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ON AUTOMORPHISMS OF SUPERSINGULAR K3 SURFACES

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1. Introduction

Let k be an algebraically closed field of positive characteristic which we take as the ground field. Let $f : X \to C$ be a morphism from a nonsingular projective surface X to a nonsingular projective curve C such that almost all fibers of f are nonsingular elliptic curves. Furthermore, we assume that f has a section O, which is a morphism from C to X such that $f \circ O = \mathrm{id}_C$. We call such a surface X an elliptic surface over k. The generic fiber E of f is an elliptic curve defined over the function field $K = k(C)$ of C with the K-rational point which we denote by O by abuse of notation. So, we can consider the Mordell-Weil group $E(K)$ consisting of all K-rational points of E which is a finitely generated abelian group with zero element O by a theorem of Lang and Néron. We define the Mordell-Weil group of an elliptic surface $f : X \to C$ as the Mordell-Weil group $E(K)$ of its generic fiber E.

In [5], Miranda and Persson classified all the rational elliptic surfaces over the complex field with finite Mordell-Weil group. W. Lang [3],[4] classified all rational elliptic surfaces over an algebraically closed field of positive characteristic under the hypothesis that the degenerate fibers are all semi-stable. They call these surfaces *extremal* rational elliptic surfaces. In the present paper, we consider extremal elliptic K3 surfaces over an algebraically closed field of characteristic $p \geq 5$. An elliptic surface defined over an algebraically closed field of positive characteristic is called *extremal* if its Mordell-Weil group is finite and its Picard number $\rho(X)$ is equal to the second Betti number $b_2(X)$. An extremal elliptic K3 surface is therefore a supersingular K3 surface in the sense that its Picard number is equal to the second Betti number.

Our main theorem is stated as follows:

Theorem 1.1. *Let k be an algebraically closed field of characteristic $p \geq 5$ and let $f : X \to \boldsymbol{P}^1$ be a supersingular elliptic $K3$ surface defined over k such that the group of sections of f is finite.*

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Then X is isomorphic to one of the following elliptic surfaces.

1) $p = 11$ *and* X *have three singular fibers of type* II, I_{11} *and* I_{11}, *with* $E(K) \cong \{0\}$.
2) $p = 7$ *and* X *have three singular fibers of type* III, I_7 *and* I_{14}, *with* $E(K) \cong \boldsymbol{Z}/2\boldsymbol{Z}$.
3) $p = 7$ *and* X *have three singular fibers of type* II^*, I_7 *and* I_7, *with* $E(K) \cong \{0\}$.
4) $p = 5$ *and* X *have three singular fibers of type* IV, I_5 *and* I_{15}, *with* $E(K) \cong \boldsymbol{Z}/3\boldsymbol{Z}$.
5) $p = 5$ *and* X *have three singular fibers of type* III^*, I_5 *and* I_{10}, *with* $E(K) \cong \boldsymbol{Z}/2\boldsymbol{Z}$.

Furthermore, all these surfaces are obtained from the extremal rational elliptic surfaces by Frobenius base extension. More precisely, the surfaces 1) *and* 3) *are obtained from the rational elliptic surface whose type of degenerate fibers is* $(\mathrm{II}^*, \mathrm{I}_1, \mathrm{I}_1)$, *the surfaces* 2) *and* 5) *are obtained from the elliptic surface whose type of degenerate fibers is* $(\mathrm{III}^*, \mathrm{I}_1, \mathrm{I}_2)$ *and the surface* 4) *is obtained from the elliptic surface whose type of singular fibers is* $(\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3)$.

Note that the above theorem asserts that supersingular K3 surfaces with only finite number of sections do not exist if $p > 11$.

Remark 1.1. Any elliptic surface in the above theorem is not semi-stable. Thus there exist no semi-stable extremal elliptic K3 surfaces if the positive characteristic is different from 2 and 3. On the other hand, there are many semi-stable extremal elliptic K3 surfaces over the complex field (cf. [6]).

As a corollary, we have

Corollary 1.2. *Let X be a supersingular K3 surface which has an elliptic fibration with a section. Then* $\mathrm{Aut}(X)$ *contains an element σ of infinite order such that σ preserves the elliptic fibration and acts trivially on* $H^0(X, \Omega_X^2)$. *Furthermore, X contains infinitely many nonsingular rational curves.*

The same result for Kummer surfaces in positive characteristic different from 2 is obtained in [12].

From Theorem 1.1, we can calculate the Mordell-Weil group of each elliptic surface in Theorem 1.1.

Corollary 1.3. *For surfaces in Theorem* 1.1, *the Mordell-Weil groups $E(K)$ are isomorphic to* 1) $\{0\}$, 2) $\boldsymbol{Z}/2\boldsymbol{Z}$, 3) $\{0\}$, 4) $\boldsymbol{Z}/3\boldsymbol{Z}$, 5) $\boldsymbol{Z}/2\boldsymbol{Z}$, *respectively.*

For the cases $p = 2$ and 3, we treat the analogous results elsewhere, and extremal elliptic surfaces which are not necessarily $K3$ surfaces are treated in the paper [2].

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2. Notations and preliminary lemmas

Let k be an algebraically closed field of characteristic $p \geq 5$, which we fix throughout the present paper. Let $f: X \to C$ be an elliptic surface with a section. The following theorem is well-known (cf. [11]).

Theorem 2.1. *Let* $\mathrm{NS}(X)$ *be the Néron-Severi group of* X, *let* T *be its subgroup generated by all the irreducible components of fibers of* f *and the section* O *and let* $E(K)$ *be the Mordell-Weil group of* X. *Then,*

$$E(K) \cong \mathrm{NS}(X)/T. \tag{1}$$

This is an immediate consequence of the structure theorem on $\mathrm{NS}(X)$ in [10].

As a corollary, one obtains a famous formula concerning the rank ρ of the Néron-Severi group $\mathrm{NS}(X)$, the rank r of the Mordell-Weil group $E(K)$ and the number m_v of irreducible components of a singular fiber $f^{-1}(v)$, $v \in C$.

Corollary 2.2. *[Formula of Shioda-Tate] We have:*

$$\rho = 2 + r + \sum_{v \in C} (m_v - 1),$$

where $v \in C$ *ranges over the points over which the fibers are singular.*

When X is extremal, the quantities on both sides of the formula (1) are finite groups and one obtains a formula in the following corollary by comparing these orders.

Corollary 2.3. *If* X *is extremal,*

$$|E(K)|^2 = \frac{\det T}{\det \mathrm{NS}(X)}, \tag{2}$$

where $\mathrm{NS}(X)$ *and* T *are considered as the lattices endowed with bilinear forms induced by the intersection pairing on* X, *and* $\det \mathrm{NS}(X)$ (resp. $\det T$) is the determinant of the lattice $\mathrm{NS}(X)$ (resp. T).

Now we introduce and fix some notations. Let $f: X \to \boldsymbol{P}^1$ be a relatively minimal elliptic K3 surface defined over k, R be the set of all points v of $\boldsymbol{P}^1$ such that

$f^{-1}(v)$ is a singular fiber, δ_v be the order of vanishing at $v \in \boldsymbol{P}^1$ of the discriminant Δ of a minimal Weierstrass equation for the surface which is determined independently of the choice of a minimal Weierstrass equation of X, m_v be the number of irreducible components of $f^{-1}(v)$ and $f_v := \delta_v - (m_v - 1)$ be the conductor of $f^{-1}(v)$ for $v \in \boldsymbol{P}^1$. Note that the conductor f_v is 0 if $f^{-1}(v)$ is a nonsingular elliptic curve, 1 if $f^{-1}(v)$ is a semi-stable elliptic curve, i.e., $f^{-1}(v)$ is of multiplicative type, and 2 if $f^{-1}(v)$ a non-semi-stable elliptic curve, i.e., $f^{-1}(v)$ is of additive type. (For the definition of the conductor in the cases $p = 2$ and 3, see [7].)

The following lemma is well-known (cf. [5]).

Lemma 2.4. *Let X be an extremal elliptic $K3$ surface. Then we have:*

$$\sum_v \delta_v = 12\chi(O_X)$$

where $\chi(O_X)$ is the Euler-Poincaré characteristic of X.

Using Lemma 2.4 and the formula of Shioda and Tate, one obtains

$$\begin{aligned} \sum_{v \in R} f_v &= \textstyle\sum_{v \in R}(\delta_v - (m_v - 1)) \\ &= 12\chi(\vee_X) - (\rho - 2 - r) \\ &= 24 - (22 - 2 - 0) = 4, \end{aligned} \tag{3}$$

where $\chi(\vee_X) = 2$, $\rho = 22$ and $r = 0$ for our extremal elliptic K3 surface X.

On the other hand, the Néron-Severi lattice $\mathrm{NS}(X)$ of a supersingular K3 surface is a p-elementary lattice with $\det \mathrm{NS}(X) = -p^{2\sigma_0}$ $(1 \leq \sigma_0 \leq 10)$ and the lattice structure is uniquely determined by its discriminant $\det \mathrm{NS}(X)$ (cf. [9]), hence by σ_0 which is called the Artin invariant (cf. [1]).

By Corollary 2.3, we have

$$-p^{2\sigma_0} = \frac{\det T}{|E(K)|^2}. \tag{4}$$

Therefore, we have:

Lemma 2.5. $\det T$ *is a perfect square. In particular it is divisible by the characteristic p.*

We further mention facts on the modular function J which will play an important role in the subsequent sections. The modulur function J is a mapping from $C(\cong \boldsymbol{P}^1$ in our case) to $\boldsymbol{P}^1$ which assigns $c \in C$ to the J-invariant $J(f^{-1}(c))$ of the fiber $f^{-1}(c)$.

Lemma 2.6. *We have*:

$$\deg J = \sum_{n \geq 1} n(\nu(\mathrm{I}_n) + \nu(\mathrm{I}_n^*))$$

where $\nu(U)$ denotes the number of the singular fibers of type U in the elliptic fibration.

Proof. The degree of J is equal to the number of its poles counted with multiplicities. Every pole of J is associated to singular fibers of type I_n and I_n^* (cf. [5]). □

3. Proof of Theorem 1.1

By definition of the conductor f_v, we have

$$0 < \#R \leq \sum_{v \in R} f_v = 4. \tag{5}$$

Assume that the characteristic of k is positive and different from 2 and 3. Thus we have the following 3 cases.

i) $R = \{v_1, \cdots, v_4\}$ with $f_{v_1} = \cdots = f_{v_4} = 1$, hence, f is a semi-stable fibration.
ii) $R = \{v_1, v_2, v_3\}$ with $f_{v_1} = 2$ and $f_{v_2} = f_{v_3} = 1$.
iii) $R = \{v_1, v_2\}$ with $f_{v_1} = f_{v_2} = 2$.

CASE i) Since f is semi-stable, let I_a, I_b, I_c, I_d be 4 singular fibers with $a \leq b \leq c \leq d$.

By (4) and Lemma 2.6, we have

$$\begin{cases} abcd = (|E(K)|p^{\sigma_0})^2 \\ a+b+c+d = 24. \end{cases} \tag{6}$$

Thus the possibilities of the quadruplet (a, b, c, d) are exhausted by the following:

$$p = 11 \quad \text{and} \quad (1,1,11,11)$$
$$p = 7 \quad \text{and} \quad (1,2,7,14),\ (1,7,7,9),\ (2,7,7,8),\ (5,5,7,7)$$
$$p = 5 \quad \text{and} \quad (1,3,5,15),\ (1,5,8,10),\ (2,2,10,10),\ (3,5,6,10),\ (5,5,7,7).$$

Lemma 3.1. *All configurations in the above list do not exist when $p \geq 5$.*

Proof. Since the modular function J is of degree 24 the mapping J is separable. Thus one can easily find that the wild ramification violates the Hurwitz formula.

More precisely, let (a, b, c, d) be one of the above configurations, $f : X \to C$ be semi-stable elliptic $K3$ surfaces having 4 singular fibers of type $\mathrm{I}_a, \mathrm{I}_b, \mathrm{I}_c, \mathrm{I}_d$, and consider the J-function $J : C \to \boldsymbol{P}^1$ of the elliptic surface. At each point of $J^{-1}(0)$, the multiplicity is divisible by 3 and at each point of $J^{-1}(1)$ the multiplicity is divisible by 2. Furthermore, $J^{-1}(\infty)$ consists of the points v_1, v_2, v_3, v_4 with multiplicity a, b, c, d, respectively.

Now apply the Hurwitz formula for the J-map to get

$$\begin{aligned}(7)\qquad -2 &\geq -2\cdot 24 + (3-1)\cdot 8 + (2-1)\cdot 12 \\ &\quad + (a-1) + (b-1) + (c-1) + (d-1) = 0.\end{aligned}$$

This is a contradiction. □

CASE ii) Since the fibers over $t = v_2$, v_3 are semi-stable, we let their types be I_a and I_b $(a \leq b)$, respectively. Then, again by (4) and Corollary 2.2, we have

$$(8)\qquad \begin{cases} ab\det T_1 = (|E(K)|p^{\sigma_0})^2 \\ (a-1) + (b-1) + (m_v - 1) = 20, \end{cases}$$

where T_1 is the lattice associated to the fiber over v_1, that is, the lattice generated by all the irreducible components of the fiber over v_1 except for the irreducible component which intersects the section O. Thus the possible configurations of the type of $f^{-1}(v_1)$ and (a, b) are as follows:

$$\begin{gathered} p = 11 \quad \text{and} \quad (\mathrm{II},\ (11,11)) \\ p = 7 \quad \text{and} \quad (\mathrm{III},\ (7,14)),\ (\mathrm{I}_4^*,\ (7,7)),\ (\mathrm{II}^*,\ (7,7)) \\ p = 5 \quad \text{and} \quad (\mathrm{IV},\ (5,15)),\ (\mathrm{I}_8^*,\ (5,5)),\ (\mathrm{III}^*,\ (5,10)). \end{gathered}$$

Note that one can obtain that the Artin invariant σ_0 is equal to 1 in each case as a corollary of the computation.

Proposition 3.2. *In the above list, surfaces with the configurations* $(\mathrm{I}_4^*,\ (7,7))$ *in* $p = 7$ *and* $(\mathrm{I}_8^*,\ (5,5))$ *in* $p = 5$ *do not exist.*

Proof. From Lemma 2.6, the degree of the J-function of these surfaces is 18 which is prime to the characteristic of k. Thus we use the Hurwitz formula for these J again, we get

$$\begin{aligned}(9)\qquad -2 &\geq -2\cdot 18 + (3-1)\cdot 6 + (2-1)\cdot 9 \\ &\quad + \begin{cases} (7-1) + (7-1) + (4-1) & (\text{when p} = 7) \\ (8-1) + (5-1) + (5-1) & (\text{when p} = 5) \end{cases} \\ &= 0\end{aligned}$$

which is a contradiction. □

Proposition 3.3. *For the other* 5 *configurations, there exist unique surfaces having these configurations up to isomorphism. Furthermore, these surfaces have inseparable J-functions, and these are obtained from extremal rational surfaces having separable J-function by Frobenius base extension.*

Proof. Since an automorphism of $\boldsymbol{P}^1$ induces an automorphism of X, the 3-transitivity of $\mathrm{PGL}(2, k) = \mathrm{Aut}(\boldsymbol{P}^1)$ implies the uniqueness.

Next, we show existence of these surfaces by exhibiting the Weierstrass equations of these generic fibers explicitly. Consider elliptic curves over $k(t) = k(\boldsymbol{P}^1)$ defined by the following Weierstrass equations:

1) $y^2 = x^3 - 3t^4x + 2t^5$ in $p = 11$
2) $y^2 = x^3 - t^3x + t^5(1-t)$ in $p = 7$
3) $y^2 = x^3 - 3t^4x + 2t^5$ in $p = 7$
4) $y^2 = x^3 + t^3(3t-1)x + t^4(1+3t-t^2)$ in $p = 5$
5) $y^2 = x^3 - t^3x + t^5(1-t)$ in $p = 5$.

Then the minimal models of the surfaces defined by these Weierstrass equations are rational elliptic surfaces whose types of degenerate fibers and degrees of J-functions are following:

1) $(\mathrm{II}^*, \mathrm{I}_1, \mathrm{I}_1)$ $\deg J = 2$
2) $(\mathrm{III}^*, \mathrm{I}_1, \mathrm{I}_2)$ $\deg J = 3$
3) $(\mathrm{II}^*, \mathrm{I}_1, \mathrm{I}_1)$ $\deg J = 2$
4) $(\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3)$ $\deg J = 4$
5) $(\mathrm{III}^*, \mathrm{I}_1, \mathrm{I}_2)$ $\deg J = 3$.

After taking the Frobenius base changes of these surfaces and taking the regular minimal models of them, we have required surfaces. The Weierstrass equations of their generic fibers which are elliptic curves over $k(u) = k(\boldsymbol{P}^1)$ and their degenerate fibers are as follows:

1) $y^2 = x^3 - 3u^8x + 2u$ with $(\mathrm{II}, \mathrm{I}_{11}, \mathrm{I}_{11})$
2) $y^2 = x^3 - ux + u^5(1-u^7)$ with $(\mathrm{III}, \mathrm{I}_7, \mathrm{I}_{14})$
3) $y^2 = x^3 - 3u^8x + 2u^5$ with $(\mathrm{II}^*, \mathrm{I}_7, \mathrm{I}_7)$
4) $y^2 = x^3 + u^3(3u^5-1)x + u^2(1+3u^5-u^{10})$ with $(\mathrm{IV}, \mathrm{I}_5, \mathrm{I}_{15})$
5) $y^2 = x^3 - u^3x + u^7(1-u^5)$ with $(\mathrm{III}^*, \mathrm{I}_5, \mathrm{I}_{10})$.

The latter part of the assertion is clear from the above construction of the surfaces. □

CASE iii) Since the determinant of the lattice associated to an additive fiber is $1, 2, 3$ or 4, the determinant of the lattice T is not divisible by the characteristic p which is greater than 3. This contradicts Lemma 2.5. Thus this case does not occur.

Thus we proved Theorem 1.1.

REMARK 3.1. All the surfaces in Theorem 1.1 are Kummer surfaces.

Proof. Since all the surfaces we are considering are supersingular K3 surfaces and have the Artin invariants σ_0 equal to 1 by the remark above Proposition 3.2, the result is follow from the fact that a supersingular K3 surface is a Kummer surface if and only if its Artin invariant is equal to 1 or 2(cf. [8]). □

Proof of Corollary 1.2. If the Mordell-Weil rank of supersingular K3 surface is positive, then the assertion is clear.

On the other hand, since those surfaces whose Mordell-Weil rank is 0 are all Kummer surfaces by the previous Proposition, it follows from the result by Ueno who proved the same statement of Corollary 1.2 for Kummer surfaces in [12] (Theorem and Corollary). □

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