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ON RATIONALITY OF LOGARITHMIC $\mathbb{Q}$-HOMOLOGY PLANES-II

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Introduction

This is the second paper in a series of three articles. In the first article, authored jointly by the second and the third author, we showed that a logarithmic $\mathbb{Q}$-homology plane which is non-rational is smooth (See [4]). In this part we will show that there are no $\mathbb{Q}$-homology planes whose projective completion is a surface of general type. As has been observed, due to [3], we need consider only those $\mathbb{Q}$-homology planes with logarithmic Kodaira dimension equal to 2. We shall continue to use notations and results of [4] and refer to it as part I. In fact even the section numbers are continuation of those in part I.

6. Listing of trees

We begin with a smooth, non-rational $\mathbb{Q}$-homology plane V, with $\overline{\kappa}(V) = 2$ and a smooth projective completion X with $\kappa(X) = 2$. Recall that $\Delta = X \setminus V$ is an MNC-divisor, and T is its dual graph. In this section our aim is to give a complete list for T. Subsequently, we eliminate all these possibilities, thus proving that there is no non-rational $\mathbb{Q}$-homology plane as above.

Proposition 1. *The dual graph T of Δ necessarily falls into one of the cases listed in Table* 1. *Entries in the column 'Weight Set' give the weights of vertices of T other than (-2). T_0 stands for the tree mentioned in Lemma* 4.4.

Lemma 6.1. *We have $\theta \leq 2$.*

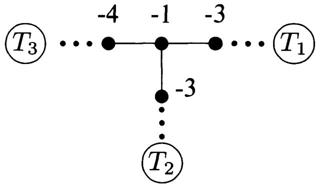
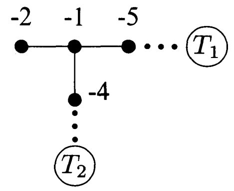
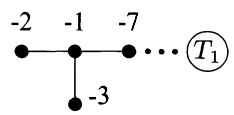
Proof. By (15) we see that $\theta \leq 3$. If possible, let $\theta = 3$. Then again by (15) we see that $\beta_2'' = 9$ and by (12) we have $\nu \geq -1$. Also, by Lemma 4.5 it follows that $\lambda < \theta$.

First consider the case $\lambda = 2$. Then $r_i = 0$ for $i \geq 4$ and

$$\sigma + \tau + e_1 + r_3 + u = 1.$$

[1]Results obtained in sections 6 and 7 are part of C.R. Pradeep's doctoral thesis

Table 1.

No.	β_2	Lemma	Weight Set	Subtree T_0	$(K+\Delta)^2$
1	8	6.4	-3		1
2	9	6.2	-3,-3		1
3	9	6.2	-4		1
4	9	6.4	-3		0
5	10	6.3	-4,-3,-3,-1		1
6	11	6.3	-5,-4,-1		1
7	12	6.3	-7,-3,-1		1

By Lemma 3.4, it follows that $e_1 = 0$. Suppose $\sigma = 1$. Then, X cannot be minimal and since $r_3 = \tau = u = 0$, we see that there is a (-1)-curve E intersecting Δ in at most two points, contradicting Lemma 3.4 again. Therefore $\sigma = 0$. Now, if $\tau = 1$, then $r_3 = u = 0$ and since $\sigma + n_1 = 0$, it follows that $X = X''$ is minimal. This contradicts the assumption $\tau = 1$. Hence $\tau = 0$. Similarly we see that $u = 0$. Hence we are left with the case $r_3 = 1$ and $\sigma + u + \tau = 0$. Let $R_3 = \{D_0\}$. Since $e_1 = 0$, we see that $D_0 \in \Delta$. Let D_1, D_2 and D_3 be the components of Δ adjacent to D_0. Since $u = 0$, it is easily seen that these components do not intersect any other component of $\mathcal{E}$. Then by Lemma 4.4 we see that the weight set of Δ is one of the following.

CASE (a) $\{-5,-3,-3,-2,\dots,-2,-1\}$ or $\{-4,-4,-3,-2,\dots,-2,-1\}$ or $\{-4,-3,-3,-3,-2,\dots,-2,-1\}$

CASE (b) $\{-6,-4,-2,\dots,-2,-1\}$ or $\{-5,-5,-2,\dots,-2,-1\}$ or $\{-5,-4,-3,-2,\dots,-2,-1\}$

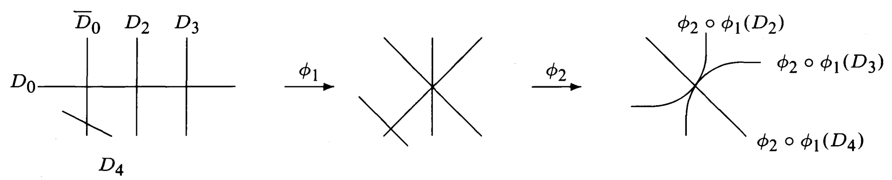
CASE (c) $\{-7,-3,-3,-2,\dots,-2,-1\}$ or $\{-8,-3,-2,\dots,-2,-1\}$.

In each of these cases $\nu < -1$ and hence the case $\lambda = 2$ does not occur.

Now, let $\lambda = 1$. By Lemma 4.1, we have $\mathcal{E} \subset \Delta$ and hence $e_1 = 0$. Also, if $r_4 = 1$ it follows that $r_3 = 0$. Thus $\mathcal{E} = R_4$ consists of exactly one (-1)-curve. It follows that, after blowing down D_0, we reach X''. Since Δ has at least four maximal twigs, it is easily seen that $\nu < -1$, which is a contradiction. Hence $r_4 = 0$. Thus we have,

$$\tau + \sigma + e_1 + u + r_3 = 2.$$

Suppose $r_3 = 2$ and let $R_3 = \{D_0, \overline{D}_0\}$. First consider the case $D_0 \cap \overline{D}_0 = \emptyset$. Let $\{D_0, D_1, D_2, D_3\}$ (resp. $\{\overline{D}_0, \overline{D}_1, \overline{D}_2, \overline{D}_3\}$) be as in Lemma 4.4 with $(D_0)^2 = (\overline{D}_0)^2 = -1\ldots$ etc.. Since the union of (-2)-curves on X'' can be contracted to rational double points, we see that the image of $D_0 + D_1 + D_2 + D_3$ on X'' has one component –say $D_3'' = C''$ with $(K''.C'') > 0$. Similarly the image of $\overline{D}_0 + \overline{D}_1 + \overline{D}_2 + \overline{D}_3$ on X'' has one component – say $\overline{D}_3'' = \overline{C}''$ with $(K''.\overline{C}'') > 0$. But, since $\lambda = 1$, it follows that $D_3'' = C'' = \overline{C}'' = \overline{D}_3''$. But then Δ has at least four maximal twigs and it is not difficult to see that $\nu < -1$ which is a contradiction. Now, we consider the case $D_0 \cap \overline{D}_0 \neq \emptyset$. Let D_2 and D_3 be the other two components of Δ intersecting D_0 and let D_4 be the other component of Δ intersecting $\overline{D}_0$. Then we have the following sequence of contractions:



If the above sequence of contractions is the maximal sequence of contractions, then one of D_2'' or D_3'' is a (-3)-curve – say $(D_3'')^2 = -3$. But then we see that $(D_0)^2 = -1$, $(\overline{D}_0)^2 = -2$, $(D_2)^2 = -4$, $(D_3)^2 = -5$, $(D_4)^2 = -3$ and all the other components of Δ are (-2)-curves. In this case it is easy to see that $\nu < -1$ which is a contradiction. Thus the above sequence is not the maximal sequence of contractions. Since $u = 0$, we see that $\phi_2 \circ \phi_1(D_2)$ or $\phi_2 \circ \phi_1(D_3)$ is not a (-1)-curve. Hence $(\phi_2 \circ \phi_1(D_4))^2 = -1$. But then $\phi_2 \circ \phi_1(D_4)$ does not intersect any other component of $\phi_2 \circ \phi_1(\Delta)$ as $R_3 = \{D_0, \overline{D}_0\}$. Also since $\tau = 0$ we see that $\mathcal{E}_2 = \emptyset$. Thus we reach X'' after contracting $\phi_2 \circ \phi_1(D_4)$. Clearly one of D_2'' or D_3'' is a (-3)-curve – say $(D_3'')^2 = -3$. Then we have $(D_0)^2 = -1$, $(\overline{D}_0)^2 = -2$, $(D_2)^2 = -5$, $(D_3)^2 = -6$, $(D_4)^2 = -2$ and all the other components of Δ are (-2)-curves. Again, it is easily seen that $\nu < -1$ which is a contradiction. Thus the case $r_3 = 2$ cannot occur.

Now, let $r_3 = 1$. Then $\sigma + \tau + u = 1$. If $\sigma = 0$, by Lemma 3.2 we see that Lemma 4.4 is applicable and as seen before it is easily seen that $\nu < -1$ which is a contradiction. Hence $\sigma = 1$. But then by Lemma 3.2, we see that $u = 1$ which is a contradiction. Thus $r_3 \neq 1$.

Finally, if $r_3 = 0$, we see that $\mathcal{E} = \emptyset$ and hence $\sigma + \tau + u = 0$ which contradicts the assumption that $\theta = 3$. This completes the proof of the lemma. $\square$

Lemma 6.2. *If* $\lambda = 2$, *then the possibilities are from* (2) *or* (3) *of Table* 1.

Proof. By (15), we have $\beta_2'' \geq 8$ and by Lemma 6.1 we have $e_1 = \sigma = \tau = r_3 = 0$. By Lemma 3.2 we have, $n_2 \leq n_1 = 0$. But $n_1 = n_2 = 0$ implies that $\beta_2 = \beta_2'' \leq 9$ and hence $\beta_2 = 8$ or $= 9$. In order to complete the proof of the lemma it is enough to show that $\beta_2 \neq 8$. Assume that $\beta_2 = 8$. Then since T should have at least three tips, it follows easily that $\nu < -1$ and that contradicts (12). $\square$

Lemma 6.3. *If* $\lambda = r_3 = 1$, *the possibilities are from* (5), (6) *or* (7) *of Table* 1.

Proof. In this case, since $\theta = 2$, by (15), we have $\beta_2'' = 8$ or $= 9$. Since $\lambda = 1$, by Lemma 4.1, we have $\mathcal{E} \subset \Delta$. Since $r_3 = 1$ and $\sigma = 0$, Lemma 4.4 is applicable. Taking into account the fact that $\lambda = 1$ we can determine the weight set for Δ precisely, in the three cases. In particular, we see that the components $\{D_0, \ldots, D_3\}$ for the subtree as in 5) 6) 7) of Table 1. Thus, in order to complete the lemma, we need to say that $\beta_2'' \neq 8$. Assuming on the contrary, we easily compute $\nu < -1$ and see that this leads to a contradiction of (12). Hence the result. $\square$

Lemma 6.4. *If* $\lambda = 1$, $r_3 = 0$, *then the possibilities are from* (1) *or* (4) *of the list.*

Proof. Since $r_3 = 0$, we have $n_1 = 0$ and hence number of (-1)–curves in S is less than or equal to $n_1 = 0$. Also, since $\lambda = 1$, by Lemma 4.1 we have $\mathcal{E} \setminus \Delta = \emptyset$. Hence $n_1 = n_2 = 0 \Rightarrow \sigma = \tau = 0$. Hence by (15), we see that $\beta_2 = \beta_2'' \geq 7$. If $\beta_2 = 7$ then, as we can see that $\nu < -1$, we arrive at a contradiction to (12). Therefore $\beta_2 = 8$ or $= 9$. Thus we are in the cases as claimed in the lemma. $\square$

This proves Proposition 1.

7. A complete list for Δ

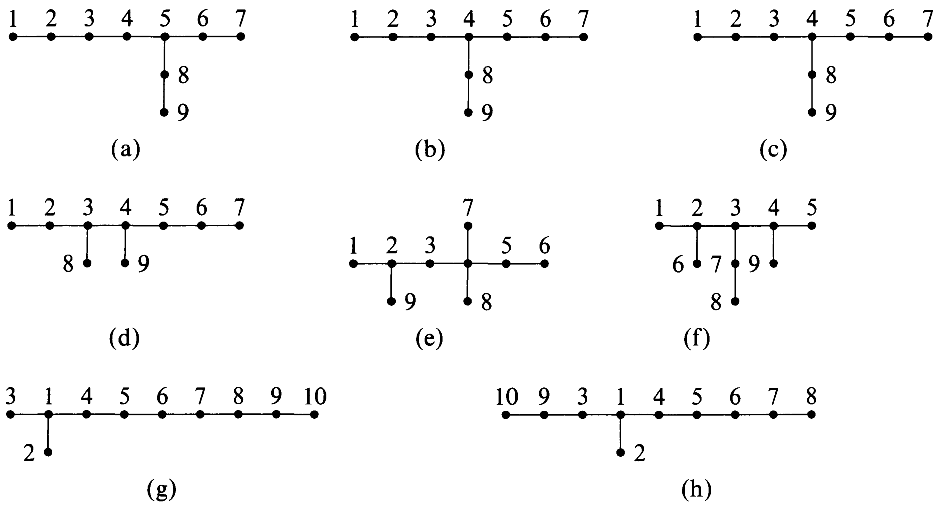
In this section we shall prove:

Proposition 2. *Let V be a $\mathbb{Q}$-homology plane with $\overline{\kappa}(V) = 2$, and let (X, Δ) be a smooth projetive completion of V with Δ a MNC divisor and $\kappa(X) = 2$. Then the dual graph of Δ is necessarily one of the following with weight set as given in Table 2. In the Table 2, the entry under 'Tree label' corresponds to the labelling given below, the entry under the column 'Weight Set' gives the weights of the corresponding vertices with the understanding that all the other vertices have weight equal to -2*

Table 2.

Tree label	Weight set	K
(a)	$w_7 = -4$	$(1/2)(3,6,9,12,15,8,1,10,5)$
(b)	$w_1 = w_9 = -3$	$(1/3)(1,6,11,16,12,8,4,9,2)$
(c)	$w_2 = -3$	$(1/2)(1,2,7,12,9,6,3,8,4)$
(d)	$w_5 = -3$	$(1/3)(4,8,12,10,3,2,1,6,5)$
(e)	$w_3 = -3$	$(1/2)(1,2,2,6,4,2,3,3,1)$
(f)	$w_2 = -3$	$(1/2)(1,2,6,6,3,1,4,2,3)$
(g)	$w_1 = -1, w_2 = -3, w_3 = -4, w_4 = -3$	$(1/3)(18,5,3,7,6,5,4,3,2,1)$
(h)	$w_1 = -1, w_2 = -3, w_3 = -3, w_4 = -4$	$(1/5)(26,7,9,5,4,3,2,1,6,3)$

and the entry under the column K gives the vector $\mathbf{x} = (x_i)$ where $K \sim \sum x_i D_i$.



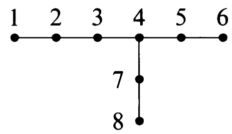
To begin with, in the following three lemmas, we will get rid of the cases (1), (6) and (7) of Table 1.

Lemma 7.1. *Case* (1) *of Table* 1 *cannot occur.*

Proof. Note that in this case we have $\theta = 1$ and $\beta_2 = 8$. Looking at the weight set, we see that if there are five (or more) maximal twigs in a tree, then at least four of the tips must be (-2)-tips. By Lemma 2.3, we conclude that $\nu < -2$, contradicting (12). Thus, the trees arising in this case must have at most four maximal twigs. First we consider the trees with exactly three maximal twigs. These arise from the partition of 7 into exactly three parts:

$$
\begin{aligned}
7 &= 1+1+5 \\
&= 1+2+4 \\
&= 1+3+3 \\
&= 2+2+3
\end{aligned}
$$

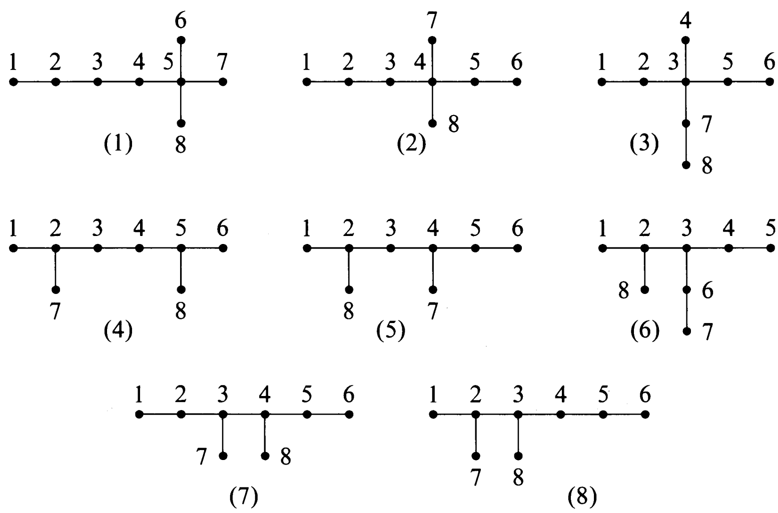
Clearly trees corresponding to partitions 1+1+5, 1+2+4 and 1+3+3 have negative definite intersection form and hence not possible. Hence we need to consider the following tree only:



By Theorem 4.1(c) we know that all the (-2)-curves on X can be contracted to finitely many rational double points. Hence $w_1 \neq -3$. If one of w_2, w_3, w_4 or w_5 is a (-3)-curve, then it is easy to see that the intersection form is negative definite which is a contradiction. If $w_6 = -3$, we solve for an expression for canonical divisor of X as follows. Since Δ generates $\mathrm{Pic}(X) \otimes \mathbb{Q}$, we have $K \sim \sum_{i=1}^{8} x_i D_i$ for some rational numbers x_i. Then

$$
(K.D_j) = \begin{cases} 0 & j \neq 6 \\ 1 & j = 6 \end{cases}
$$

We now solve for x_i to obtain $K \sim 3D_1 + 6D_2 + 9D_3 + 12D_4 + 7D_5 + 2D_6 + 8D_7 + 4D_8$. But then $p_g(X) \neq 0$ which is absurd. Hence we need to consider trees with exactly four maximal twigs. Following are all the eight-vertex trees with exactly four maximal twigs (see [1]).



Now, let us consider each of these trees individually. Consider Tree (4) first. If all the curves were (-2)-curves, then it is easily seen that this tree becomes negative semidefinite. Hence, with one of the curve being a (-3)-curve it is negative definite. So this tree is excluded.

In all of the other trees, in view of Lemma 2.4, if all the tips were (-2)-curves then $\nu < -2$, and hence we get a contradiction to (12). So we can assume that one of the tips is the (-3)-curve.

Tree 1: By Theorem 4.1, it follows that w_6, w_7 or w_8 is the (-3)-curve. In any case we see that $\nu = bk(T) = -(1/2) - (1/2) - (1/3) - (4/5) = -(32/15) < -2$ which contradicts (12). Hence Tree (1) cannot occur.

Tree 2: Again by Theorem 4.1 it follows that w_7 or w_8 is the (-3)-curve. But then, $\nu = bk(T) = -(1/2) - (1/3) - (2/3) - (3/4) < -2$ which contradicts (12). Hence tree (2) cannot occur.

Tree 3: This tree is imposible by Theorem 4.1 no matter which tip is the (-3)-curve.

Tree 5: By Theorem 4.1, we see that either w_1, w_8 or w_7 is the (-3)-curve. (The first two cases are similar.) In each case, we solve for an expression for canonical divisor of X as before. Since Δ generates $\mathrm{Pic}(X) \otimes \mathbb{Q}$, we have $K \sim \sum_{i=1}^{8} x_i D_i$ for some $x_i \in \mathbb{Q}$. For definiteness, say $w_1 = -3$, then

$$(K.D_j) = \begin{cases} 0 & j \neq 1 \\ 1 & j = 1. \end{cases}$$

We now solve for x_i and obtain:

$$K \sim D_1 + 4D_2 + 5D_3 + 6D_4 + 4D_5 + 2D_6 + 3D_7 + 2D_8.$$

Since K is an effective $\mathbb{Z}$-divisor $p_g(X) \neq 0$. This is absurd. Hence w_1, $w_8 \neq -3$. If $w_7 = -3$, we see that the adjacency matrix has determinant equal to zero, which is absurd. Hence Tree (5) cannot occur.

Tree 6: This tree is impossible because of Theorem 4.1, no matter which tip is the (-3)-curve. Hence Tree (6) cannot occur.

Tree 7: Here, we see that $w_7 = -3$. But then, $\nu = bk(T) = -(1/2) - (1/3) - (2/3) - (2/3) < -2$, contradicting (12). Hence Tree (7) cannot occur.

Tree 8: Here we see that w_1 or w_7 or $w_8 = -3$. In any case $\nu = bk(T) = -(1/2) - (1/2) - (1/3) - (3/4) < -2$ contradicting (12). Hence Tree (8) cannot occur.

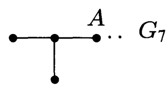
Hence Case 1 of Table 1 cannot occur. □

Lemma 7.2. *Case* 6 *of Table* 1 *cannot occur.*

Proof. Note that in this case $\theta = 2$. First we narrow down the possibilities of eleven-vertex trees which might arise in this case using bark considerations. Next, as in the previous lemma, we study the individual trees to prove the lemma.

Looking at the weight set, we see that if there are five (or more) maximal twigs in a tree, at least four of the tips must be (-2)-tips. Then by Lemma 2.3 we see that $\nu = bk(\Delta) < -2$ contradicting (12). Thus we need to consider trees with at most four maximal twigs. Trees with exactly three maximal twigs are easy to construct and they are precisely the trees numbered 1, 2, 3 and 4 in the list given below.

Now consider those trees with exactly four maximal twigs. If all the four tips are (-2)-curves then it is easily seen that $\nu = bk(\Delta) < -2$. So, there is no need to consider this case. But then either the (-4)-curve or the (-5)-curve has to be a tip and hence T_2 or T_1 is empty. We shall denote the non empty one by G_7 (in order to remind us that it has exactly seven vertices) and denote the vertex to which it is attached to T_0 by A.

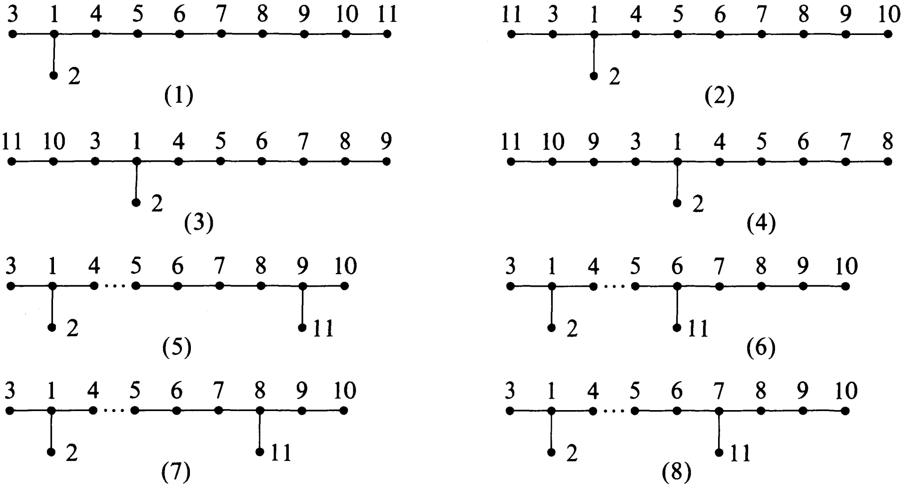


Observe that G_7 cannot have more than two connected components, as otherwise T would have more than four maximal twigs. Also we see that T cannot have two $[2,2]$ twigs, as otherwise $\nu = bk(\Delta) \leq -(1/2) - (1/5) - (2/3) - (2/3) < -2$ contradicting (12). First we consider the case when G_7 has two connected components, Γ_1, Γ_2 say. Clearly both these components must be linear and when attached, the vertex of Γ_i adjacent to A must be a tip of Γ_i. Possible partitions of the 7 vertices are $1+6$, $2+5$ and $3+4$. In view of the fact that T cannot have two $[2,2]$ twigs, the partitions $2+5$ and $3+4$ are ruled out. The partition $1+6$ yields two possibilities for the set of maximal twigs of T, viz.,

$$\{[2],[2],[5],[2,2,2,2,2,2]\}\{[2],[2],[4],[2,2,2,2,2,2]\}.$$

In either of these cases, clearly $\nu = bk(\Delta) < -2$ contradicting (12). Hence G_7 has to be connected.

Suppose G_7 is linear. Then one of the non-tips is adjacent to A. Again, since two $[2,2]$ twigs are forbidden, we only get $[2]$, $[2,2,2,2,2]$ as twigs from G_7. But then $bk(\Delta) \leq -(1/2) - (1/5) - (1/2) - (5/6) < -2$. So, we can assume that G_7 is not linear. Clearly, G_7 cannot have more than three twigs, and hence, has precisely three twigs. This amounts to considering partitions of 6 into exactly three parts. Possibilities are $1+1+4$, $1+2+3$ and $2+2+2$. Also we easily see that the vertex adjacent to vertex A - after attaching G_7 - must be a tip of G_7. Since T cannot have two $[2,2]$ twigs, the tree corresponding to partition $2+2+2$ cannot arise. Also for the same reason, in case of partition $1+2+3$, there are only two possible attachings. In the list given below tree numbers 7 and 8 correspond to these cases. In case of partition $1+1+4$, there are two possible attachings. Trees numbered 5 and 6 in the list below correspond to these cases.



Now, as in the previous lemma we consider each of these trees individually. In all these trees we have $w_1 = -1$ and $w_j = -2$ for all $j \neq 3$, 4.

Tree 1: If $w_3 = -4$, $w_4 = -5$, we have $K \sim -30D_1 - 15D_2 - 8D_3 - 8D_4 - 7D_5 - 6D_6 - 5D_7 - 4D_8 - 3D_9 - 2D_{10} - D_{11}$ and hence $-K$ is an effective $\mathbb{Z}$-divisor. But then $|nK| = \emptyset$, a contradiction and hence this case cannot occur. If $w_3 = -5$, $w_4 = -4$ we see that $K \sim 12D_1 + 6D_2 + (9/5)D_3 + (16/5)D_4 + (14/5)D_5 + (12/5)D_6 + 2D_7 + (8/5)D_8 + (6/5)D_9 + (4/5)D_{10} + (2/5)D_{11}$ and hence $K^2 = -1/5$, which is absurd. Hence Tree (1) cannot occur.

Tree 2: If $w_3 = -4$, $w_4 = -5$, we have $K \sim (120/11)D_1 + (60/11)D_2 + (28/11)D_3 + (21/11)D_4 + (18/11)D_5 + (15/11)D_6 + (12/11)D_7 + (9/11)D_8 + (6/11)D_9 + (3/11)D_{10} + (14/11)D_{11}$ and hence $K^2 = -(1/11)$ which is absurd and hence this case cannot occur. Let $w_3 = -5$, $w_4 = -4$. Let $B := D - D_1$. We note that by Theorem 11.4 of [2], we have $\overline{\kappa}(X \setminus B) = 2$. We apply Kobayashi's inequality to the pair (X, B). We reach the log-canonical model (X_c, B_c) of (X, B) by contracting all the three connected components of B (Since $\beta_2(X_c) = 1$, we easily see that we indeed reach the log-canonical model of (X, B) after contracting B). Thus $B_c = \emptyset$. But then, the three isolated singularities on X_c are log-terminal singularities of the pair $(X_c, \emptyset)$ and hence $(X_c, \emptyset)$ is free from LCS-singularities. It is easy to see that the orders of the local fundamental groups of these singularities are 2, 9 and 22. Also, we see that $K \sim (15/2)D_1 + (15/4)D_2 + D_3 + (7/4)D_4 + (3/2)D_5 + (5/4)D_6 + D_7 + (3/4)D_8 + (1/2)D_9 + (1/4)D_{10} + (1/2)D_{11}$ which implies that $K_c = (15/2)D_1''$. By the process of diagonalization, we see that $(D_1'')^2 = (4/99)$. Hence $(K_c)^2 = (225/99)$. Thus, the Kobayashi's inequality (3) applied to the pair (X, B) yields

$$(K_c + B_c)^2 = \frac{225}{99} \leq 3\left\{3 - 3 + \frac{1}{2} + \frac{1}{9} + \frac{1}{22}\right\} = \frac{195}{99}$$

which is absurd. Hence this tree cannot occur.

Tree 3: If $w_3 = -4$, $w_4 = -5$, we have $K \sim 8D_1 + 4D_2 + (9/5)D_3 + (6/5)D_4 + D_5 + (4/5)D_6 + (3/5)D_7 + (2/5)D_8 + (1/5)D_9 + (6/5)D_{10} + (3/5)D_{11}$ and hence $K^2 = -(4/5)$ which is absurd and hence this case cannot occur. Similarly $w_3 = -5$, $w_4 = -4$ implies that $K \sim (160/23)D_1 + (80/23)D_2 + (21/23)D_3 + (36/23)D_4 + (30/23)D_5 + (24/23)D_6 + (18/23)D_7 + (12/23)D_8 + (6/23)D_9 + (14/23)D_{10} + (7/23)D_{11}$ and $K^2 = -(25/23)$, which is absurd. Hence Tree (3) cannot occur.

Tree 4: If $w_3 = -4$, $w_4 = -5$, we see that $\nu = -(1135/546) < -2$ contradicting (12). Thus this tree cannot occur. If $w_3 = -5$, $w_4 = -4$, we have $K \sim (90/13)D_1 + (45/13)D_2 + (12/13)D_3 + (20/13)D_4 + (16/13)D_5 + (12/13)D_6 + (8/13)D_7 + (4/13)D_8 + (9/13)D_9 + (6/13)D_{10} + (3/13)D_{11}$ and hence $K^2 = -(14/13)$ which is absurd. Thus this case cannot occur. Hence Tree (4) cannot occur.

Tree 5: If $w_3 = -4$, $w_4 = -5$, we see that the adjacency matrix is not invertible and hence this case cannot occur. If $w_3 = -5$, $w_4 = -4$ then $K \sim 8D_1 + 4D_2 + D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9 + D_{10} + D_{11}$. Since K is an effective $\mathbb{Z}$-divisor, $p_g(X) \neq 0$, which is absurd. Hence Tree (5) cannot occur.

Tree 6: If $w_3 = -4$, $w_4 = -5$, we see that $\nu = -(1/2) - (1/2) - (1/4) - (4/5) < -2$ contradicting (12). If $w_3 = -5$, $w_4 = -4$ we see that $K \sim (44/13)D_1 + (22/13)D_2 + (1/13)D_3 + (8/13)D_4 + (14/13)D_5 + (20/13)D_6 + (16/13)D_7 + (12/13)D_8 + (8/13)D_9 + (4/13)D_{10} + (10/13)D_{11}$ and $K^2 = -(25/13)$ which is absurd. Hence Tree (6) cannot occur.

Tree 7: If $w_3 = -4$, $w_4 = -5$, we have $K \sim 10D_1 + 5D_2 + 2D_3 + 2D_4 + 3D_5 + 4D_6 + 5D_7 + 6D_8 + 4D_9 + 2D_{10} + 3D_{11}$. As above, this case also cannot occur. If $w_3 = -5$, $w_4 = -4$ we see that $K \sim 4D_1 + 2D_2 + (1/5)D_3 + (4/5)D_4 + (6/5)D_5 + (8/5)D_6 + 2D_7 + (12/5)D_8 + (8/5)D_9 + (4/5)D_{10} + (6/5)D_{11}$ and $K^2 = -(9/5)$ which is absurd. Hence Tree (7) cannot occur.

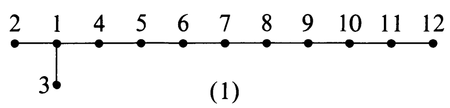
Tree 8: If $w_3 = -4$, $w_4 = -5$, we have $K \sim 6D_1 + 3D_2 + D_3 + D_4 + 2D_5 + 3D_6 + 4D_7 + 3D_8 + 2D_9 + D_{10} + 2D_{11}$. Since K is an effective $\mathbb{Z}$-divisor, $p_g(X) \neq 0$, a contradiction. If $w_3 = -5$, $w_4 = -4$, then on the smooth minimal model X'', the configuration of (-2)-curves do not form a negative definite system and hence this case cannot occur. Hence Tree (8) cannot occur. Thus we have proved the lemma. □

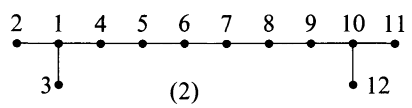
Lemma 7.3. *Case* 7 *of Table* 1 *cannot occur.*

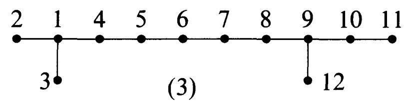
Proof. Note that in this case $\theta = 2$. Here too the strategy is to narrow down the possibilities for the trees using bark considerations first and then study each of the remaining possibilities.

Five (or more) maximal twigs imply that $\nu = N^2 \leq -(1/3) + 4(-1/2) < -2$ contradicting (15). Thus, we need to consider trees with at most four maximal twigs. All the trees arising in this case are got by attaching a eight-vertex tree $G_8 = (T_1)$, at the (-7)-curve to T_0. There is only one tree with exactly three twigs. It is the tree number 1.

Next we observe that in case of four twigs, we cannot have two $[2,2]$ twigs. Also, it cannot have a $[2,2,2]$ twig. This consideration alone gives us that G_8 has to be connected and non linear. Also, G_8 has to have exactly three twigs and the vertex adjcent to the (-7)-vertex - after attaching G_8 - has to be a tip of G_8. Such G_8 arise from partitions of 7 into exactly three parts. Possible partitions are $1+1+5$, $1+2+4$, $1+3+3$ and $2+2+3$. In view of above observations we easily see that following trees numbered 2 and 3 are all the possible attachings.







As above we study each tree individually. In all the trees we have $w_1 = -1$, $w_3 = -3$, $w_4 = -7$ and $w_j = -2$ for all $j \neq 1$, 3, 4.

Tree 1: We have $K \sim -50D_1 - 25D_2 - 17D_3 - 9D_4 - 8D_5 - 7D_6 - 6D_7 - 5D_8 - 4D_9 - 3D_{10} - 2D_{11} - D_{12}$ and hence $K^2 = -12$. But $K^2 = 10 - \beta_2 = -2$. Thus this case cannot occur.

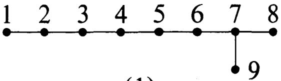
Tree 2: We see that the adjacency matrix is not invertible and hence this case cannot occur.

Tree 3: We have $K \sim 10D_1 + 5D_2 + 3D_3 + D_4 + 2D_5 + 3D_6 + 4D_7 + 5D_8 + 6D_9 + 4D_{10} + 2D_{11} + 3D_{12}$. Since K is an effective $\mathbb{Z}$-divisor, $p_g(X) \neq 0$ which is absurd. Hence this case cannot occur. Thus we have proved the lemma. $\square$

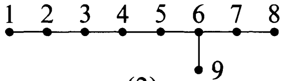
Now we are left with Cases 2, 3, 4 and 5 of the table (and the trees with three maximal twigs in Case 1). In Case 2, 3 and 4 we need to consider trees with nine vertices. In Case 5, if the tree has five (or more) twigs, we see that at least two of them are (-3) and (-4) tips and (at least) three are (-2) tips. But then, by Lemma 2.3, we have $\nu = bk(\Delta) < -2$ contradicting (12). Hence trees arising in Case 5 can have at most four twigs. Below, we have included the list of all non linear trees with 9 vertices (see List 1), and a list of all non linear trees with 10 vertices and having at most four maximal twigs (see List 2).We have made use of the list of all trees with at most ten vertices given in [1]. Also, in Case 5, the fact that the (-1)-curve intersects exactly three other curves is used to eliminate some trees with 10 vertices.

As above, we compute the canonical divisor for surfaces with these trees as dual graph of divisor at infinity. Consider the following adjacency matrix $A = (a_{ij})$ of a tree T.

LIST 1: TREES WITH NINE VERTICES



(1)



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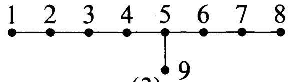

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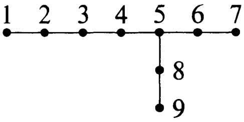

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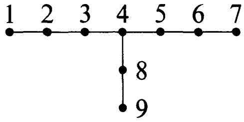

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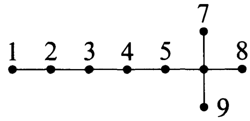

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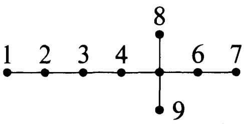

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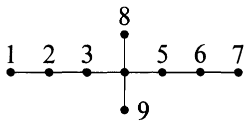
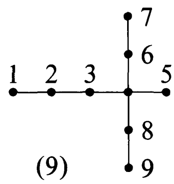


(8)



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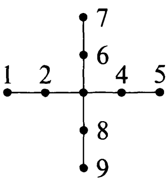
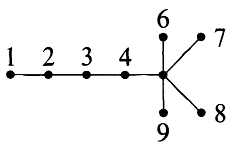


(10)



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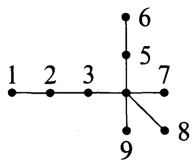

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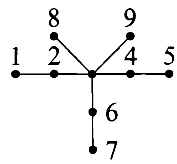

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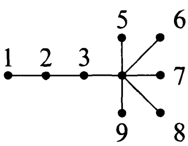

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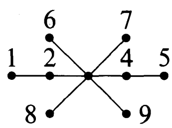

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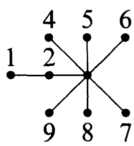

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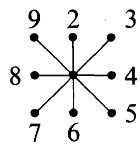

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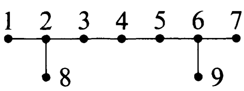

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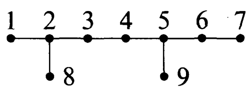

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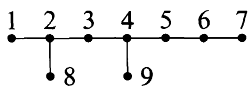

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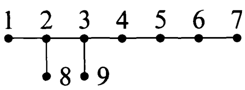

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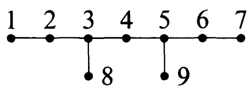

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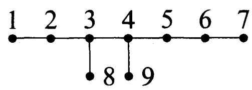

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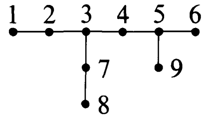

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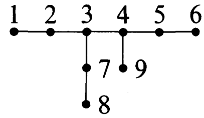

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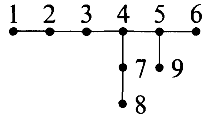

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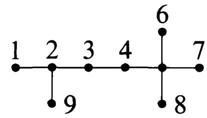

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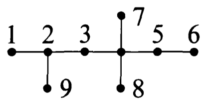

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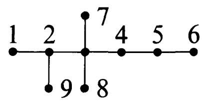

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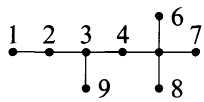

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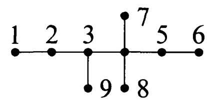

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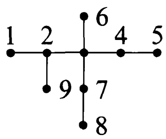

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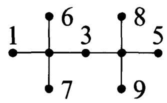

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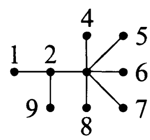

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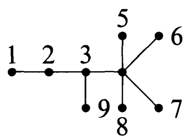

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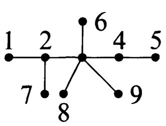

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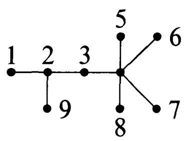

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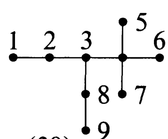

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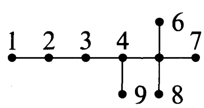

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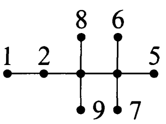

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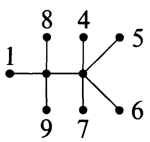

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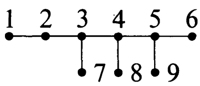

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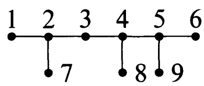

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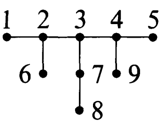

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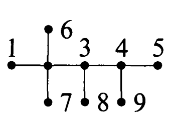

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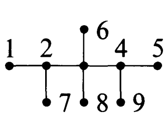
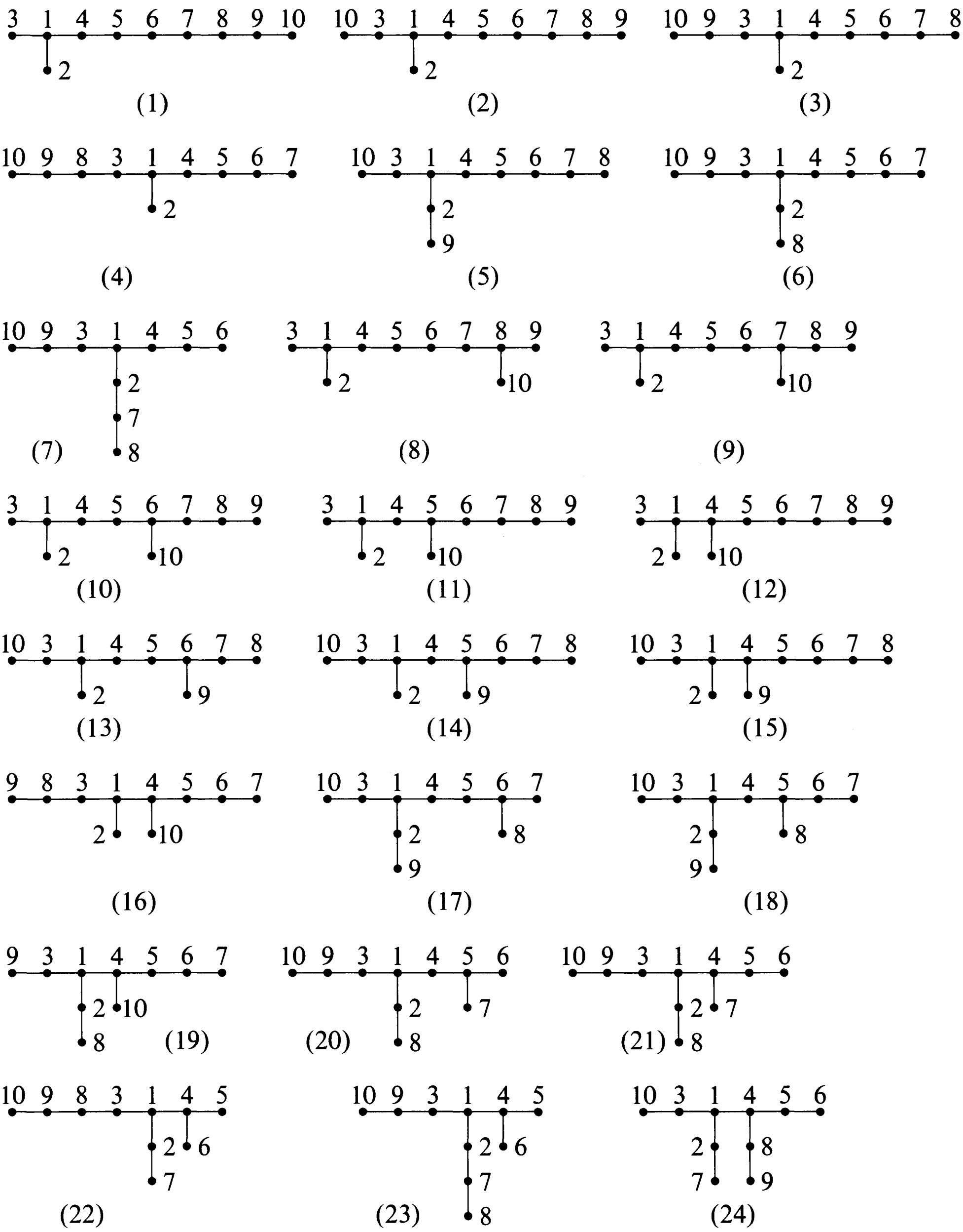


(46)

LIST 2: TREES WITH TEN VERTICES



$$a_{ij} = \begin{cases} 1 & \text{if vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } i \neq j \text{ and } v_i \text{ and } v_j \text{ are not adjacent} \\ w_i & \text{if } i = j \end{cases}$$

where w_i is the self intersection number of D_i, the curve corresponding to the vertex v_i. Since Δ generates $\mathrm{Pic}X \otimes \mathbb{Q}$ for some rational numbers x_i, we have, $K \sim \sum_{i=1}^{\beta_2} x_i D_i$. Set $\mathbf{x} = (x_1, \ldots, x_{\beta_2})^t$, be the column vector. Then we have,

$$((K.D_1), \ldots, (K.D_{\beta_2}))^t = A\mathbf{x}.$$

Hence, to find K, we need to solve the above equation for $\mathbf{x}$. Then,

$$K^2 = \sum x_i^2 D_i^2 + 2\left(\sum_{i<j} x_i x_j D_i D_j\right).$$

By Nöther's formula we know that $K^2 = 10 - \beta_2$. Using the *Mathematica* program *kay.m*, we compute K for each of these trees with all possible combinations for weights, with all allowable weight sets. If K turns out to be an effective $\mathbb{Z}$-divisor, the program rejects the case immediately, and goes to the next case. Otherwise, the program computes K^2. If this value is not equal to $10 - \beta_2$, then again such a case is rejected. The remaining cases have been collected in the following Tables 3a and 3b.

In these tables, entry in the column named 'Tree No.' refers to the numbering of trees in the list given below. As in Table 1, we give only weights other than -2, in the column named 'Weight Set'. Also in Table 2b, $\mathbf{w}$ stands for (w_1, w_2, w_3, w_4). Entry under the column K gives the vector $\mathbf{x} = (x_i)$, where $K \sim \sum_{i=1}^{\beta_2} x_i D_i$. The last column gives the bark.

It is easily seen that the only trees for which $bk(\Delta)$ satisfies (7) are those included in the statement of Proposition 2. This completes the proof of Proposition 2.

8. Completion of the proof

In this section we shall finally eliminate each of the eight trees listed in Proposition 2.

Tree (a): We have the relation $2K \sim 3D_1 + 6D_2 + 9D_3 + 12D_4 + 15D_5 + 8D_6 + D_7 + 10D_8 + 5D_9$, which gives a double cover of X ramified precisely over D_1, D_3, D_5, D_7 and D_9. Let $g : X \to \underline{X}$ be the contraction of these curves, $g(D_i) = y_i$, $i = 1, 3, 5, 7, 9$. We then have a double covering $h : Y \to \underline{X}$ ramified precisely over y_i. We observe that $\pi_1^{loc}(y_i) = \mathbb{Z}_2$ for $i = 1, 3, 5, 9$ and $\pi_1^{loc}(y_7) = \mathbb{Z}_4$. Since $h : Y \to \underline{X}$ is a double cover, we deduce that $h^{-1}(y_i)$ are smooth points except for $i = 7$ and $h^{-1}(y_7)$ is a singularity of type A_1. Let $\tilde{X} \to Y$ be the minimal resolution and f be the composite of $\tilde{X} \to Y \to \underline{X}$. Write $g(D_j) = \underline{D_j}$ and let $\tilde{D}_j$

Table 3a. (for nine-vertex trees)

Tree No.	Weight Set	K	$bk(\Delta)$
4	$w_7 = -4$	$(1/2)(3, 6, 9, 12, 15, 8, 1, 10, 5)$	$-184/105$
5	$w_1 = w_9 = -3$	$(1/3)(1, 6, 11, 16, 12, 8, 4, 9, 2)$	$-221/140$
5	$w_2 = -3$	$(1/2)(1, 2, 7, 12, 9, 6, 3, 8, 4$	$-49/24$
17	$w_1 = -3$	$(1/2)(1, 1, 1, 1, 1, 1, 1, 1, 1)$	-4
23	$w_5 = -3$	$(1/3)(4, 8, 12, 10, 3, 2, 1, 6, 5)$	$-31/14$
25	$w_1 = w_7 = -3$	$(1/5)(1, 8, 15, 18, 12, 6, 4, 2, 9)$	$-13/6$
25	$w_2 = -4$	$(1/4)(1, 2, 15, 18, 12, 6, 10, 5, 9)$	$-101/42$
26	$w_2 = w_6 = -3$	$(1/5)(1, 2, 10, 18, 14, 3, 12, 6, 7)$	$-17/8$
28	$w_3 = -3$	$(1/2)(1, 2, 2, 6, 4, 2, 3, 3, 1)$	$-8/3$
29	$w_2 = w_6 = -3$	$(1/2)(1, 2, 6, 4, 2, 0, 3, 3, 1)$	$-17/7$
40	$w_4 = w_5 = -3$	$(1/2)(2, 4, 6, 2, 0, 1, 1, 3, 3)$	-3
41	$w_2 = w_4 = -3$	$(1/3)(1, 2, 6, 1, 3, 3, 3, 1, 1)$	$-10/3$
41	$w_3 = -3$	$(1/2)(2, 4, 2, 1, 1, 1, 1, 2, 2)$	$-7/2$
44	$w_1 = w_7 = -3$	$(1/3)(1, 6, 8, 8, 4, 3, 2, 1, 4)$	$-73/30$
44	$w_2 = -3$	$(1/2)(1, 2, 6, 6, 3, 1, 4, 2, 3)$	$-8/3$
45	$w_1 = w_4 = -3$	$(1/4)(2, 10, 8, 2, 1, 5, 5, 4, 1)$	$-17/6$

be the proper transform of $\underline{D_j}$ in $\tilde{X}$ for $j = 2, 4, 6, 8$. Also, let $E = f^{-1}(y_7)$. Then $\tilde{D}_j$ (and E) are rational curves and $\cup\tilde{D}_j \cup E$ supports an ample divisor. Hence the images of these irreducible components in the Albanese of $\tilde{X}$ generate the Albanese torus of $\tilde{X}$. Thus Albanese of $\tilde{X}$ is trivial and hence $q(\tilde{X}) = 0$.

Now we have $g^\star(\underline{D_2}) = (1/2)(D_1+2D_2+D_3)$, $g^\star(\underline{D_4}) = (1/2)(D_3+2D_4+D_5)$, $g^\star(\underline{D_6}) = (1/4)(2D_5 + 4D_6 + D_7)$ and $g^\star(\underline{D_8}) = (1/2)(D_5 + 2D_8 + D_9)$. From this

Table 3b. (for ten-vertex trees)

Tree No.	Weight Set	K	$bk(\Delta)$
1	$\mathbf{w} = (-1,-3,-4,-3)$	$(1/3)(18,5,3,7,6,5,4,3,2,1)$	$-87/60$
3	$\mathbf{w} = (-1,-3,-3,-4)$	$(1/5)(26,7,9,5,4,3,2,1,6,3)$	$-625/336$
13	$\mathbf{w} = (-1,-3,-3,-4)$	$(1/4)(19,5,6,4,5,6,4,2,3,3)$	$-21/10$

it follows that $(\underline{D_2})^2 = (\underline{D_4})^2 = (\underline{D_8})^2 = -1$, $(\underline{D_6})^2 = -5/4$, $(\underline{D_2}.\underline{D_4}) = (\underline{D_4}.\underline{D_6}) = (\underline{D_4}.\underline{D_8}) = (\underline{D_6}.\underline{D_8}) = 1/2$. Going to a double cover and a resolution we can now see that $(\tilde{D_2})^2 = (\tilde{D_4})^2 = (\tilde{D_7})^2 = (\tilde{D_8})^2 = -2$, $(\tilde{D_6})^2 = -3$, $(\tilde{D_2}.\tilde{D_4}) = (\tilde{D_4}.\tilde{D_6}) = (\tilde{D_4}.\tilde{D_8}) = (\tilde{D_6}.\tilde{D_8}) = 1$. Similar consideration yields the expression $\tilde{K} \sim 3\tilde{D}_2 + 6\tilde{D}_4 + 4\tilde{D}_6 + 5\tilde{D}_8 + 2E$. In particular $p_g(\tilde{X}) > 0$ and $(\tilde{K})^2 = 4$. On the other hand, $1 = e(V) = e(X \setminus \Delta) = e(\underline{X} \setminus g(\Delta))$. Hence $e(\tilde{X} \setminus f^{-1}(g(\Delta))) = 2$. Therefore $e(\tilde{X}) = 2 + e(f^{-1}(g(\Delta))) = 2 + 6 = 8$. Hence $\chi(\tilde{X}) = (1/12)((\tilde{K})^2 + e(\tilde{X})) = 1 = 1 + p_g - q > 1$, a contradiction.

In each of the other cases, the proof is similar (in fact it is even simpler in the sense that, there is no need to resolve any singularity). We have $nK = \sum_i x_i D_i$, which defines a n-fold cover of X for which the ramification curves are precisely those D_is for which $x_i \not\equiv 0 (\mathrm{mod}\ n)$. Observe that $n = 2$, 3 or 5. We contract these ramification curves to some normal singularities on a normal variety $g : X \to \underline{X}$. We have listed below, $g^\star(\underline{D_j})$ where $\underline{D_j}$ are the irreducible components of $g(\Delta)$ and the (non-zero) intersection numbers $(\underline{D_j}.\underline{D_k})$. Consider the n-fold cover $f : \tilde{X} \to \underline{X}$. Observe that in each of these cases $\tilde{X}$ is already smooth. The irreducible components $\tilde{D_j}$ of $f^{-1}(g(\Delta))$ are all simply connected rational curves. The intersection numbers $(\tilde{D_j}.\tilde{D_k})$ are computed. Likewise expression for $\tilde{K}$ is also obtained. As before it is then easy to see that $p_g(\tilde{X}) > 0$ and $q(\tilde{X}) = 0$. Likewise we can compute $e(f^{-1}(g(\Delta)))$ and $e(\tilde{X})$, from which, we see that $\chi(\tilde{X}) = 1$, which is a contradiction in each of these cases.

Tree (b): For this tree we first contract D_1, D_3, D_4, D_6, D_7 and D_9. Then we have $g^\star(\underline{D_2}) = (1/3)(D_1 + 3D_2 + 2D_3 + D_4)$, $g^\star(\underline{D_5}) = (1/3)(D_3 + 2D_4 + 3D_5 + 2D_6 + D_7)$, $g^\star(\underline{D_8}) = (1/3)(D_3 + 2D_4 + 3D_8 + D_9)$. From this we see that $(\underline{D_2})^2 = (\underline{D_8})^2 = -1$, $(\underline{D_5})^2 = -2/3$, $(\underline{D_2}.\underline{D_5}) = (\underline{D_2}.\underline{D_8}) = 1/3$ and $(\underline{D_5}.\underline{D_8}) = 2/3$. Thus we have $(\tilde{D_2})^2 = (\tilde{D_8})^2 = -3$, $(\tilde{D_5})^2 = -2$, $(\tilde{D_2}.\tilde{D_5}) = (\tilde{D_2}.\tilde{D_8}) = 1$, $(\tilde{D_5}.\tilde{D_8}) = 2$ and $\tilde{K} \sim 2\tilde{D}_2 + 4\tilde{D}_5 + 3\tilde{D}_8$ which implies that $(\tilde{K})^2 = 5$. As before we see that $e(\tilde{X}) = 3 + (1 + 3) = 7$ and hence $\chi(\tilde{X}) = 1$ which is a contradiction.

Tree (c): For this tree we first contract D_1, D_3, D_5 and D_7. Then we have $g^\star(\underline{D_2}) = (1/2)(D_1 + 2D_2 + D_3)$, $g^\star(\underline{D_4}) = (1/2)(D_3 + 2D_4 + D_5)$ and $g^\star(\underline{D_6}) = (1/2)(D_5 + 2D_6 + D_7)$. From this we see that $(\underline{D_2})^2 = -2$, $(\underline{D_4})^2 = (\underline{D_6})^2 = -1$, $(\underline{D_2}.\underline{D_4}) = (\underline{D_4}.\underline{D_6}) = 1/2$. Thus we have $(\tilde{D_2})^2 = -4$, $(\tilde{D_4})^2 = (\tilde{D_6})^2 = -2$. It is easy to see that the proper transform of $\underline{D_8}$ (and $\underline{D_9}$) consists of two irreducible components which we denote by $\tilde{D_{8,1}}$ and $\tilde{D_{8,2}}$ (similarly we have $\tilde{D_{9,j}}$ for $j = 1$, 2). Also we have $(\tilde{D_{8,j}})^2 = (\tilde{D_{9,j}})^2 = -2$ for $j = 1$, 2. We see that $(\tilde{K}) \sim \tilde{D}_2 + 6\tilde{D}_4 + 3\tilde{D}_6 + 4\tilde{D_{8,1}} + 4\tilde{D_{8,2}} + 2\tilde{D_{9,1}} + 2\tilde{D_{9,2}}$ which implies that $(\tilde{K})^2 = 2$. As before we see that $e(\tilde{X}) = 2 + (1 + 7) = 10$ and hence $\chi(\tilde{X}) = 1$ which is a contradiction.

Tree (d): For this tree we first contract D_1, D_2, D_4, D_6, D_7 and D_9. Then we have $g^\star(\underline{D_3}) = (1/3)(D_1 + 2D_2 + 3D_3 + 2D_4 + D_9)$, $g^\star(\underline{D_5}) = (1/3)(2D_4 + 3D_5 + 2D_6 + D_7 + D_9)$. From this we see that $(\underline{D_3})^2 = -2/3$, $(\underline{D_5})^2 = -5/3$, $(\underline{D_3}.\underline{D_5}) = 2/3$. Thus we have $(\tilde{D}_3)^2 = -2$, $(\tilde{D}_5)^2 = -5$, $(\tilde{D}_3.\tilde{D}_5) = 2$. Also we have $(\tilde{D_{8,j}})^2 = -2$ for $j = 1, 2, 3$. We see that $\tilde{K} \sim 4\tilde{D}_3 + \tilde{D}_5 + 2\tilde{D_{8,1}} + 2\tilde{D_{8,2}} + 2\tilde{D_{8,3}}$ which implies that $(\tilde{K})^2 = 3$. As before we see that $e(\tilde{X}) = 3 + (1 + 5) = 9$ and hence $\chi(\tilde{X}) = 1$ which is a contradiction.

Tree (e): For this tree we first contract D_1, D_7, D_8 and D_9. Then we have $g^\star(\underline{D_2}) = (1/2)(D_1 + 2D_2 + D_9)$, $g^\star(\underline{D_4}) = (1/2)(2D_4 + D_7 + D_8)$. From this we see that $(\underline{D_2})^2 = (\underline{D_4})^2 = -1$. Thus we have $(\tilde{D}_2)^2 = (\tilde{D}_4)^2 = (\tilde{D_{5,j}})^2 = (\tilde{D_{6,j}})^2 = -2$, $(\tilde{D_{3,j}})^2 = -3$ for $j = 1$, 2. We see that $\tilde{K} \sim \tilde{D}_2 + \tilde{D_{3,1}} + \tilde{D_{3,2}} + 3\tilde{D}_4 + 2\tilde{D_{5,1}} + 2\tilde{D_{5,2}} + \tilde{D_{6,1}} + \tilde{D_{6,2}}$ which implies that $(\tilde{K})^2 = 2$. As before we see that $e(\tilde{X}) = 2 + (1 - 1 + 8) = 10$ and hence $\chi(\tilde{X}) = 1$ which is a contradiction.

Tree (f): For this tree we first contract D_1, D_5, D_6 and D_9. Then we have $g^\star(\underline{D_2}) = (1/2)(D_1 + 2D_2 + D_6)$, $g^\star(\underline{D_4}) = (1/2)(2D_4 + D_5 + D_9)$. From this we see that $(\underline{D_2})^2 = -2$, $(\underline{D_4})^2 = -1$. Thus we have $(\tilde{D}_2)^2 = -4$, $(\tilde{D_{i,j}})^2 = (\tilde{D}_4)^2 = -2$, for $i = 3, 7, 8$ and $j = 1$, 2. We see that $\tilde{K} \sim \tilde{D}_2 + 3\tilde{D_{3,1}} + 3\tilde{D_{3,2}} + 3\tilde{D}_4 + 2\tilde{D_{7,1}} + 2\tilde{D_{7,2}} + \tilde{D_{8,1}} + \tilde{D_{8,2}}$ which implies that $(\tilde{K})^2 = 2$. As before we see that $e(\tilde{X}) = 2 + (1 - 1 + 8) = 10$ and hence $\chi(\tilde{X}) = 1$ which is a contradiction.

Tree (g): For this tree we first contract D_2, D_4, D_6, D_7, D_9 and D_{10}. Then we have $g^\star(\underline{D_1}) = (1/3)(3D_1 + D_2 + D_4)$, $g^\star(\underline{D_5}) = (1/3)(D_4 + 3D_5 + 2D_6 + D_7)$, $g^\star(\underline{D_8}) = (1/3)(D_6 + 2D_7 + 3D_8 + 2D_9 + D_{10})$. From this we see that $(\underline{D_1})^2 = -1/3$, $(\underline{D_5})^2 = -1$, $(\underline{D_8})^2 = -2/3$, $(\underline{D_1}.\underline{D_5}) = (\underline{D_5}.\underline{D_9}) = 1/3$. Thus we have $(\tilde{D}_1)^2 = -1$, $(\tilde{D_{3,j}})^2 = -4$, $(\tilde{D}_5)^2 = -3$, $(\tilde{D}_8)^2 = -2$, $(\tilde{D}_1.\tilde{D}_5) = (\tilde{D}_5.\tilde{D}_9) = 1$ for $j = 1, 2, 3$. We see that $\tilde{K} \sim 6\tilde{D}_1 + \tilde{D_{3,1}} + \tilde{D_{3,2}} + \tilde{D_{3,3}} + 2\tilde{D}_5 + \tilde{D}_8$ which implies that $(\tilde{K})^2 = 2$. As before we see that $e(\tilde{X}) = 3 + (1 + 6) = 10$ and hence $\chi(\tilde{X}) = 1$ which is a contradiction.

Tree (h): For this tree we contract all the components of Δ excepet D_4. By diagonalization process it is easy to see that $(\underline{D_4})^2 = 1$ which implies that $(\tilde{D}_4)^2 = 5$. Since $\tilde{K} \sim \tilde{D}_4$, we see that $(\tilde{K})^2 = 5$. As before we see that $e(\tilde{X}) = 5 + (1 + 1) = 7$ and hence $\chi(\tilde{X}) = 1$ which is a contradiction.

Thus we have proved:

Theorem 8.1. *There does not exist any $\mathbb{Q}$-homology plane with $\bar{\kappa}=2$ whose smooth projective completion is of general type.*

References

[1] T. Harary: *Graph Theory*, Narosa/Addison-Wesley, 1988.

[2] S. Iitaka: *Algebraic Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[3] M. Miyanishi and T. Sugie: *Homology Planes and quotient singularities*, J. Math. Kyoto Univ. **31** (1991), 755–788.

[4] C.R. Pradeep and A.R. Shastri: *On rationality of logarithmic Q-homology planes-I*, To appear in Osaka J. Math.

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