hence $\tilde{D}$) is connected corresponds to the case when X is a smooth $\mathbb{Q}$-homology plane. The additional number of connected components in $\tilde{D}$ make life easier, and hence one can prove the following theorem with less efforts:

Theorem 1.1. *With X and Δ as in Question* C, *assume further that Δ is not connected. Then X is rational.*

In the present article, we shall complete the proof of this theorem thereby proving:

Corollary. *There are no non-smooth non-rational logarithmic $\mathbb{Q}$-homology planes.*

In Section 2, we collect some basic definitions and results relevant to our analysis. In Section 3, we obtain an auxiliary inequality from Kobayashi's inequality. Another key role in our proof is played by a result due to Gurjar and Miyanishi which states that a logarithmic $\mathbb{Q}$-homology plane is strongly minimal. In particular, it does not contain any contractible curve and all log exceptional curves for (X, Δ) are contained in Δ. We shall see that this is applicable even in the slightly general situation of the Question C also. Later sections will be devoted to getting rid of various cases involved.

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2. Preliminaries

For the details in this section we refer to [2]. Let Y be a smooth projective surface. As in [2], we call a $\mathbb{Q}$-divisor F on Y *pseudo-effective* if $(H.F) \geq 0$ for every ample divisor H on Y. The Zariski-Fujita decomposition of $K_Y + D$, in case $K_Y + D$ is pseudo-effective, is as follows:

(a) There exists a $\mathbb{Q}$-effective decomposition of $K_Y + D$, $K_Y + D \approx P + N$, where $\approx$ denotes numerical equivalence.

(b) P is numerically effective i.e., $(P.A) \geq 0$ for any irreducible curve A.

(c) $N = 0$ or the intersection form on the irreducible components of N is negative definite.

(d) $(P.D_i) = 0$ for every irreducible component D_i of N.

For a proof of the following, we refer to [6].

Theorem 2.1. *Let (Y, D) be a normal completion of a quasi-projective surface Z such that $\overline{\kappa}(Z) \geq 0$. Then $K_Y + D$ is pseudo-effective and if $P + N$ is the Zariski-Fujita decomposition of $K_Y + D$, the following holds*:

(a) $\overline{\kappa}(Z) = 0$ *iff* $P \sim 0$.

(b) $\overline{\kappa}(Z) = 1$ *iff* $P \not\sim 0$ *and* $(P)^2 = 0$.

(c) $\overline{\kappa}(Z) = 2$ *iff* $(P)^2 > 0$.

In our study we need to estimate the value of $(N)^2$, where N is the negative part of a certain *log-canonical divisor* $(K_Y + D)$. For this, we make use of the *theory of peeling* as developed by M. Miyanishi, S. Tsunoda, T. Fujita et al. (See [8] or [2].) For our purpose, the relevant definitions and results are found in Sections 3 and 6 of [2]. This is summarized in Section 10 of [4] which we reproduce below.

Let Y be a non-singular projective surface and D a reduced curve on Y. We shall assume that all the irreducible components of D are smooth rational curves and hence we shall drop the term 'rational' from Fujita's terminology. Recall that D is said to be NC (normal crossings) if all the components of D are smooth and D has at worst ordinary double points. By a $(-n)$-*curve,* we mean a non-singular rational curve D_0 with $(D_0)^2 = -n$. We call D a MNC (minimal with normal crossings) if it is NC and blowing down of any (-1)-curve in D disturbs the NC condition. We shall assume that D is a MNC-curve. For any component D_0 of D the branching number $\beta(D_0)$ is defined by $(D_0.(D - D_0))$. We call D_0 a *tip* if $\beta(D_0) = 1$. A sequence Γ of components $\{D_1, \ldots, D_r\}$, $r \geq 1$, is called a *twig* of D if $\beta(D_1) = 1$, $\beta(D_j) = 2$ and $(D_{j-1}.D_j) = 1$ for $2 \leq j \leq r$. We denote Γ by $[w_1, \ldots, w_r]$ where $w_i = -(D_i)^2$. We call Γ a maximal twig if there is a (unique) component D_0 of D such that $(D_r.D_0) = 1$ and $\beta(D_0) \geq 3$. Then D_0 is called the branching component of Γ. For any twig Γ we denote the curve $D_2 \cup \ldots \cup D_r$ by $\overline{\Gamma}$. By convention $\overline{\Gamma} = \emptyset$ if $r = 1$. A sequence Γ is called a *club* if $\beta(D_1) = \beta(D_r) = 1$, $\beta(D_j) = 2$, $2 \leq j \leq r - 1$.

A connected component Γ of D is called a *fork* if

(a) Γ has a unique component D_0 with $\beta(D_0) = 3$.

(b) Γ has three maximal twigs $\Gamma_1, \Gamma_2, \Gamma_3$ with D_0 as the branching component.

(c) Γ is negative definite.

(d) $d(\Gamma_1)^{-1} + d(\Gamma_2)^{-1} + d(\Gamma_3)^{-1} > 1$ where $d(-)$ denote the discriminant.

Now, assume that $K+D$ is pseudo-effective, so that by 6.13 of [2], all clubs and maximal twigs of D are negative definite. If Γ is a maximal twig of D, then *bark* of Γ denoted $Bk(\Gamma)$ is an element $N_1 \in \mathbb{Q}(\Gamma)$ such that $(N_1.D_i) = ((K+D).D_i)$ for $1 \leq i \leq r$. If Γ is a club of D then $Bk(\Gamma)$ is an element $N_2 \in \mathbb{Q}(\Gamma)$ such that $(N_2.D_i) = ((K+D).D_i)$, for all $D_i \in \Gamma$. For an isolated club $\Gamma = \{D_1\}$, we have $Bk(\Gamma) = 2(-(D_1)^2)^{-1}D_1$. For any curve D we define $Bk(D) = \sum Bk(\Gamma_j)$ where summation runs over all maximal twigs and clubs of D.

For a fork D we define *thicker bark* denoted $Bk^{\star}(D)$, as an element $N \in \mathbb{Q}(D)$ such that $(N.D_i) = ((K+D).D_i)$, for every irreducible component D_i of D. Finally for any connected component Λ of D which is not a fork we define $Bk^{\star}(\Lambda) = Bk(\Lambda)$. Following [4], we introduce the notation $bk(D)$ (resp. $bk^{\star}(D)$) for the rational number $(Bk(D).Bk(D))$ (resp. $(Bk^{\star}(D).Bk^{\star}(D))$).

We will need the following result proved in [11].

Lemma 2.1. *Let D be a contractible $\mathbb{Q}$-divisor. Let $F \in \mathbb{Q}(D)$ and let $(F.D_j) \leq 0$ for any irreducible component D_j of D. Then F is effective.*

Following is a result proved in [2].

Lemma 2.2. *Let D be a $\mathbb{Q}$-divisor and $\Gamma = [-(D_1)^2, \ldots, -(D_r)^2]$ a twig of D. Let $N_1 = \sum n_i D_i \in \mathbb{Q}(\Gamma)$ be the bark of Γ. Then $n_1 = d(\overline{\Gamma})/d(\Gamma)$, $n_r = d(\Gamma)^{-1}$, $(N_1)^2 = -n_1$ and $0 < n_i < 1$ for $1 \leq i \leq r$.*

Proof. By definition, we have $N_1 \in \mathbb{Q}(\Gamma)$ and $(N_1.D_j) = ((K+D).D_j) = ((K+D_j+D-D_j).D_j) = -2+\beta(D_j) \leq 0$. Thus by Lemma 2.1 above, we see that N_1 is an effective divisor. Also, $-n_j$ is the $(1,j)^{th}$ entry of the inverse matrix of the adjacency matrix of Γ, which is the $r \times r$ matrix with $(i,j)^{th}$ entry $(D_i.D_j)$. Hence we get $n_1 = d(\overline{\Gamma})/d(\Gamma)$ and $n_r = d(\Gamma)^{-1}$. That $(N_1)^2 = -n_1$ is clear from the definition of N_1. Clearly both n_1 and n_r are less than 1. Now, assume that $M = \mathrm{Max}_i\{n_i\} \geq 1$. Take the least i such that $n_i = M$. Since $1 < i < r$, we have $0 = (N_1.D_i) = n_{i-1} + n_i(D_i)^2 + n_{i+1} < M(2+(D_i)^2) \leq 0$. This contradiction proves that $M < 1$. Thus we have proved the lemma. □

REMARK 2.1. The fact that $bk(\Gamma) = -d(\overline{\Gamma})/d(\Gamma)$ implies that $bk(\Gamma)$ may be obtained by the method of continued fractions. Henceforth we shall use this fact freely.

The following lemma is useful when estimating $bk(D)$.

Lemma 2.3. *Let Γ denote the set of twigs of D, a non-linear tree with at least five vertices. If any of the following is a subset of Γ, then $bk(D) < -1$:*

$\{[2],[2]\},\{[2],[3],[4]\},\{[3],[3],[3]\},\{[3],[3],[4],[6]\},$
$\{[3],[3],[5],[5]\},\{[3],[2,2]\},\{[2,2],[3,2]\},$
$\{[n],[n\times 2]\}$, *where* $[n\times 2]$ *denotes a twig with* n *vertices each of weight* 2.

Proof. Note that D has to have at least three tips. Further, in case D has exactly three tips, all three cannot be maximal twigs. Then, by Lemma 2.2 we see that $bk(D) < -1$. Observe that $bk([3,2]) = -2/5$ and $bk([n\times 2]) = -n/n+1$. □

Lemma 2.4. (a) *Let* $\{L_1,\dots,L_k\}$ *be the tips of a graph* T *with the weight of* L_i *being* w_i *then* $bk(T) \le \sum_{i=1}^{k} 1/w_i$.

(b) *If* T *is a fork, then* $bk^\star(T) < bk(T)$. *Moreover,* $bk^\star(T) < -1$.

(c) *Let* $T = [-w_1,\dots,-w_r]$ *be a club. If* $r = 1,2,3$ *or* 4, *then* $bk(T) = 4/w_1$, $(w_1+w_2-2)/(w_1w_2-1)$, $w_2(w_1+w_3)/(w_1w_2w_3-w_1-w_3)$ *or* $(w_1w_2w_3+w_2w_3w_4-w_1-w_2-w_3-w_4-2)/(1-w_1w_2-w_3w_4-w_1w_4+w_1w_2w_3w_4)$ (resp.).

(d) *If* T *is the dual graph of a minimal resolution of a quotient singularity, then* $bk^\star(T) < -3/n$, *where* $n = d(T)$ *is the order of the local fundamental group of the singularity.*

Proof. (a) First, let $k > 2$. i.e., T is not a club. Let Γ be a twig (with tip L_1). By Lemma 2.2 we see that $bk(\Gamma) = -d(\overline{\Gamma}/d(\Gamma)$. Thus

$$bk(\Gamma) = -\frac{d(\overline{\Gamma})}{-w_1 d(\overline{\Gamma}) - d(\overline{\overline{\Gamma}})} \le -\frac{d(\overline{\Gamma})}{-w_1 d(\overline{\Gamma})} = \frac{1}{w_1}$$

and hence $bk(T) \le \sum 1/w_i$.

Now, let $k \le 2$. i.e., T is a club. If T consists of exactly one component, then $bk(T) = 4/w_1$ and hence (a) holds. Hence, let $T = [T_1,\dots,T_r]$ with $r > 1$. As in the definitions above let $N_1 = \sum \lambda_i T_i$ and $N_2 = \sum \mu_i T_i$. Then $N_2 = Bk(T)$, and N_1 has the numerical property of $Bk(T)$ if T were a maximal twig with T_1 as its tip. Hence again by Lemma 2.2 above, we see that $\lambda_1 = -(N_1)^2 \ge -1/w_1$. On the other hand we have $((N_2-N_1).T_i) \le 0$ for every component T_i and hence by Lemma 2.1, $N_2 \ge N_1$. In particular $\mu_1 \ge \lambda_1 \ge -1/w_1$. By symmetry, we get $\mu_r \ge -1/w_r$. Now $bk(T) = (N_2)^2 = -(\mu_1+\mu_2) \le (1/w_1 + 1/w_r)$. Hence we have proved (a) of the lemma.

(b) To prove (b), let T be a fork with the three maximal twigs Γ_1, Γ_2 and Γ_3 and let D be the corresponding curve. Let $N = Bk(T)$ and $N_i = Bk(\Gamma_i)$, $i = 1,2,3$. Let $G = N - (N_1+N_2+N_3)$. Then we have $(G.L) = 0$ for every component L of $\sum \Gamma_i$ and $(G.D_0) = ((K+D).D_0) - (d(\Gamma_1)^{-1} + d(\Gamma_2)^{-1} + d(\Gamma_3)^{-1}) < 1 - 1 = 0$. Hence $G > 0$ by Lemma 2.1 above. Now $(N)^2 = (N_1)^2 + (N_2)^2 + (N_3)^2 + (G)^2$ and $(G)^2 = (G.(nD_0))$ where $n \ge 0$ is the coefficient of D_0 in N. Since $(G.D_0) < 0$, it follows that $(G)^2 < 0$. Hence $bk^\star(D) = (N)^2 < (N_1)^2 + (N_2)^2 + (N_3)^2 < bk(D)$. The second part follows easily from condition (d) in the definition of the fork. This proves (b) of the lemma.

(c) One can compute the barks directly from the definition.

(d) If T is a fork, then $n > 6$ and as seen above $bk^\star(T) < -1$. If T is an isolated tip, then $bk^\star(T) = -4/n$. Finally let T be a club. Then let us consider T as maximal twig of a larger tree with one of its leaves as the leader. Then $bk^\star(T) < bk(T) = -d(\bar{T})/d(T) = -d(\bar{T})/n$. Now, unless $T = [2,2]$, by choosing the leader appropriately, we see that $d(\bar{T}) \geq 3$. Of course if $T = [2,2]$ then we directly see that $bk^\star(T) = -2$ and $n = d(T) = 3$. Thus in all cases, we see that $bk^\star(T) < -3/n$. □

DEFINITION. Two trees T and T' are said to be isomorphic if there exists a map $f : T \to T'$ such that whenever T_1T_2 is an edge in T, then $f(T_1)f(T_2)$ is an edge in T'. Let T be a tree in which vertices $T_1, \ldots T_n$ are ordered in some fixed order. Let the n-tuple $(w_1, \ldots, w_n)$ denote the ordered weight set of T. For any tree T' isomorphic to T we say $T \leq T'$ if $w_i \leq w_i'$ where w_i' is the weight of $f(T_i)$ under the isomorphism $f : T \to T'$.

Lemma 2.5. *If T and $\tilde{T}$ are isomorphic trees such that $T \leq \tilde{T}$, then $bk(T) \geq bk(\tilde{T})$.*

Proof. Given a T, let T' be obtained by decreasing one of the weights on T, by 1. It is enough to prove that $bk(T') \geq bk(T)$. In fact, if the change in the weight is made on a component not contained in support of the $Bk(T)$, then this is obvious. So, let us assume that the weight of C_j has been reduced by one where C_j is one of the components of $Bk(T)$. For simplicity, we shall consider the case when C_j is inside a maximal twig $\Gamma = [C_1, \ldots, C_k]$ say. Let $A := Bk(\Gamma) = \sum \alpha_i C_i$ and $\tilde{A} = Bk(\Gamma')$. Let E be a vertex with $(E^2) = -1$ and $(E.C_j) = 1$ and of course $(E.C) = 0$ for all other components of T. We can think of T' as corresponding to the proper transform of T under a blow-up on the vertex C_j. Recall that A and $\tilde{A}$ are defined by the property

$$(A.C_i) = (K + T).C_i, \quad \forall \quad i = 1, \ldots, k$$

and

$$(\tilde{A}.C_i') = (K' + T').C_i', \quad \forall \quad i = 1, \ldots, k.$$

Taking the total transforms we have,

$$(A' + \alpha_j E).C_i = (K' + T' + 2E).C_i', \quad i \neq j$$

and

$$(A' + \alpha_j E).(C_j' + E) = (K' + T' + 2E).(C_j' + E).$$

This is the same as

$$(A.C_i) = (A'.C'_i) = ((K' + T').C'_i) = (\tilde{A}.C'_i), \quad i \neq j,$$

and

$$(A.C_j) = (A'.C'_j) + \alpha_j = ((K' + T').C'_j) + 2 - 1 + 1 - 2 = (\tilde{A}.C'_j).$$

In particular, since $\alpha_j \geq 0$, we have,

$$(A'.C'_i) \leq (\tilde{A}.C'_i), \quad \forall \quad i = 1, \dots, k.$$

Therefore,

$$(A'.\tilde{A}) \leq (\tilde{A})^2.$$

Further, from the above observations it follows that $(A)^2 = (\tilde{A}.A') \leq (\tilde{A})^2$. In fact, since $\alpha_j > 0$, strict inequality holds. Other cases can be considered similarly. □

REMARK 2.2. In light of this lemma, we see that increase in the weight of a vertex in a maximal twig (so that all the weights are still less than -1) reduces the value of $bk(D)$. Hence Lemma 2.3 is valid for any set of such increased weights. In the sequel, Lemma 2.3 will be put to use with this additional sense.

Importance of bark stems from the following crucial result due to Fujita (see Thm. 6.20 of [2]).

Theorem 2.2. *Let Y be a non-singular projective surface and D a MNC-curve on Y with all its irreducible components rational. If $K + D$ is pseudo-effective and $K + D = P + N$ its Zariski-Fujita decomposition, then $N = Bk^\star(D)$ unless there exists a (-1)-curve E not in D satisfying one of the following:*

(a) $(D.E) = 0$.

(b) $(D.E) = 1$ *and E meets a component of $Bk^\star(D)$.*

(c) $(D.E) = 2$ *and E meets precisely two components of D, one of which is a tip of a club of D.*

REMARK 2.3. As remarked in [4], even though the word *precisely* is not mentioned in Fujita's theorem, it is obvious from the proof given there. We shall refer to these conditions on (Y, D) as *Fujita's conditions*. In application, when $Y = X$ and $D = \Delta$ of the Question C, we observe that by contracting all exceptional curves violating Fujita's condition we can pass on to a surface pair $(\bar{X}, \bar{\Delta})$ which satisfies all the conditions of the Question C again. Therefore, without loss of generality, we can assume that the given pair (X, Δ) itself satisfies Fujita's conditions. However, we note that, in carrying out these operations, at each step, the number of connected components of Δ goes down at most by one whereas the second betti number goes up by one in comparison with that of the surface. Alternatively, in Lemma 3.3, we shall prove that indeed, the pair (X, Δ) satisfies Fujita's conditions. This observation is crucial, in obtaining simpler proof of the Theorem 1.1, as claimed.

Now, we shall state an inequality due to R. Kobayashi. For the details we refer to [5]. We recall the definitions of *log-canonical* and *log-terminal singularities* of a pair (Y, D). Let (Y, D, p) be a germ of a normal surface pair, i.e., (Y, p) is a germ of a normal surface and D is a finite union of branch loci $D = \sum(1 - 1/z_i)D_i$ where $2 \leq z_i \leq \infty$ are integers and each component D_i passes through the point p. Let $\mu : (\overline{Y}, \overline{D}, E) \to (Y, D, p)$ be the resolution of p such that $\overline{D}$ is a MNC curve and E is the exceptional set. Let $E = \cup E_j$ with E_j being the irreducible components of E. It is known that

$$\mu^\star(K_Y + D) = K_{\overline{Y}} + \overline{D} + \sum_j a_j E_j$$

where a_j are defined by the equations (see [9]):

$$\left(K_{\overline{Y}} + \overline{D} + \sum_j a_j E_j\right).E_k = 0 \qquad \text{for each } k. \tag{1}$$

A germ of the normal surface pair (Y, D, p) is called *log-canonical* (resp. *log-terminal*) singularity if $a_j \leq 1$ for all j (resp. $a_j < 1$ for all j and $z_i < \infty$ for all i). For a normal surface pair (Y, D) with at worst log-canonical singularities, we write $LCS(Y, D)$ for all log-canonical singularities of (Y, D) which are not log-terminal.

We now recall the definition of *log-minimal* and *log-canonical models* for a pair (Y, D), where Y is a projective surface with at worst log canonical singularities and $D = \sum_i(1 - 1/z_i)D_i$ $(z_i = 2, 3, \ldots, \infty)$, a divisor on Y.

DEFINITION. An irreducible curve E on Y is a log-exceptional curve of the first kind (resp. log-exceptional of the second kind) if $E^2 < 0$ and $((K_Y + D).E) < 0$ (resp. if $E^2 < 0$ and $((K_Y + D).E) = 0$).

Given a smooth surface Y and a reduced effective divisor D on Y, by successive contractions of log-exceptional curves of the first kind, we arrive at log-minimal model (Y_m, D_m). This pair is characterized by the following two properties:

(a) (Y_m, D_m) is log-minimal, i.e., it contains no log-exceptional curve of the first kind;

(b) there exists a bimeromorphic holomorphic mapping $f : (Y, D) \to (Y_m, D_m)$ such that $D_m = f_\star(D)$ and $K_Y + D = f^\star(K_m + D_m) + \sum_i a_i E_i$, $a_i > 0$

for all i, where K_m is the canonical divisor of Y_m and $E = \cup E_i$ is the exceptional set of f.

By contracting all log-exceptional curves of the second kind in the log-minimal model (Y_m, D_m) we arrive at log-canonical model (Y_c, D_c). This pair is characterized by the following two properties:

(a) (Y_c, D_c) is log-canonical, i.e., it contains no log-exceptional curve of the first or second kind;

(b) there exists a bimeromorphic holomorphic mapping $g : (Y_m, D_m) \to (Y_c, D_c)$ such that $D_c = g_\star(D_m)$ and $g^\star(K_c + D_c) = K_m + D_m$ where K_c denotes the canonical divisor of Y_c.

It is known that log-canonical model of (Y, D) is unique if $\overline{\kappa}(Y \setminus D) = 2$ (see Fact 2, page 350 of [5]). Observe that, all the exceptional curves of f need not be characterized by property in the above definition, on the surface Y itself, but could satisfy this property after successive contractions.

In all our study we shall deal with only those divisors D such that $\overline{\kappa}(X \setminus D) = 2$. Hence we always have unique log-canonical model. The following result is proved by R. Kobayashi (see Theorem 1 and Theorem 2 of [5]).

Theorem 2.3. *Let (Y, D) be a normal surface pair with $\overline{\kappa}(Y \setminus D) = 2$. Suppose (Y, D) has at worst log-canonical singularities. Let (Y_c, D_c) be the log-canonical model for (Y, D) where $D_c = \sum_i (1 - 1/z_i) D_{c,i}$ is the image divisor of D. Define $Y_0 := Y_c \setminus (\cup_{z_i = \infty} D_{c,i}) \setminus LCS(Y_c, D_c)$ and $D^0_{c,i} := D_{c,i} \cap Y_0$. Then*

$$(2) \quad (K_c + D_c)^2 \leq 3 \left\{ e(Y_0) + \sum_i \left(\frac{1}{z_i} - 1 \right) (e(D^0_{c,i}) - d_i) + \sum_p \left(\frac{1}{|\Gamma(p)|} - 1 \right) \right\}$$

where, $e(-)$ denotes the topological euler number, d_i is the number of singularities of (Y_c, D_c) lying over $D^0_{c,i}$ and $|\Gamma(p)|$ is the order of the local fundamental group $\Gamma(p)$ of a log-terminal singular point p of (Y_c, D_c).

In the application of this theorem, we always have a situation in which $Y = X$ is simply connected, D is an (integral) reduced effective divisor, i.e., $z_i = \infty$ for all i. Thus any singularity which lies in the image divisor D_c of D is going to be LCS. Moreover, since we begin with log projective surfaces, all singularities outside D_c are goint to be log terminal. Let $\{p_1, \ldots, p_s\}$ be the set of such singularities, and let γ_i denote the order of the local fundamental group at p_i. Then (2) reduces to:

$$(3) \qquad (K_c + D_c)^2 \leq 3 \left\{ e(X_{c0}) - s + \sum_{i=1}^{s} \frac{1}{\gamma_i} \right\}.$$

Let us introduce the notation $b_i(-)$ denote the i^{th} betti number of a topological space, and let $\beta_i = b_i(X)$, $b_i = b_i(D)$. Since X is simply connected, we have,

$\beta_1 = \beta_3 = 0$. Also, $\beta_0 = \beta_4 = 1$. Hence, $e(X) = 2 + \beta_2$. In order, to relate $e(X)$, $e(X_{c0})$ and $e(D)$, let us make the following technical definition:

DEFINITION. We say (Y, D) is *log-content* if all the log-exceptional curves for the pair (Y, D) are contained in D.

Let now (X, D) be log-content. Let l be the number of connected components of D_c. Then $s + l = b_0$. (This follows because, tacitly we are assuming that D is a MNC curve.) Let c be the number of irreducible components in the exceptional set. Then it follows that $b_2(X_c) = b_2(X) - c$, $b_1(D_c) = b_1$, $b_2(D_c) = b_2(D) - c$. Therefore, $e(X_c \setminus D_c) = e(X_c) - e(D_c) = 1 + (\beta_2 - c) + 1 - (l - b_1 + b_2 - c) = 2 + (\beta_2 - b_2) + b_1 - l$. Thus we may rewrite (3) as

$$(4) \qquad (K_c + D_c)^2 \leq 3\left\{2 + (\beta_2 - b_2) + b_1 - b_0 + \sum_{i=1}^{s} \frac{1}{\gamma_i}\right\}.$$

We will need the following important result due to Sakai relating the Zariski-Fujita decomposition and the log-minimal model (see [10]).

Lemma 2.6. *Let $f : (Y, D) \to (Y_m, D_m)$ be the bimeromorphic holomorphic mapping contracting the log-exceptional curves of the first kind. Then $P = f^\star(K_m + D_m)$ where $K_Y + D = P + N$ is the Zariski-Fujita decomposition of the log-canonical devisor $K_Y + D$.*

As a consequence of this lemma, we see that

$$\begin{aligned} (K_c + D_c)^2 &= (g^\star(K_c + D_c))^2 \\ &= (K_m + D_m)^2 \\ &= (f^\star(K_m + D_m))^2 \\ &= (P)^2. \end{aligned}$$

Thus, if $K + D = P + N$ is the Zariski-Fujita decomposition of $K + D$, we may rewrite (4) as

$$(5) \qquad 0 < (P)^2 = (K_c + D_c)^2 \leq 3\left\{2 + (\beta_2 - b_2) + b_1 - b_0 + \sum_{i=1}^{s} \frac{1}{\gamma_i}\right\}.$$

Lemma 2.7. *For X as above and any NC divisor D with all its irreducible components rational, we have*

$$(K + D)^2 = 10 - \beta_2 + 2(b_1 - b_0) + (K.D).$$

Proof. If D is connected and simply connected, then it follows that, $(K + D.D) = -2$. From this, we derive that if $b_1(D) = l$, then $(K + D.D) = -2 + 2l$. Taking sum over all the connected components, we have, in general, $(K + D.D) = 2(b_1 - b_0)$. Therefore, using Nöther's formula, and the simply connectedness of X we have,

$$(6)\quad (K+D)^2 = (K)^2 + (K.D) + ((K+D).D) = 10 - \beta_2 + 2(b_1 - b_0) + (K.D).$$

Thus we have proved the lemma. □

Using $(K + D)^2 = P^2 + N^2$ in the inequality (5), we get

$$0 < 10 - \beta_2 + 2(b_1 - b_0) + (K.D) - (N)^2 \leq 3\left(2 + (\beta_2 - b_2) + b_1 - b_0 + \sum_{i=1}^{s} \frac{1}{\gamma_i}\right).$$

Introducing the notation, $(N)^2 + 3\sum_{i=1}^{s} 1/\gamma_i = \nu = \nu(D)$, we get,

$$(7)\qquad -\nu \leq 4\beta_2 - 3b_2 + b_1 - b_0 - (K.D) - 4.$$

3. Auxiliary inequality

As seen in the previous section, we have been lead to the problem of estimating $(K.D)$. This is done in this section, more of less exactly as in [4], with a few improvements, which we shall recall here, for the sake of completeness, and for fixing up the notation.

From now onwards we shall assume that X is smooth simply connected projective surface which is not rational and D is a MNC divisor on X satisfying Fujita's conditions and such that (X, D) is log-content. Our aim here is to derive a number of auxiliary inequalities, resulting from Kobayashi's inequality.

As a consequence of non-rationality of X we have the following result.

Lemma 3.1. (a) *Any two (-1)-curves on X are disjoint.*
(b) *For any irreducible component C of D, we have $(C)^2 < 0$.*
(c) *There is at least one branching curve in D.*

We will use this lemma tacitly throughout our study.

Let $\pi : X \to X''$ be a composition of contractions of (-1)-curves, where X'' is the smooth minimal model for the function field of X. Let $D'' := \pi(D)$. Let $\mathcal{E}$ be the exceptional set for π. Write $\pi = \phi_m \circ \phi_{m-1} \cdots \circ \phi_1$ where each ϕ_j is a contraction of the (-1) curve E_j. Let $\psi_0 = Id_X$ and $\psi_j = \phi_j \circ \cdots \circ \phi_1$ for $j \geq 1$. By Lemma 3.1(a), we see that any two (-1) curves in X are disjoint. Hence we can arrange $\phi_j s$ in such a way that if $\pi_1 = \phi_{n_1} \circ \cdots \circ \phi_1$, then

(a) $D' := \pi_1(D)$ has all the components still smooth and

(b) for each $j > n_1$, $(E_j.C) \geq 2$ for at least one component C of $\psi_{j-1}(D)$.

Let $X' = \pi_1(X)$, $\pi_2 := \phi_n \circ \cdots \circ \phi_{n_1+1} : X' \to X''$ and let $\mathcal{E}_i$ be the exceptional set for π_i, $i = 1, 2$. Write $m = n_1 + n_2$. Clearly $b_2(\mathcal{E}_i) = n_i$ and $b_2(\mathcal{E}) = m = n_1 + n_2$. We introduce some more notations. The integer $\beta(E_j) := (\psi_{j-1}(D) - E_j).E_j$ is the branching number of E_j w.r.t. $\psi_{j-1}(D)$. Let $\beta(C_i) := \beta(\psi_{j-1}(C_i))$ for any component C_i of $\mathcal{E}$ where j is such that $\psi_{j-1}(C_i)$ is a (-1) curve. Let
$R_i := \cup\{L_j \in \mathcal{E}_1 \mid \beta(L_j) = i\}$, $r_i := b_2(R_i)$, $i \geq 2$, $S := \mathcal{E}_2 \cap D'$,
$e_1 := n_1 - b_2(\mathcal{E}_1 \cap D)$, and $\sigma := n_2 - \sum_{E' \in S}(E'^2 + 2)$.

Now, let $D = \{D_r\}$, $D' = \{D'_s\}$ and $D'' = \{D''_t\}$. Let $\{P_{t,i}\}$ be all the singular points of D''_t-including the infinitely near ones-and let the multiplicities at these be $m_{t,i} (\geq 2)$. We define

$$\tau := \sum_{t,i} m_{t,i} - 2n_2 \quad \text{and} \quad \lambda := \sum_t K''.D''_t.$$

For $1 \leq j \leq n_1$, let now ϕ_j contract $\psi_{j-1}(L_j)$ where $L_j \subset \mathcal{E}_1 \cap R_i$, for some $i \geq 2$. Then clearly,

$$(8) \qquad \sum_r ((\psi_{j-1}(D_r))^2 + 2) = \begin{cases} \sum_r ((\psi_j(D_r))^2 + 2) - i & \text{if } L_j \not\subset D, \\ \sum_r ((\psi_j(D_r))^2 + 2) - i + 1 & \text{if } L_j \subset D. \end{cases}$$

Here we take $(\psi_{j-1}(D_r))^2 = 0$ (resp. $(\psi_j(D_r))^2 = 0$) if $\psi_{j-1}(D_r)$ (resp. $\psi_j(D_r)$) is a point. By the adjunction formula we have

$$-(K.D) = \sum_r (D_r^2 + 2) = \sum_r (\psi_0(D_r))^2 + 2)$$

and

$$-(K'.D') = \sum_s ((D'_s)^2 + 2) = \sum_s ((\pi_1(D_s))^2 + 2) = \sum_r ((\psi_{n_1}(D_r))^2 + 2).$$

Now by repeated application of (8) for $j = 1, \ldots, n_1$ and the fact that $n_1 = \sum_{i \geq 2} r_i$, we obtain

$$-(K.D) = -(K'.D') - \sum_{i \geq 2} ir_i + b_2(\mathcal{E}_1 \cap D)$$

and hence,

$$-(K.D) = -(K'.D') - \sum_{i \geq 1} ir_{i+2} - n_1 - e_1.$$

Now observe that each D'_t is a smooth model for D''_t. However, it is possible that a number of points have been further blown-up on the minimal smooth model of D''_t to arrive at D'_t. Let us denote this number by u_i and let $u = \sum_t u_t$. On the other hand, by genus formula, we have for each t,

$$((D''_t)^2 + 2) + (D''_t.K'') = \sum_i m_{t,i}(m_{t,i} - 1)$$

and

$$(D'_t)^2 + 2 = (D''_t)^2 + 2 - \sum_i m_{t,i}^2 - u_t = -\sum_i m_{t,i} - (D''_t.K'') - u_t. \tag{9}$$

Recalling the definition of τ, λ and u we get

$$\sum_t ((D'_t)^2 + 2) = -\tau - 2n_2 - \lambda - u.$$

Since

$$-(K'.D') = \sum_s ((D'_s)^2 + 2) = \sum_t ((D'_t)^2 + 2) + \sum_{E' \in S} ((E')^2 + 2),$$

we have

$$-(K.D) = -\lambda - \tau - \sigma - e_1 - \sum_{i \geq 1} i r_{i+2} - m - u.$$

Let us put

$$\theta = \lambda + \tau + \sigma + e_1 + u + \sum_{i \geq 1} i r_{i+2}. \tag{10}$$

Then we have,

$$-(K.D) = -\theta - m. \tag{11}$$

Substituting this in (7) and noting that $\beta''_2 := b_2(X'') = \beta_2 - m$, we get,

$$-\nu \leq \beta''_2 - 4 - \theta + 3(\beta_2 - b_2) + b_1 - b_0. \tag{12}$$

Since each term in the expression for θ is non negative, we get bound on each of these terms and some inter-relation. The idea is to show that, these relations are not compatible and hence to arrive at a contradiction. This part of the proof is quite a detailed case by case study. The following qualitative observations most of which are taken from [4], aid us in this task considerably.

Lemma 3.2.

(a) *To each* (-1)*-curve* E_i' *in* S *there exist* $D_t' \subset D'$ *such that* $(E_i'.D_i') \geq 2$ *and some point* $x_i \in E_i' \cap D_i'$ *such that either* $b_2(\pi_1^{-1}(x_i)) \geq 2$ *and* $\pi_1^{-1}(x_i)$ *contains a curve* $L_i \in R_3 \cup R_4$ *or* $\pi_1^{-1}(x_i) = \{L_i\}$ *for some* (-1)*-curve* L_i *which is not in* D. *In particular, if* $p :=$*number of* (-1)*-curves in* S, *then* $p \leq n_1$.

(b) *If* $\sigma = 0$ *then* $\mathcal{E}_2 = S$ *and* S *is a disjoint union of* (-1)*-curves and hence* $n_2 = p \leq n_1$.

(c) *If* $\sigma = 1$ *then either*

(i) $\mathcal{E}_2 \setminus D' = \{E_1'\}$ *and* S *is a disjoint union of* (-1) *curves or*

(ii) $\mathcal{E}_2 = S$ *consists of a disjoint union of* (-1) *curves and* $\{E_1', E_2'\}$ *with* $(E_1')^2 = -1$, $(E_2')^2 = -2$ *and* $(E_1'.E_2') = 1$.

(d) $\sigma + n_1 = 0$ *implies that* $X = X''$ *is minimal; in particular,* $\tau = 0 = u$.

(e) $u = 0$ *implies that each* D_t' *is the minimal smooth model of* D_t''.

Proof. (a) Let E_i' be any (-1)-curve in S. By definition of $\mathcal{E}_2$, there exists $D_i' \subset D'$ with $(D_i'.E_i') \geq 2$. Since D is a system of divisors with normal crossings and $E_i' \subset D'$, it follows that for all $x \in D_i' \cap E_i'$ such that $(D_i'.E_i')_x \geq 2$ we have the said property for $\pi_1^{-1}(x_i)$. If $(D_i'.E_i')_x = 1$ for all $x \in D_i' \cap E_i'$ then $D_i' \cup E_i'$ is not simply connected. Since D is simply connected, it follows that for some $x \in D_i' \cap E_i'$ we have $(D_i'.E_i')_x \geq 2$ and so we are done.

(b) By definition we have

$$\sigma = n_2 - \sum_{E' \subset S} ((E')^2 + 2).$$

Since $b_2(S) \leq n_2$ and $(E')^2 \leq -1$ for each $E' \subset S$, $\sigma = 0$ implies that $(E')^2 = -1$ for all $E' \in S$ and $b_2(S) = n_2$. Hence $\mathcal{E}_2 = S$ and $\mathcal{E}_2$ is a disjoint union of (-1)-curves.

(c) If $\sigma = 1$ then either $b_2(S) = n_2 - 1$ or $b_2(S) = n_2$. In the former case, we further have $(E')^2 = -1$ for all $E' \in S$ as before. In the latter case $\mathcal{E}_2 = S$ and all except one curve, say E_1', in S are (-1)-curves and $(E_1')^2 = -2$. The rest of the claim of the lemma is obvious.

(d) This is an easy consequence of (b).

(e) This is an easy consequence of the definition of u. □

We shall now prove:

Lemma 3.3. *Let* (X, Δ) *be as in Question* C. *Then the following holds:*

(a) (X, Δ) *is log-content.*

(b) $b_2(\Delta) = \beta_2$.

(c) *Components of* Δ *form a* $\mathbb{Q}$*-basis for Pic* $(X) \otimes \mathbb{Q}$.

(d) *The intersection form on* Δ *has exactly one positive eigen value.*

(e) *The connected component of Δ that supports the positive eigen value has non-linear dual graph (if X is non-rational).*

(f) *Δ has at most two connected components.*

(g) *(X, Δ) satisfies Fujita's conditions.*

Proof. We shall first prove the assertions (a) to (f) assuming that (X, Δ) satisfies Fujita's conditions. Then we shall show that, indeed, (X, Δ) satisfies Fujita's conditions.

To prove (a), we make use of the results of [3]. In order to apply Lemma 4 and Lemma 6 of [3] directly for our situation the only trouble is that $X \setminus \Delta$ is not necessarily affine. However what is needed instead is that $X \setminus \Delta$ does not contain any complete curve and this follows from the hypothesis that the irreducible components of Δ generate $Pic(X) \otimes \mathbb{Q}$. Hence, if (X, Δ) were not log-content, by Lemmas 4 and 6 of [3] we see that there exists a (-1)-curve E on X not contained in Δ such that E intersects Δ in at most two distinct connected components of Δ transversally. This violates Fujita's conditions and hence (a) follows.

Now, assume that $b_2(\Delta) > \beta_2$. Since (X, Δ) is log-content we may apply inequality (5) to the pair (X, Δ),

$$(13) \qquad 0 < P^2 \leq 3\left\{2 + \beta_2 - b_2 + b_1 - b_0 + \sum_{i=1}^{s} \frac{1}{\gamma_i}\right\} \leq 3\left\{1 - b_0 + \sum_{i=1}^{s} \frac{1}{\gamma_i}\right\}.$$

Now, recall that $\gamma_i \geq 2$ for all i and $0 \leq s \leq b_0 - 1$. Hence, we obtain

$$0 < 3(1 - b_0 + (b_0 - 1)/2),$$

which is absurd. Thus $b_2 = \beta_2$ proving (b). The above inequality now yields

$$(14) \qquad 0 < P^2 \leq 3\left\{2 + \beta_2 - b_2 + b_1 - b_0 + \sum_{i=1}^{s} \frac{1}{\gamma_i}\right\} \leq 3\left\{2 - b_0 + \sum_{i=1}^{s} \frac{1}{\gamma_i}\right\}.$$

Hence $0 < 3(2-b_0+(b_0-1)/2)$ which impies that $b_0 \leq 2$. This proves (f). Statements (c), (d) and (e) are all straight forward consequences of (b).

Now, we shall prove that (X, Δ) satisfies Fujita's conditions. Assuming on the contrary, we see that there exists a (-1)-curve E as in Theorem 2.2. Let $\Delta_1 = \Delta + E$. Clearly $b_2(\Delta_1) > \beta_2$. Contracting all such curves we still obtain a smooth surface pair $(\overline{X}, \overline{\Delta})$ which satisfies the hypothesis of Question C. Also, $b_2(\overline{\Delta}) > \beta_2(\overline{X})$ contradicting the assertion (b) above. This contradiction proves that (X, Δ) satisfies Fujita's conditions.

This completes the proof of the lemma. □

As an immediate consequence we obtain a result due to R.V. Gurjar and M. Miyanishi (see [3]).

Corollary. *On a logarithmic $\mathbb{Q}$-homology plane V of Kodaira dimension 2, we can have at most one singularity.*

Finally, for $D = \Delta$, by Lemma 2.4(d) and the fact that $b_1 = 0$, it follows that $-\nu > 0$. Since $b_2 = \beta_2$, etc., from (12), we get,

$$0 < -\nu(\Delta) \leq \beta_2'' - 4 - \theta - b_0. \tag{15}$$

Since, the terms on the r.h.s. are integers, it follows that,

$$0 \leq \beta_2'' - 5 - \theta - b_0. \tag{16}$$

In the sequel, we shall denote the connected component of Δ which has exactly one positive eigen value by Δ_∞.

Lemma 3.4. *Suppose (X, Δ) are as in Question* C. *Then*

(a) *X does not contain any simply connected curve E meeting Δ in less than two points.*

(b) *If equality is reached in* (16), *then X does not contain any (-1)-curve E which intersects Δ in at most two points.*

Proof. Put $\Delta_1 = \Delta + E$. Since E intersects Δ in less than two points, each connected component of Δ_1 is simply connected. Hence, Δ_1 satisfies all the conditions of the Question C, in place of Δ. Therefore, by Lemma 3.3, we have, $b_2(\Delta_1) = \beta_2(X) = b_2(\Delta)$, which is absurd. This proves (a).

In (b), it follows that, either $b_0(\Delta_1) = b_0(\Delta)$, $b_1(\Delta_1) = 1$, or $b_0(\Delta_1) = b_0(\Delta) - 1$, $b_1(\Delta_1) = 0$. Of course, $b_2(\Delta_1) = \beta_2 + 1$. Moreover, it is easily seen that $\kappa(X, K + \Delta_1) = 2$, and (X, Δ_1) is log-content. Therefore, we can apply (7) to (X, Δ_1) to obtain:

$$-\nu(\Delta_1) \leq 4\beta_2 - 4 - 3b_2(\Delta_1) + b_1(\Delta_1) - b_0(\Delta_1) - (K.\Delta_1).$$

On the other hand, equality in (16) implies that $\theta = \beta_2'' - 5 - b_0(\Delta)$, and hence

$$(K.\Delta_1) = (K.\Delta) + (K.E) = \theta + m - 1 = \beta_2'' - 5 - b_0(\Delta) + m - 1.$$

Substituting this in the above inequality, we get $-\nu(\Delta_1) \leq 0$. But since Δ_∞ should have at least three tips, the corresponding connected component of Δ_1 should have at least one tip. Hence $-\nu(\Delta_1) > 0$. This contradiction proves the lemma. □

4. Some general remarks in the two cases

In this section, we shall establish some general properties of the configurations of Δ. Most of these can be found in [4] but the proofs given there are not always valid any more in our situation. First we recall some general facts about minimal surface of general type and derive some restrictions on λ.

Theorem 4.1. *Let Y be a minimal surface of general type. Then the following holds:*

(a) $(K_Y)^2 > 0$ *and hence the second betti number* $\beta(Y) \leq 9$.

(b) *For any irreducible curve D on Y, we have* $(K_Y.D) \geq 0$ *and* $(K_Y.D) = 0$ *iff D is a (-2)-curve.*

(c) *All the (-2)-curves on Y can be contracted to finitely many rational double point singularities.*

(d) *The number of (-2)-curves is at most equal to $(\rho(Y) - 1)$, where $\rho(Y)$ denotes the Picard number of Y.*

(e) *There cannot exist three (-2)-curves intersecting (transversally) at a single point.*

(f) *There cannot exist two (-2)-curves intersecting tangentially at a single point.*

A proof of (c) may be found in [10]. Statements (e) and (f) follow easily from (c). For the other results we refer to Section 2, chapter VII of [1].

Lemma 4.1. *If $\mathcal{E} \not\subset \Delta$, then $\lambda \geq 2$.*

Proof. For, amongst the curves contracted by π, there is at least one curve which is not in Δ and hence $b_2(\Delta'') > \beta_2''$. On the other hand, at most $\beta_2'' - 1$ curves on X'' can be (-2)-curves. Hence the lemma. $\square$

Now, we shall assume that X is an elliptic fibre space. So, let $\phi : X \to C$ be an elliptic fibration and $\phi'' : X'' \to C$ be the corresponding minimal elliptic fibration. Clearly $C = P^1$. Since X'' is a minimal elliptic surface, we have, $\beta_2'' = 10$ and hence (16) may be written as

$$-\nu \leq 6 - \theta - b_0. \tag{17}$$

Since r.h.s is an integer and $-\nu$ is positive rational number, we see that

$$\theta \leq 5 - b_0. \tag{18}$$

Recall that an irreducible curve contained in a fibre of ϕ is called a vertical component. Otherwise, it is called an horizontal component. Note that if Δ has no horizontal components, then every component of Δ is in a fibre and hence the intersection form of the adjacency matrix of Δ has no positive eigen value contradicting Lemma 3.3. Hence Δ has at least one horizontal component. i.e., $\lambda \geq 1$. We will need the following result about multiplicity of the multiple fibres, as a consequence of non-rationality of X. The proof is exactly as in [4], and we reproduce it below for the sake of completeness.

Lemma 4.2. *There are exactly two multiple fibres of ϕ'', viz., m_1P_1'' and m_2P_2''. Also $\{m_1, m_2\} = \{2,3\}$ or $\{2,5\}$. If $(K''.H'') \leq 2$ for some horizontal component, then $\{m_1, m_2\} = \{2,3\}$.*

Proof. Let $\{m_iP_i''\}_1^r$ be the multiple fibres of ϕ''. By the simply connectivity of X'' it follows that $r \leq 2$ and m_1 and m_2 are coprime. Since $p_g = q = 0$, we have the canonical bundle formula

$$K'' = \phi''^{-1}(\mathcal{O}_{\boldsymbol{P}^1}(-1)) \otimes [P_1'']^{(m_1-1)} \otimes \cdots [P_r'']^{(m_r-1)}.$$

Thus if $r \leq 1$, it follows that $|nK''| = \emptyset$ for all n, and hence $|nK| = \emptyset$ for all n, contradicting our basic assumption. Hence $r = 2$.

Now, if P'' denotes a general fibre of ϕ'', we have the linear equivalence

$$P'' \sim m_1P_1'' \sim m_2P_2''$$

and hence

$$K'' \sim (m_2 - 1)P_2'' - P_1'' \sim (m_1 - 1)P_1'' - P_2''.$$

Thus K'' is numerically equivalent to the $\mathbb{Q}$-divisor mP'' where $m = (m_1m_2 - m_1 - m_2)/m_1m_2$. Now let H'' be a horizontal component in Δ''. Then $1 \leq (K''.H'') = m(P''.H'')$. Since m_1 and m_2 are coprime it follows that $m_1m_2 - m_1 - m_2$ divides $(K''.H'')$. On the other hand by (18), we have $\lambda \leq 4$ and hence $(K''.H'') \leq 4$. Hence $\{m_1, m_2\} = \{2,3\}$ or $\{2,5\}$, and if $(K''.H'') \leq 2$ for some H'', then $\{m_1, m_2\} = \{2,3\}$, as claimed. □

Now we show that Δ contains at least two horizontal components.

Lemma 4.3. *There are at least two horizontal components in Δ. In particular, $\lambda \geq 2$ and $\{m_1, m_2\} = \{2,3\}$.*

Proof. Assume that there is only one horizontal component in Δ. Let us denote it by H. Let H be an h-fold section for ϕ. Then as observed above we know that h is an integral multiple of m_1m_2. Let S_i, $1 \leq i \leq k$, be the simply connected fibres and T_j, $1 \leq j \leq l$ be the fibres with $b_1(T_j) = 1$. Since Δ is simply connected it follows that no T_j is completely contained in Δ. Hence,

$$b_2(\Delta) \leq \sum_i b_2(S_i) + \sum_j (b_2(T_j) - 1) + 1. \tag{19}$$

On the other hand, by the well known addition formula for euler characteristic, we have

$$(20) \qquad 2 + b_2(\Delta) = e(X) = \sum_i e(S_i) + \sum_j e(T_j) = \sum_i (b_2(S_i) + 1) + \sum_j b_2(T_j).$$

Putting these two together, we get $k + l \leq 3$.

Since irreducible components of Δ are linearly independent, it follows that no two fibres of ϕ are completely contained in Δ. On the other hand, suppose no fibre is completely contained in Δ. Then

$$b_2(\Delta) \leq \sum_i (b_2(S_i) - 1) + \sum_j (b_2(T_j) - 1) + 1 < \rho(X)$$

which is a contradiction. Therefore there is a unique fibre, say S_1, contained in Δ. Then

$$(21) \qquad b_2(\Delta) \leq b_2(S_1) + \sum_{i \geq 2} (b_2(S_i) - 1) + \sum_j (b_2(T_j) - 1) + 1.$$

Combining this with (20) we have; $2k + l \leq 4$. Of course, we have $k \geq 1$, $l \geq 0$.

Now suppose $k = 1$. If $l = 0$ then

$$e(S_1) = e(X) = 2 + b_2(\Delta) = 2 + 1 + b_2(S_1) = 2 + e(S_1)$$

which is absurd. So $l \geq 1$.

The idea of the proof is the following. Let $W := X \setminus \Delta$. We eliminate all possibilities for k and l by considering the restricted fibration $\phi^* : W \to \mathbb{C}$ and using the fact that

$$(22) \qquad e(W) = e(\mathbb{C})e(F_x) + \sum_s (e(F_s) - e(F_x))$$

where F_x is a generic fibre of ϕ^* and F_s a singular fibre of ϕ^*. Also note that

$$e(W) = e(X) - e(\Delta) = 2 + b_2(X) - (b_0(\Delta) + b_2(\Delta)) \leq 1.$$

Denote the closure of $T_i \setminus \Delta$ by C_i. Let $A_i = T_i \cup H = B_i \cup C_i$ where B_i is the union of all components of A_i not in C_i. Then we have $e(A_i) = e(B_i) + e(C_i) - \eta_i$ where η_i is the number of points in $B_i \cap C_i$. Also

$$(23) \qquad b_2(C_i) = 1 + b_2(T_i) - b_2(B_i).$$

Let H intersect T_i in α_i points. Then $b_1(A_i) = \alpha_i$ and hence

$$(24) \qquad e(A_i) = 1 - \alpha_i + b_2(A_i).$$

Since Δ is simply connected and has at most two connected components, we see that B_i is also simply connected and has at most two connected components. Thus $e(B_i) \leq 2 + b_2(B_i)$ and hence

$$(25) \qquad \begin{aligned} e(C_i) - \eta_i &= e(A_i) - e(B_i) \geq 1 - \alpha_i + b_2(A_i) - (2 + b_2(B_i)) \\ &= b_2(C_i) - \alpha_i - 1. \end{aligned}$$

Now suppose $l = 1$. Then

$$e(S_1) + e(T_1) = e(X) = 2 + b_2(\Delta) = 2 + b_2(S_1) + b_2(B_1) = 1 + e(S_1) + b_2(B_1).$$

Using this in (23) we get

$$b_2(C_1) = 1 + b_2(T_1) - (e(T_1) - 1) = 2.$$

Let P_i, $i = 1, 2$ denote the multiple fibres of ϕ. If T_1 is not one of them, then we see that H intersects T_1 in at most h points, i.e., $\alpha_1 \leq h$. Hence by (25), we have $e(C_1) - \eta_1 \geq b_2(C_1) - h - 1 = 1 - h$. Therefore $e(T_1 \setminus \Delta) = e(C_1) - \eta \geq 1 - h$. On the other hand, H intersects P_i in at most h/m_i points and hence $e(P_i \setminus \Delta) = e(P_i \setminus H) \geq -h/m_i$. The set of singular fibres of ϕ^* includes $T_1 \setminus \Delta$ and $P_i \setminus \Delta$, $i = 1, 2$. By (22) we have

$$1 \geq e(W) \geq -h + (1 - h + h) + \left(-\frac{h}{m_1} + h\right) + \left(-\frac{h}{m_2} + h\right) \geq 2,$$

which is absurd.

Now, if T_1 happens to be one of the multiple fibres, say $T_1 = P_1$, then H intersects T_1 in at most h/m_1 points and hence we get $e(C_1) - \eta_1 \geq 1 - h/m_1$. Hence by (22) we have

$$1 \geq e(W) \geq -h + \left(1 - \frac{h}{m_1} + h\right) + \left(-\frac{h}{m_2} + h\right) \geq 2,$$

which is absurd.

Let us now consider the case $k = 1$, $l = 2$. We know that at least one component each from T_1 and T_2 is not contained in Δ. But then we see that (21) is an equality and hence exactly one component of T_i is not contained in Δ. Since Δ has at most two connected components, it is easily seen that at least one of $B_i s$ $(i = 1, 2)$ is connected – say B_1 is connected. Then as in (25) we have $e(C_1) - \eta_1 = 1 - \alpha_1$ and $e(C_2) - \eta_2 \geq -\alpha_2$. Now, again consider the restricted fibration $\phi^* : W \to \mathbb{C}$. Assuming that T_i, $i = 1, 2$ is not a multiple fibre we see that

$$1 \geq e(W) \geq -h + (1 - \alpha_1 + h) + (-\alpha_2 + h) + \left(-\frac{h}{m_1} + h\right) + \left(-\frac{h}{m_2} + h\right)$$

$$\begin{aligned}&\geq -h+(1-h+h)+(-h+h)+\left(-\frac{h}{m_1}+h\right)+\left(\frac{h}{m_2}+h\right)\\&\geq 2\end{aligned}$$

which is absurd. A similar argument when one or both T_i are multiple fibres eliminates this case completely.

Finally, let $k = 2$. Then $l = 0$. Let S_2 be the second simply connected fibre. As above we see that exactly one component of S_2 – say C – is not contained in Δ. As before we define $A = S_2 \cup H = B \cup C$. Clearly $b_1(A) \leq h - 1$. Also, both B and C are simply connected. Thus C intersects B (and hence Δ) in at most h distinct points. Hence $e(C \setminus \Delta) \geq 2 - h$. By (22) we see that

$$1 \geq e(W) \geq -h + (2-h+h) + \left(-\frac{h}{m_1}+h\right) + \left(-\frac{h}{m_2}+h\right) \geq 3$$

which is absurd.

This eliminates the last case also. Therefore, there must be at least two horizontal components in Δ.

By (18) we know that $\lambda \leq 4$. Since there are at least two horizontal components and at least one of them, say H'' in Δ'' has the property $(K''.H'') \leq 2$. But then by Lemma 4.2 we see that $\{m_1, m_2\} = \{2, 3\}$. This completes the proof of the lemma. □

For the rest of this section, X could be either a surface of general type or an elliptic surface. We need to handle the situation of the following lemma quite often.

Lemma 4.4. *Let $D_0 \in R_3$, and let D_1, D_2, D_3 be the components that meet D_0. Then the image of one of D_i, $(i = 1, 2, 3)$ in X'' is a curve such that $\alpha := (K'' \cdot D_i'') > 0$. Let us denote one such component by D_3. Further, assume that D_1, D_2, D_3 are not in R_3 and do not intersect any other component of R_3. Then, the weight set $W = \{(D_0)^2, (D_1)^2, (D_2)^2, (D_3)^2\}$ is one of the following:*

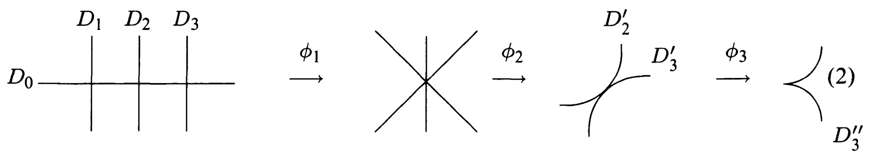
(1) $W = \{-1, (D_1'')^2 - 1, (D_2'')^2 - 1, -\alpha - 3\}$.

(2) $W = \{-1, -2, (D_2'')^2 - 2, -\alpha - 4\}$ *and D_1 is an isolated tip in Δ.*

(3) $W = \{-1, -2, -3, -\alpha - 6\}$ *and D_1 is an isolated tip.*

Proof. Observe that after contracting D_0 the image of the other three curves intersect in the same point. It follows that not all of them will be contracted in X''.

Since D_i do not intersect the rest of R_3 at worst we have a sequence of possible contractions as indicated by the following diagram.



We consider the thre possibilities one by one:

(a) After contracting D_0, none of the other three curves may be contracted. Now D''_i intersect transversally at a single point. Not all these three curves can be (-2)-curves, for otherwise,

(i) if X'' is a surface of general type then as they cannot be contracted to a rational double point contradicting Theorem 4.1.

(ii) if X'' is elliptic, they will form a full fibre F'' and for any horizontal component H'' in Δ it is easy to see that $(F''.H'') \neq 6$ which is a contradiction.

Therefore, it follows that $K''.D''_i > 0$, for some $i = 1, 2, 3$.

(b) It may happen that we can contract one more curve say D_1 also and then the image of D_2 and D_3 cannot be contracted. They meet at a single point tangentially. For exactly the same reason as in (a) both of them cannot be (-2)-curves.

(c) It may happen that we can contract the image of one of D_2 or D_3 and then the remaining curve is a cuspidal curve. Then if X'' is a surface of general type by Theorem 4.1, we see that $(K''.D''_3) > 0$. In elliptic case $(K''.D''_3) > 0$ for the same reason as in (a).

The case (a), it is clear that weight-set is clearly as indicated in (1). In case (b), let us say $(D_1)^2 = -2$. Then $(D_2)^2 < -3$ and D_1 will be a tip. This gives W as in (2). In case (c), we can even contract D_2 also. Hence it follows that $(D_2)^2 = -3$. Also the image of D_3 has a ordinary cusp. Therefore, by the genus formula $(K''.D''_3) + (D''_3)^2 = 0$ and hence $(D''_3)^2 = -\alpha$. It follows that W is as indicated in (3). □

We shall end this section with a typical step towards our goal.

Lemma 4.5. *If there is an equality in* (16), *then* $\lambda < \theta$.

Proof. Clearly, $\lambda \leq \theta$. If possible let $\lambda = \theta$ and let (16) be an equality. Then we see that $\sigma + n_1 = 0$, which by Lemma 3.2, implies that $X = X''$ is a minimal surface.

Suppose that X is a surface of general type. Since there is an equality in (16), we have $\theta = \beta''_2 - 5 - b_0 \leq 4 - b_0 \leq 3$. By (15) we see that $-\nu(\Delta) \leq 1$. Hence if we can show that $\nu < -1$ we obtain a contradiction. In view of Lemma 2.5 we only need to consider the case $\theta = \lambda = 3$. Therefore the weight set for Δ is one of the following:

$$\{-5,-2,\ldots,-2\},\{-4,-3,-2,\ldots,-2\},\{-3,-3,-3,-2,\ldots,-2\}.$$

By (15) we see that $b_0 = 1$ and hence $\Delta = \Delta_\infty$. Then it is easy to see that $\nu = bk(\Delta_\infty) < -1$. Hence X cannot be a surface of general type.

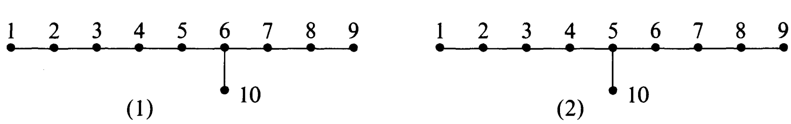
Now, suppose that X is of elliptic type. Equality in (16) implies that $\theta = 5 - b_0 \leq 4$. As above, we only need to consider the case $\theta = \lambda = 4$ and show that $\nu(\Delta) < -1$. Since there are at least two horizontal components in Δ, possible weight set for Δ are the following:

(a) $\{-3,-3,-3,-3,-2,\ldots,-2\}$
(b) $\{-4,-3,-3,-2,\ldots,-2\}$
(c) $\{-5,-3,-2,\ldots,-2\}$ or
(d) $\{-4,-4,-2,\ldots,-2\}$.

By (15) we see that $b_0 = 1$ and hence $\Delta = \Delta_\infty$. Since Δ has at least three tips, in each of these cases we see that in the worst case the weight set of the tips are $\{-3,-3,-3\}$, $\{-4,-3,-3\}$, $\{-5,-3,-2\}$ and $\{-4,-4,-2\}$ respectively. In all cases except in the case of $\{-4,-3,-3,-2,\ldots,-2\}$, we see that $\nu(\Delta) = bk(\Delta_\infty) < -1$. In case the weight set is $\{-4,-3,-3,-2,\ldots,-2\}$ if $\Delta = \Delta_\infty$ has four (or more) tips or if it has a (-2)-tip, we see that $\nu = bk(\Delta_\infty) < -1$. Hence we need to consider only ten vertex trees with exactly three tips and whose weight set of the tips is $\{-4,-3,-3\}$. If every maximal twig has at least two components, then $\nu = bk(\Delta_\infty) < -2/5 - 2/5 - 2/7 < -1$ and hence not possible. Hence at least one of the maximal twigs contain exactly one irreducible component. Such trees arise from partitions of 9 into exactly three parts with at least one of the summands equal to 1. Following are such partitions:

$$\begin{aligned} 9 &= 1+1+7 \\ &= 1+2+6 \\ &= 1+3+5 \\ &= 1+4+4. \end{aligned}$$

Since Δ is free from (-1)-curves, trees corresponding to the partitions $1+1+7$ and $1+2+6$ are negative definite and hence cannot occur. Following are the trees corresponding to the remaining partitions.



We study each of these trees individually and eliminate them.

Tree 1: If $w_1 = -4$, $w_9 = -3$ and $w_{10} = -3$, then $\nu = -5/16 - 3/7 - 1/3 = -361/336 < -1$ and hence this combination cannot occur.
If $w_1 = -3$, $w_9 = -3$ and $w_{10} = -4$, then the tree is negative definite and hence this combination cannot occur.
If $w_1 = -3$, $w_9 = -4$ and $w_{10} = -3$, then $\nu = -5/11 - 3/10 - 1/3 = -359/330 < -1$ and hence this combination cannot occur.

Tree 2: If $w_1 = -4$, $w_9 = -3$ and $w_{10} = -3$, then $\nu = -4/13 - 4/9 - 1/3 = -127/117 < -1$ and hence this combination cannnot occur.
If $w_1 = -3$, $w_9 = -3$ and $w_{10} = -4$, then $\nu = -4/9 - 4/9 - 1/4 = -41/36 < -1$ and hence this combination cannot occur.

This completes the proof of the lemma. □

5. The singular case

We shall complete the proof of the Theorem 1.1, in this section. By the assumption, and from Lemma 3.3, we have now $b_0 = 2$, $s = 1$. From (16), we have,

$$\theta \leq \beta_2'' - 7. \tag{26}$$

We shall make two subsections here to deal with the two major cases.

5.1. The general type case

In this subsection, we shall consider the case when X is a surface of general type. Then the above inequality reduces to $\theta \leq 2$. Also, we know that $\lambda \geq 1$. Clearly $r_i = 0$, $i \geq 4$. While estimating ν we should remember that Δ has two connected components, one corresponding to the resolution of the single quotient singularity.

Suppose $\lambda = 1$. Then by Lemma 4.1, it follows that $\mathcal{E} \subset \Delta$. In particular, $e_1 = 0$. Assume that $\mathcal{E}$ is non empty, i.e., X is not minimal. Then, it is necessary that $r_3 = 1$, and $\sigma + \tau = 0$. Therefore, by Lemma 4.4, π may consist of at most three contractions. Also, since $\lambda = 1$ here, it follows that $\alpha = 1$ and the weight set of Δ will consist of W along with some -2 curves. From this, it is not difficult to see that $-\nu > 1$, which is a contradiction.

On the other hand, if $X = X''$ is minimal, the weight set for the dual graph of Δ consists of one (-3)-curve and all other (-2)-curves. Clearly Δ_∞ consists of at most 8 irreducible curves. If all the components of Δ_∞ are (-2)-curves, by Theorem 4.1 we see that the intersection form of Δ_∞ is negative definite contradicting Lemma 3.3. In view of Lemma 2.4(d) and Lemma 2.3 we see that if Δ_∞ has four (or more) tips, then $\nu(\Delta) < -2$ which is a contradiction. Thus we need to consider the case when Δ_∞ has exactly three tips. If Δ_∞ has 7 (or less) irreducible curves, then the intersection form on Δ_∞ is negative definite which is a contradiction. Thus Δ_∞ contains exactly seven (-2)-curves and one (-3)-curve. Then $\Delta_s := \Delta - \Delta_\infty$

contains exactly one (-2)-curve. Clearly the contribution of Δ_s to $\nu(\Delta)$ is equal to $-2+1/2=-3/2$ and the contribution of Δ_∞ to $\nu(\Delta)$ is at most $2(-1/2)-1/3=-4/3$ and hence $\nu(\Delta)<-2$ which is a contradiction.

The case $\lambda=2$ cannot occur because of Lemma 4.5. This proves that X cannot be of general type.

5.2. The singular elliptic case

Here we have, $\theta\leq 3$ but also $\lambda\geq 2$. As in the general type case, we must remember here that Δ has two connected components and one of them corresponds to the resolution of the quotient singularity.

Here again by Lemma 4.5, the case $\lambda=3$ does not occur. Let now $\lambda=2$. Then $\sigma+\tau+r_3+e_1+u\leq 1$.

Consider the case $r_3=1$. Then $\sigma+\tau+e_1+u=0$ and hence, it follows that, π may be a contraction of at most three curves, as discussed is Lemma 4.4. In view of Lemma 4.4, depending on whether the two horizontal components are adjacent to the (-1)-curve or not we have the following weight set for Δ:

(1) $\{-4,-4,-3,-2,\ldots,-2,-1\}$ or $\{-4,-3,-3,-2,\ldots,-2,-1\}$.

(2) $\{-5,-5,-2,\ldots,-2,-2,-1\}$ or $\{-5,-4,-3,-2,\ldots,-2,-1\}$.

(3) $\{-7,-3,-3,-2,\ldots,-2,-1\}$ or $\{-8,-3,-2,\ldots,-2,-1\}$.

In any of these cases, it is fairly easy to see that $-\nu>1$, which leads to a contradiction.

Therefore, we now consider the case $\lambda=2$, $r_3=0$. Suppose that $e_1=1$. Then the lone component E of $\mathcal{E}_1\setminus\Delta$ has to intersect Δ in two transversal points. This contradicts the Lemma 3.4. Therefore $e_1=0$.

Suppose that $\sigma=1$. Then it follows that $\Delta=\Delta'$, $\mathcal{E}_2=\{E\}$ which has to intersect a component of Δ say D_1 with multiplicity at least two. Also it cannot intersect D_1 with multiplicity ≥ 3 since $\tau=0$. Nor it can intersect any component of Δ with multiplicity ≥ 2. Finally, it cannot intersect any other component at all since $u=0$. Hence, E intersects Δ in one point or in exactly two transversal points, contradicting Lemma 3.4. Therefore $\sigma=0$.

Thus we have shown that when $\lambda=2$, it must be that $e_1=r_3=\sigma=0$. It follows that $\tau=0=u$ and $X=X''$ is minimal. Clearly the weight set is $\{-3,-3,-2,\ldots,-2\}$. If we can show that $-\nu>2$ we arrive at a contradiction to (15).

If the connected component Δ_∞ has four tips then this is easy to see. So, we have to consider only the case when Δ_∞ has precisely three tips. Observe that Δ_∞ cannot be consisting of only (-2)-curves, for then it will be contained inside a fibre and hence cannot have a positive eigen value. Observe that Δ_∞ has at most 9 irreducible components. Since Δ_∞ has at least one (-3)-curve it is easy to see that if it has 7 (or less) irreducible components, then Δ_∞ is negative definite contradicting

Lemma 3.3. Let Δ_∞ have 8 irreducible components. Then $\Delta_s = \Delta - \Delta_\infty$ consists of exactly two irreducible components.

If $\Delta_s = [2,2]$ we have the contribution of Δ_s to $\nu(\Delta)$ equal to $-2+1/3 = -5/3$ and that of Δ_∞ is at most $2(-1/3) - 1/2 = -7/6$ and hence $\nu(\Delta) < -2$. If $\Delta_s = [2,3]$ then we have the contribution of Δ_s to $\nu(\Delta)$ equal to $-4/3 + 1/5 = -17/15$ and that of Δ_∞ is at most $-1/3 + 2(-1/2)$ and hence $\nu(\Delta) < -2$. Now let Δ_∞ have exactly 9 irreducible components. Then $\Delta_s = [2]$ or $[3]$. In either case it is easy to see that $\nu(\Delta) < -2$.

This completes the proof that X cannot be of elliptic type and thereby completes the proof of the Theorem 1.1.

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