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GENERALIZATIONS OF THEOREMS OF FULLER

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Let R be a right artinian ring and e a primitive idempotent of R. In [6, Corollary 3.2 and Theorem 3.4] (also see Anderson-Fuller [1, Theorem 31.3].) K. Fuller showed that the following conditions are equivalent.

(1) eR is an injective right R-module.

(2) There exists a primitive idempotent f of R such that

  (2$*$) $S(eR) \cong T(fR)$ and $S(Rf) \cong T(Re)$, where $S(M)$ and $T(M)$ denote the socle and the top of M, respectively.

(3) There exists a primitive idempotent f of R such that

  (3ℓ) $\ell_{eR}(r_{Rf}(I)) = eI$ for each left ideal I, and

  (3r) $r_{Rf}(\ell_{eR}(K)) = Kf$ for each right ideal K of R, where $r_{Rf}(I) = \{a \in Rf | Ia = 0\}$ and $\ell_{eR}(K) = \{b \in eR | bK = 0\}$.

Let R be a semiprimary ring and e and f primitive idempotents of R. Then (eR, Rf) is called an i-pair in [3] if the above condition (2$*$) is satisfied. In [3, Theorem 1, Proposition 4 and Corollary 1], Y. Baba and K. Oshiro extended these results to semiprimary rings to show the following statements.

(a) If R is a semiprimary ring, then the condition (1) is satisfied if and only if both (2) and (3r) are satisfied.

(b) If R is a semiprimary ring satisfying (2) and the condition ($*$) below, then (1) is satisfied.

($*$) The lattice $\{r_{Rf}(X) | X \subseteq eR\}$ satisfies the ascending chain condition. Moreover, in [3, Theorem 2], they showed the following statement (c).

(c) If R is a semiprimary ring and (eR, Rf) is an i-pair for primitive idempotents e and f of R, then the following are equivalent.

(c1) Rf is artinian as a right fRf-module.

(c2) eR is artinian as a left eRe-module.

(c3) eR is an injective right R-module and Rf is an injective left R-module.

In this note, for a right R-module M with $S(M) \cong T(fR)$ and $P = \mathrm{End}M$, we consider a pair $({}_PM, Rf_{fRf})$ instead of an i-pair $({}_{eRe}eR, Rf_{fRf})$ and give generalizations of the results (a), (b) and (c) above (in Sections 1 and 2). In particular, in Section 1, for a module N_Q, we give some properties for the pair $({}_PM, N_Q)$, which are very similar to Theorem 1.1 in Morita-Tachikawa [11]. Moreover, in Section 3, by applying results obtained in Sections 1 and 2, we give elementary proofs of

Theorems 1 and 2 in Baba [2], which are related to some results in Fuller [6].

Throughout this note we always assume that every ring has an identity and every module is unitary. In particular, R always stands for a semiprimary ring with the Jacobson radical J. For a ring H, by M_H ($_HM$) we stress what M is a right (left) H-module. Let M be a module. Then $L \leq M$ (resp. $L < M$) means that L is a submodule of M (resp. $L \leq M$ and $L \neq M$). By $S(M)$, $T(M)$ and $E(M)$, we denote the socle, the top and an injective hull of M, respectively, and by $|M|$ we denote the composition length of M. Assume every homomorphism always operates from opposite side of scalar. "Acc" ("dcc") means the ascending (descending) chain condition. We denote the set of primitive idempotents of R by $\mathrm{Pi}(R)$.

1. Colocal pairs of modules

Let P and Q be rings and $_PM$, N_Q and $_PU_Q$ be a left P-module, a right Q-module and a P-Q-bimodule, respectively. Let $\varphi : M \times N \to U$ be a P-Q-bilinear map, i. e., a map satisfying the following properties:

(1) $\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y)$,

(2) $\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2)$,

(3) $\varphi(px, yq) = p\varphi(x, y)q$;

for any $x, x_1, x_2 \in M, y, y_1, y_2 \in N, p \in P$ and $q \in Q$.

Then, we say that $(_PM, N_Q)$ is a pair with respect to φ or simply a pair.

Let $(_PM, N_Q)$ be a pair with respect to φ. Then for $x \in M, y \in N$ and for $_PX \leq_P M, Y_Q \leq N_Q$, by xy we denote the element $\varphi(x, y)$, and by XY we denote the P-Q-subbimodule of $_PU_Q$ generated by $\{xy | x \in X, y \in Y\}$. Moreover, for $A \subseteq M$ and $B \subseteq N$, we define submodules $r(A)$ $(= r_N(A))$ of N_Q and $\ell(B)$ $(= \ell_M(B))$ of $_PM$, as follows: $r(A) = \{y \in N | Ay = 0\}$ and $\ell(B) = \{x \in M | xB = 0\}$, and we call $r(A)$ (resp. $\ell(B)$) the right (resp. left) annihilator of A (resp. of B).

Let $(_PM, N_Q)$ be a pair and put $U = MN$. For submodules $X' \leq X \leq {_PM}, Y' \leq Y \leq N_Q$ with $XN' = X'Y = 0$, we have a pair $(_PX/X', Y/Y'_Q)$ by defining $(x + X')(y + Y') = xy$. This is called a pair induced from (M, N). For an arbitrary ring H, we call an H-module V colocal if V has the (non-zero) smallest submodule. We call a pair $(_PM, N_Q)$ colocal if the module U $(= MN)$ is colocal both as a left P-module and as a right Q-module. Note, in case $(_PM, N_Q)$ is a colocal pair, we have $S(_PU) = S(U_Q)$. We call a pair (M, N) left faithful (resp. right faithful) if $\ell(N) = 0$ (resp. $r(M) = 0$), and a pair (M, N) faithful if it is left and right faithful. We denote the class of right annihilator submodules in N_Q by $Ar(M, N)$; that is $Ar(M, N) = \{Y \leq N_Q | Y = r\ell(Y)\}$, and similarly $A\ell(M, N) = \{X \leq {_PM} | X = \ell r(X)\}$, and the lattice of submodules of $_PM$ (resp. N_Q) by $\mathrm{Lat}(_PM)$ (resp. $\mathrm{Lat}(N_Q)$). We say that a pair $(_PM, N_Q)$ satisfies r-ann (resp. ℓ-ann) if $Ar(M, N) = \mathrm{Lat}(N_Q)$ (resp. $A\ell(M, N) = \mathrm{Lat}(_PM)$).

Let P be a ring, Q a subring of R, M a P-R-bimodule and I a left ideal of R which is also an R-Q-bimodule. In this case, unless otherwise stated, by the notation

$({}_PM, I_Q)$ we always mean a pair with respect to the bilinear map $\varphi : M \times I \to MI$ defined by $\varphi(m,a) = ma; m \in M, a \in I$. In case P is a subring of R and Q is a ring, for a right ideal K of R which is also a P-R-bimodule and for an R-Q-bimodule N, we consider the pair $({}_PK, N_Q)$ in the same way.

Lemma 1.1. *Let $({}_PM, N_Q)$ be a colocal pair, and $Y' < Y \leq N_Q$ with $Y' = r\ell(Y')$. If $(Y/Y')_Q$ is simple, then ${}_P(\ell(Y')/\ell(Y))$ is also simple and $Y = r\ell(Y)$.*

Proof. Put $U = MN$, $X = \ell(Y)$ and $X' = \ell(Y')$. By the assumption, there exists an element $y \in Y$ such that $Y = yQ + Y' \leq N_Q$. From $r\ell(Y') = Y' < Y \leq r\ell(Y)$, we obtain $X = \ell(Y) < \ell(Y') = X'$. For any $x \in X'$, the left multiplication map $\hat{x} : (Y/Y')_Q \to xY_Q$ by x is an epimorphism, so we have $xY_Q \leq S(U_Q)$. Hence $X'Y = S(U_Q) = S({}_PU)$, which shows that ${}_PX'Y$ is simple. On the other hand, the map $\eta : {}_PX'/X \to {}_PX'Y$ defined by $\eta(x + X) = xy$ is a monomorphism. Thus ${}_P(\ell(Y')/\ell(Y))$ $(= {}_P(X'/X))$ is simple. By the same argument, it follows that $(r\ell(Y)/r\ell(Y'))_Q$ is simple. Hence we have $Y = r\ell(Y)$ from $r\ell(Y') = Y' < Y \leq r\ell(Y)$. □

Lemma 1.2. *Let $({}_PM, N_Q)$ be a colocal pair, and Y and Z submodules of N_Q with $Z = r\ell(Z) \leq Y_Q$. If $|(Y/Z)_Q| < \infty$, then $Y = r\ell(Y)$.*

In particular, if $({}_PM, N_Q)$ is right faithful and $|Y_Q| < \infty$, then $Y = r\ell(Y)$.

Proof. The assertion is immediate from Lemma 1.1 by induction on the length $|(Y/Z)_Q|$. □

Lemma 1.3 (See [11, Theorem 1.1] (or [15, Theorem 1.1])). *Let $({}_PM, N_Q)$ be a colocal pair, and put $M' = \ell(N) \leq M$ and $N' = r(M) \leq N$. Then $|(N/N')_Q| < \infty$ if and only if $|{}_P(M/M')| < \infty$.*

Moreover, in case the above conditions are satisfied, we have $X = \ell r(X)$ (resp. $Y = r\ell(Y)$) *for any X with $M' \leq X \leq {}_PM$* (resp. *for any Y with $N' \leq Y \leq N_Q$*), *and $|{}_P(M/M')| = |(N/N')_Q|$.*

Proof. We denote Y/N' (resp. X/M') by $\overline{Y}$ (resp. $\overline{X}$). If $|(N/N')_Q| = n$ and $\overline{N'} = \overline{Y_0} < \overline{Y_1} < \cdots < \overline{Y_n} = \overline{N}$ is a composition series of $\overline{N_Q} = (N/N')_Q$, then for $X_i = \ell(Y_i)$, $\overline{M'} = \overline{X_n} < \overline{X_{n-1}} < \cdots < \overline{X_0} = \overline{M}$ is a composition series of $\overline{M}$ by Lemma 1.1 and in particular $|{}_P(M/M')| = |(N/N')_Q| = n$. It follows from Lemma 1.2 that $X = \ell r(X)$ and $Y = r\ell(Y)$. □

REMARK 1. Let $({}_PM, N_Q)$ be a colocal pair and put $U = {}_PMN_Q$, $M' = \ell_M(N)$ and $N' = r_N(M)$. Then the following condition $(**)$ is satisfied.

$(**)$ ${}_PU_Q$ - dual takes simple left P-modules and simple right Q-modules to simples or zero.

In order to show this, let $K = xQ$ be a simple right Q-module. If $0 \neq {}_P\mathrm{Hom}_Q(K_Q, {}_PU_Q)$, then $\alpha(x)Q = \alpha(K) = S(U_Q) = S({}_PU) = P\alpha(x)$ for any $\alpha \in {}_P\mathrm{Hom}_Q(K_Q, {}_PU_Q)$. Hence $P\alpha(x) \geq P\beta(x)$ for any $0 \neq \alpha, \beta \in {}_P\mathrm{Hom}_Q(K_Q, {}_PU_Q)$ and consequently $P\alpha \geq P\beta$, which implies ${}_P\mathrm{Hom}_Q(K_Q, {}_PU_Q)$ is simple.

On the other hand, by the proof of Morita-Tachikawa [11, Theorem 1.1], in case the condition (∗∗) is satisfied, we have that $|{}_P(M/M')| < \infty$ if and only if $|(N/N')_Q| < \infty$. Thus Lemma 1.3 is obtained as a corollary to [11, Theorem 1.1].

Theorem 1.4 (See [3, Lemma 3 and Proposition 5]). *Let Q be a semiprimary ring. Assume $({}_PM, N_Q)$ is a colocal pair and put $M' = \ell(N) \leq M$ and $N' = r(M) \leq N$. Then the following conditions are equivalent*:
(1) *$Ar(M, N)$ satisfies acc, (or equivalently $A\ell(M, N)$ satisfies dcc).*
(2) $|(N/N')_Q| < \infty$.
(3) $|{}_P(M/M')| < \infty$.

Moreover, in case the above conditions are satisfied, we have $X = \ell r(X)$ (resp. *$Y = r\ell(Y)$) for any X with $M' \leq X \leq {}_PM$* (resp. *for any Y with $N' \leq Y \leq N_Q$*), *and $|{}_P(M/M')| = |(N/N')_Q|$.*

Proof. The implication (2) ⇒ (1) is trivial and the equivalence (2) ⟺ (3) follows from Lemma 1.3. Hence we only show the implication (1) ⇒ (2). Assume $|(N/N')_Q| = \infty$. Then we can take an infinite-chain $N' = Y_0 < Y_1 < Y_2 < \cdots < N_Q$ of submodules of N_Q such that $|(Y_i/N')_Q| = i$ for any $i \geq 0$. By Lemma 1.2, $Y_i = r\ell(Y_i)$ for any $i \geq 0$. Hence, from the assumption we have $Y_n = Y_{n+1} = \cdots$ for some $n \geq 0$, which is a contradiction. □

We call a pair $({}_PM, N_Q)$ right (resp. left) finite provided the lattice $Ar({}_PM, N_Q)$ (resp. $A\ell({}_PM, N_Q)$) satisfies acc and $({}_PM, N_Q)$ finite provided $({}_PM, N_Q)$ is left finite and right finite. As a special case of Theorem 1.4, we have the following corollary.

Corollary 1.5. *Let Q be a semiprimary ring and $({}_PM, N_Q)$ a right finite faithful colocal pair. Then it holds that $|{}_PM| = |N_Q| < \infty$ and $({}_PM, N_Q)$ satisfies r-ann and ℓ-ann.*

2. Indecomposable injective modules

As mentioned in the introduction, we assume that R always stands for a semiprimary ring with the Jacobson radical J.

Lemma 2.1 (See [6, Lemma 1.1]). *Let M be a right R-module and f a prim-*

itive idempotent of R and put $Q = fRf$. Consider the following conditions.

(1) $S(M) \cong T(fR)$.

(2) $\ell_M(Rf) = 0$.

(3) $\ell_M(I) = \ell_M(If)$ *for any* $I_R \leq R_R$.

(4) $S(Mf_Q) = S(M_R)f_Q$.

Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *hold.*

In particular, if M is injective with $S(M) \cong T(fR)$ (i.e. $M \cong E(T(fR_R))$), then $r_{Rf}(M) = 0$ and $\ell_M(Rf) = 0$.

Proof. The implication (1) $\Rightarrow$ (2) follows from $T(fR) \cong S(M) \leq xR$ for any $(0 \neq)\ x \in M$. (2) $\Rightarrow$ (3) is easily seen from $If = IRf$. We show the implication (3) $\Rightarrow$ (4). $S(M)f \subseteq S(Mf)$ is clear. Since $S(Mf_Q)Jf = S(Mf_Q)fJf = 0$, we have $S(Mf)J = 0$ from $\ell_M(J) = \ell_M(Jf)$. Therefore we have $S(Mf) \subseteq S(M)f$ and consequently $S(Mf_Q) = S(M_R)f$.

We assume M_R is injective with $S(M) \cong T(fR)$. Then we have $\ell_M(Rf) = 0$ from the implication (1) $\Rightarrow$ (2). If $0 \neq a \in Rf$, then we have a non-zero map $\theta : aR \to M$. Hence by the injectivity of M, $xa = \theta(a) \neq 0$ holds for some $x \in M$. Thus $r_{Rf}(M) = 0$. □

Let L_R be a simple right R-module and $f \in \mathrm{Pi}(R)$. Then note that Lf_{fRf} is a simple right fRf-module or zero (cf. Baba [2, Lemma 1]).

Let M be a right R-module. Then we call M quasi-injective if for any submodule L of M, any homomorphism $\theta : L \to M$ can be extended to some endomorphism of M. By [9, Theorem 1.1], M is quasi-injective if and only if $HM = M$, where $H = \mathrm{End}E(M_R)$. Hence in case M is quasi-injective, we have a surjective ring homomorphism $H \to \mathrm{End}(M_R)$ ($\alpha \mapsto \alpha|_M$ for any $\alpha \in H$) and we denote the map by ρ_M. As easily seen, any quasi-injective right R-module M is colocal if and only if M is end-local (i. e., $\mathrm{End}M_R$ is a local ring.). By Harada [8], a module M is called simple-injective if for any modules L and N with $L \leq N$, any homomorphism $\theta : L \to M$ with a simple image $\theta(I)$ can be extended to some homomorphism $\varphi : N \to M$. The following lemma shows that Proposition 1 in Baba-Oshiro [3] is also verified in case M is not necessarily projective.

Lemma 2.2 (See [3, Proposition 1]). *If M is an end-local and simple-quasi-injective right R-module, then M is colocal.*

Proof. See the proof of the implication (1) $\Rightarrow$ (2), (i) in [14, Lemma 1, 2] in which L_1 and L_2 are simple. □

Lemma 2.3 ([3, Proposition 2]). *Let M be a colocal right R-module. If M is R-simple-injective, then M is injective.*

Lemma 2.4. *Let M be a right R-module, and put $P = \mathrm{End}M$ and $Q = fRf$ ($\cong \mathrm{End}_R Rf$); $f \in \mathrm{Pi}(R)$. Then the following are equivalent.*
(1) $({}_PM, Rf_Q)$ *is a left faithful colocal pair.*
(2) ${}_PMf$ *is colocal and* $S(M_R) \cong T(fR_R)$.
Moreover, in case the conditions are satisfied, any endomorphism α of $S(M_R)$ can be exended to some endomorphism of M.

Proof. (1) $\Rightarrow$ (2). Since, by the assumption, $xRf \neq 0$ for any $0 \neq x \in S(M_R)$, we have $S(M_R) = \oplus_{i \in I} L_i$ with $L_i \cong T(fR_R)$ for each $i \in I$. But $S(M_R)f_Q$ ($= S(Mf_Q)$) is simple by Lemma 2.1 and L_if_Q is also simple for any i, so I is a set consisting of a single element. This shows $S(M_R) \cong T(fR_R)$.

(2) $\Rightarrow$ (1). This is immidiate from the implication (1) $\Rightarrow$ (2), (4) in Lemma 2.1. We assume that (1) and (2) are satisfied and let $\alpha : S(M_R) \to S(M_R)$ be a map. Clearly $S(M_R) = xR$ holds for some $x = xf \in S(M_R)$. Then $\alpha(x) \in S(M_R)f_Q = xQ = Px$, which implies $\alpha(x) = \varphi(x)$ for some $\varphi \in P$. □

Lemma 2.5. *Let M be an injective* (resp. *quasi-injective*) *right R-module with $S(M_R) \cong T(fR_R)$; $f \in \mathrm{Pi}(R)$. Then $({}_PM, Rf_Q)$ is a faithful* (resp. *left faithful*) *colocal pair, where $P = \mathrm{End}M$ and $Q = fRf$.*

Proof. Assume that M_R is quasi-injective with $S(M_R) \cong T(fR_R)$. By Lemma 2.1, $S(Mf_Q) = S(M_R)f_Q$ is simple and the pair $({}_PM, Rf_Q)$ is left faithful. We show that ${}_PMf$ is colocal. Let $0 \neq x = xf \in S(Mf_Q)$ and $0 \neq y = yf \in Mf_Q$. Since $(xfJ)Rf = x(fJf) = 0$, we have $xfJ = 0$ by Lemma 2.1, which shows $r_{fR}(y) \leq fJ = r_{fR}(x)$. Hence the map $\theta : yR \to M$ with $\theta(yc) = xc$ ($c \in R$) is well-defined. Therefore θ is extended to some $\varphi \in \mathrm{End}M = P$, and in particular $x = \varphi(y)$. Thus we have $Px \leq Py$. This shows that ${}_PMf$ is colocal. In case M_R is injective, it follows from Lemma 2.1 that $r_{Rf}(M) = 0$, so $({}_PM, Rf_Q)$ is faithful. □

REMARK 2. Let e be a primitive idempotent of R such that eR_R is quasi-injective and assume the lattice $Ar(R, R)$ satisfies acc. Then $S(eR_R) \cong T(fR)$ for some $f \in \mathrm{Pi}(R)$, and by Lemma 2.5, $({}_{eRe}eR, Rf_{fRf})$ is a right finite left faithful colocal pair. Hence by Theorem 1.4, ${}_{eRe}eR$ is artinian. Thus in [14, Proposition 2.7], without using torsion theory we can prove that R is a left artinian ring.

As an immediate consequence of Lemma 2.5, we have Corollary 2.6 below, which was obtained by Baba-Oshiro [3] (by Fuller [6] in case R is one-sided artinian). The corollary is useful and its proof is simple. So we give a proof directly in spite of [3], [6] and Lemma 2.5. The proof is similar to that of the implication (3) $\Rightarrow$ (2) in Kato [10, Lemma 2].

Corollary 2.6. (Baba-Oshiro [3, Proposition 4] (and Fuller [6, Theorem 3.1] for a right artinian ring R)). *Let e and f be primitive idempotents of R. If eR is an injective right R-module with $S(eR_R) = aR_R; a = eaf$, then $S(_RRf) = {}_RRa \cong T(_RRe)$. (That is: If eR is an injective right R-module with $S(eR_R) \cong T(fR_R)$, then $S(_RRf) \cong T(_RRe)$.)*

Proof. It is clear that $r_{fR}(b) \leq fJ = r_{fR}(a)$ for any $0 \neq b \in Rf$. Hence the map $\theta : bR \to eR$ with $\theta(bc) = ac$ $(c \in R)$ is well-defined. Therefore by the injectivity of eR_R we have $a = hb$ for some $h \in eR$. So $a \in Rb$, which implies that $S(_RRf) = Ra$ is simple. □

The following theorem is a slight generalization of Baba-Oshiro [3, Theorem 1]. But for the sake of completeness, we give a proof.

Theorem 2.7 (See [3, Theorem 1]). *Let M be an indecomposable right R-module. Then the following conditions are equivalent.*

(1) *M is injective.*

(2) *$(_PM, Rf_Q)$ is a faithful colocal pair satisfying r-ann for some $f \in \mathrm{Pi}(R)$, where $P = \mathrm{End}M_R$ and $Q = fRf$.*

Proof. By Lemmas 2.4 and 2.5, we may assume that $(_PM, Rf_Q)$ is a faithful colocal pair with $S(M_R) \cong T(fR_R); f \in \mathrm{Pi}(R)$. Then by lemma 2.1, $\ell(I) = \ell(If)$ is satisfied for any $I \leq R_R$.

(2) $\Rightarrow$ (1). It suffices to show that M_R is R-simple-injective by Lemma 2.3. Let $I \leq R_R$, and $\theta : I \to M$ a homomorphism with a simple image $\theta(I) = S(M_R)$ and put $K = Ker\theta$. Then θ induces an isomorphism $\overline{\theta} : I/K \to S(M_R)$. Since $Kf < If \leq Rf_Q$, it holds that $r\ell(Kf) = Kf < If = r\ell(If)$ by the assumption. Hence $\ell(K) = \ell(Kf) > \ell(If) = \ell(I)$, so there exists an element $x \in \ell(K) - \ell(I)$. Let $\hat{x} : R \to M$ be the left multiplication map by x and $\eta : I \to M$ the restriction map to I of $\hat{x}$. Then η induces an isomorphism $\overline{\eta} : I/K \to S(M_R)$. If $\alpha : S(M_R) \to S(M_R)$ is the automorphism with $\alpha\overline{\eta} = \overline{\theta}$ (i.e. $\alpha = \overline{\theta}\overline{\eta}^{-1}$), then α is extended to an endomprphism φ of M_R by Lemma 2.4, which shows $\varphi\eta = \theta$. Hence $\theta : I \to M$ is extended to a map $\varphi\hat{x} : R \to M$, so M_R is R-simple-injective.

(1) $\Rightarrow$ (2). Assume that there exists a submodule Lf of Rf_Q with $Lf < r\ell(Lf)$. Then $LfR < r\ell(Lf)R$. Put $I = r\ell(Lf)R$. Since R is a semiprimary ring, we can take a maximal submodule K of I_R with $LfR \leq K < I_R$. Then $\ell(K) = \ell(I)$ holds since $\ell(LfR) = \ell(Lf) = \ell r\ell(Lf) = \ell(I)$. On the other hand we have $Kf < If$, because $Kf = If$ implies $I = IfR = KfR \leq K$, which is a contradiction. Hence $(I/K)f \neq 0$, so we have an isomorphism $\alpha : I/K \to S(M_R)$. Let $\theta : I \to M$ be a composition map $\theta = \mu\alpha\lambda$, where $\lambda : I \to I/K$ and $\mu : S(M_R) \to M_R$ are canonical maps. By the assumption, there exists $x \in M$ such that $\theta(a) = xa$ for any

$a \in I$. From $\theta(I) \neq 0$ and $\theta(K) = 0$, we have $x \in \ell(K) - \ell(I)$, which contradicts $\ell(K) = \ell(I)$. □

The following theorem shows that in case (M, Rf) is finite, the converse of Lemma 2.5 holds.

Theorem 2.8 (See [3, Theorem 1 and Corollary 1]). *Let M be a right R-module. If $({}_PM, Rf_Q)$ is a right finite faithful* (resp. *left faithful*) *colocal pair for some $f \in \mathrm{Pi}(R)$, where $P = \mathrm{End}M_R$ and $Q = fRf$, then M_R is injective* (resp. *quasi-injective*) *with $S(M_R) \cong T(fR_R)$.*

Proof. Assume $({}_PM, Rf_Q)$ is a right finite left faithful colocal pair. Then by Lemma 2.4 $S(M_R) \cong T(fR_R)$. In case that $({}_PM, Rf_Q)$ is faithful, M is injective from Corollary 1.5 and Theorem 2.7. Putting $I = r_R(M)$, then $If = r_{Rf}(M)$. Hence the pair $({}_PM, Rf_Q)$ induces a right finite faithful colocal pair $({}_PM, Rf/If_Q)$. Moreover M can be regarded as a right $\overline{R}$-module canonically and it holds that $P \cong \mathrm{End}M_{\overline{R}}, {}_{\overline{R}}Rf/If \cong {}_{\overline{R}}\overline{R}\bar{f}$ and $Q/fIf \cong \bar{f}\overline{R}\bar{f}$ (canonically), where $\overline{R} = R/I$ and $\bar{f} = f + I \in \mathrm{Pi}(\overline{R})$. Hence, considering the pair $({}_PM_{\overline{R}}, {}_{\overline{R}}\overline{R}\bar{f}_{\bar{f}\overline{R}\bar{f}})$, then by the same argument as above, $M_{\overline{R}}$ is injective and consequently M_R is quasi-injective. □

REMARK 3. Let M_R be a right R-module with $P = \mathrm{End}M_R$. Consider the following conditions:

(1) $M = \ell_{E(M)}r_R(M)$,

(2) $M_{\overline{R}}$ is injective, where $\overline{R} = R/r_R(M)$,

(3) M_R is quasi-injective.

Then by [7, Theorem 1.2], (1) $\Longleftrightarrow$ (2) $\Rightarrow$ (3) hold, and by [4, Theorem 19.14] (or [5, Corollary 5.6A]), in case ${}_PM$ is finitely generated, (3) $\Rightarrow$ (2) holds.

But in this note (e.g., in the proof of Theorem 3.5), we consider a colocal module M_R (or a colocal module M_R with $|{}_PM| < \infty; P = \mathrm{End}M_R$). In this case, above implications follow from Theorems 2.7, 2.8 and their proofs.

Proposition 2.9 (See [3, Theorems 1 and 2]). *Let M be an indecomposable right R-module and $({}_PM, Rf_Q)$ a faithful colocal pair, where $f \in \mathrm{Pi}(R)$, $P = \mathrm{End}M_R$ and $Q = fRf$. Then the following are equivalent.*

(1) *The pair $({}_PM, Rf_Q)$ is right finite.*

(2) *The pair $({}_PM, Rf_Q)$ satisfies r-ann and ℓ-ann.*

(3) *M_R is injective and the pair $({}_PM, Rf_Q)$ satisfies ℓ-ann.*

Proof. The implication "(1) $\Rightarrow$ (2)" follows from Corollary 1.5, and "(2) $\Longleftrightarrow$ (3)" follows from Theorem 2.7.

(2) $\Rightarrow$ (1). By the equivalence (2) $\Longleftrightarrow$ (3), M_R is injective. First we shows that ${}_PM$ is linearly compact. The proof is the almost same as Mueller [12, Lemma 4]. Let $(x_i, X_i)_{i\in I}$ be a finitely solvable family of ${}_PM$. Then by the assumption, $X_i = \ell_M r_{Rf}(X_i)$, so $X_i = \ell_M r_R(X_i)$ because of $r_{Rf}(X_i) = r_R(X_i)f$ and $\ell_M(K) = \ell_M(Kf)$ for any $K \leq R_R$. Put $Y_{iR} = r_R(X_i)$ and consider a map $\theta : \sum_{i\in I} Y_i \to M$ with $\theta(\sum_{i\in F} y_i) = \sum_{i\in F} x_i y_i$, where F is a finite subset of I and $y_i \in Y_i$. By the assumption, for any finite subset F of I, there exists an element $x \in M$ such that $x_i - x \in X_i$. Then $(x_i - x)y_i \in X_i Y_i = 0$, so $\sum_{i\in F} x_i y_i = x\sum_{i\in F} y_i$, which shows θ is well-defined. Since M is injective, there exists an element $x_0 \in M$ such that $x_i y_i = x_0 y_i$ for any $y_i \in Y_i$ and any $i \in I$. Hence $(x_i - x_0)Y_i = 0$, and consequently $x_i - x_0 \in \ell_M r_R(X_i) = X_i$. Thus ${}_PM$ is linearly compact.

By the assumption, we have $\mathrm{Lat}({}_PM) = A\ell({}_PM, Rf_Q)$ and $\mathrm{Lat}(Rf_Q) = Ar({}_PM, Rf_Q)$, so $\mathrm{Lat}({}_PM)$ is anti-isomorphic to $\mathrm{Lat}(Rf_Q)$ by the correspondence $X \to Y$; where $X = \ell_M(Y)$ and $Y = r_{Rf}(X)$. Since Q is semiprimary, Rf_Q has the upper Loewy series $Rf = Y_0 > Y_1 > \cdots > Y_n = 0$. Then, $0 = \ell(Y_0) < \ell(Y_1) < \cdots < \ell(Y_n) = M$ is the lower Loewy series of ${}_PM$, and $\ell(Y_i)/\ell(Y_{i-1})$ is a semisimple left P-module for each $i = 1, \ldots, n$. Since ${}_PM$ is linearly compact, so is ${}_P\ell(Y_i)/\ell(Y_{i-1})$ (see e.g. [13, Proposition 2.2]). Thus each module ${}_P\ell(Y_i)/\ell(Y_{i-1})$ has a finite composition length (see e.g. [13, Lemma 2.3]), and hence $|{}_PM| < \infty$. □

Corollary 2.10 (Baba-Oshiro [3, Theorem 2]). *Let (eR, Rf) be an i-pair and $P = eRe$, $Q = fRf$, where $e, f \in \mathrm{Pi}(R)$. Then the following are equivalent.*

(1) *${}_PeR$ is artinian.*

(2) *Rf_Q is artinian.*

(3) *Both eR_R and ${}_RRf$ are injective.*

3. Application of colocal pairs

In this section, we give elementary proofs of Theorems 1 and 2 in Baba [2]. "Quasi-projective" for a module is defined as the dual notion to "quasi-injective". See [16] for the definition of a quasi-projective module and its characterization. Note that a right R-module M_R is end-local and quasi-projective if and only if $M_R \cong eR/eI$ for some primitive idempotent e of R and for some two sided ideal I of R.

Let $({}_PM, N_Q)$ be a pair and put $\overline{P} = P/\ell_P(M)$ and $\overline{Q} = Q/r_Q(N)$. Then we have a pair $({}_{\overline{P}}M, N_{\overline{Q}})$ naturally. It is clear that $({}_PM, N_Q)$ is colocal if and only if so is $({}_{\overline{P}}M, N_{\overline{Q}})$. Hence note that we may identify $({}_PM, N_Q)$ with $({}_{\overline{P}}M, N_{\overline{Q}})$ through the canonical maps $P \to \overline{P}$ and $Q \to \overline{Q}$.

Lemma 3.1 ([2, Theorem 1]). *Let $E = Rf/If \cong E(T({}_RRe))$ for some left ideal I of R, and put $P = eRe$ and $Q = fRf$ where $e, f \in \mathrm{Pi}(R)$. If E is quasi-*

projective, then the following hold.
(1) $r_{Rf}(eR) = If$.
(2) $({}_PeR, Rf_Q)$ *is a left faithful colocal pair.*

Proof. (1) Since Rf/If is quasi-projective and injective with $S({}_RRf/If) \cong T({}_RRe)$, we have $If(fRf) = If$ and $\ell_{eR}(Rf/If) = 0$ by Lemma 2.1. Hence $eIf(Rf/If) = 0$, so $(eR)If = eIf = 0$. Thus we have $r_{Rf}(eR) \geq If$. On the other hand, $S(Rf/If) \cong T({}_RRe)$ implies $r_{Rf/If}(eR) = 0$ by Lemma 2.1. If $eRa = 0$ for an element $a = af \in Rf$, then $eR(a+If) = 0+If$ in Rf/If. Hence $a+If = 0+If$ in Rf/If and $a \in If$, so $r_{Rf}(eR) \leq If$. Thus we have $r_{Rf}(eR) = If$.

(2) By (1), ${}_PeE_Q \cong {}_PeRf_Q$ holds, hence we can identify the pair $({}_PeR, E_Q)$ with the pair $({}_PeR, Rf/If_Q)$ induced from $({}_PeR, Rf_Q)$. Moreover we may assume $Q/fIf = \mathrm{End}_RE$ since $E = Rf/If$ is quasi-projective. It follows from Lemma 2.5 that the pair $({}_PeR, E_Q)$ is faithful colocal, so $({}_PeR, Rf_Q)$ is left faithful colocal. □

Lemma 3.2 ([2, Theorem 1]). *Let* $({}_PeR, Rf_Q)$ *be a right (or left) finite left faithful colocal pair with* $P = eRe$ *and* $Q = fRf$ *where* $e, f \in \mathrm{Pi}(R)$, *and put* ${}_RE = {}_RE(T({}_RRe))$. *Then the following hold.*
(1) ${}_RE$ *is quasi-projective with* $T({}_RE) \cong T({}_RRf)$.
(2) eR_R *is quasi-injective with* $S(eR_R) \cong T(fR_R)$.

Proof. (1) Putting $I = r_R(eR)$, then $If = r_{Rf}(eR)$. Since ${}_RRf/If$ is quasi-projective, we can regard Q/fIf as End_RRf/If. Moreover $({}_PeR, Rf/If_Q)$ is a finite faithful colocal pair since ${}_Pe(Rf/If)_Q \cong {}_PeRf_Q$. It follows from Theorem 2.8 that ${}_RRf/If$ is an injective module with $S({}_RRf/If) \cong T({}_RRe)$. Thus we have $E \cong {}_RRf/If$, which implies (1).

(2) By Theorem 2.8. □

Theorem 3.3 (Baba [2, Theorem 1]). *Let* e *and* f *be primitive idempotents of* R *and put* $E = E(T({}_RRe))$, $P = eRe$ *and* $Q = fRf$. *If* $Ar({}_PeR, Rf_Q)$ *satisfies acc or dcc, then the following conditions are equivalent.*
(1) ${}_RE$ *is quasi-projective with* $T({}_RE) \cong T({}_RRf)$.
(2) eR_R *is quasi-injective with* $S(eR_R) \cong T(fR_R)$.
(3) $({}_PeR, Rf_Q)$ *is a left faithful colocal pair.*
(4) ${}_PeRf$ *is colocal and* $S(eR_R) \cong T(fR_R)$.

Proof. This is an immediate consequence of Lemmas 2.4, 2.5, 3.1 and 3.2. □

Lemma 3.4. *Let* $({}_{P'}eR, Rf_Q)$ *be a right (or left) finite colocal pair with*

$eS({}_RRf) \neq 0$, where $e, f \in \mathrm{Pi}(R), P' = eRe$ and $Q = fRf$. Put $K = ReS({}_RRf)$, $E = E(T(fR_R))$ and $P = \mathrm{End}E_R$. Then the following hold.

(1) *${}_RK$ is a unique simple submodule of ${}_RRf$ satisfying $K \cong T({}_RRe)$.*

(2) *There exists a local quasi-injective submodule M of E_R such that $({}_PM, Rf_Q)$ is a finite left faithful colocal pair, $T(M_R) \cong T(eR_R)$ and $MK \neq 0$.*

Proof. (1) Since $S({}_{P'}eRf)$ is simple, we have $S({}_{P'}eRf) = eS({}_RRf)$. If $S({}_RRf) = \oplus_{i \in A} K_i$ with simple submodules K_i, then $eS({}_RRf) = \oplus_{i \in A} eK_i$. Hence there exists only one index $i \in A$ such that $eS({}_RRf) = eK_i$. Thus $K = ReS({}_RRf) = K_i$ is simple.

(2) Putting $I = \ell_R(Rf)$, then we have $eI = \ell_{eR}(Rf)$ and $eIf = eIRf = 0$. Hence $({}_{P'}eR/eI, Rf_Q)$ is a finite left faithful colocal pair with $P'/\ell_{P'}(eR/eI)(= eRe/eIe) \cong \mathrm{End}eR/eI_R$. It follows from Theorem 2.8 that eR/eI_R is quasi-injective. Since $S(eR/eI_R) \cong T(fR_R) \cong S(E_R)$, there exists a submodule M of E_R with $M \cong eR/eI$. Then M_R is quasi-injective and we have the surjective ring homomorphism $\rho_M : P \to \mathrm{End}M_R$. Therefore $({}_PM, Rf_Q)$ is a finite left faithful colocal pair with $T(M_R) \cong T(eR_R)$. Moreover if $MReS({}_RRf) = 0$, then $(eR/eI)eS({}_RRf) = 0$, so $eS({}_RRf) \leq eI$ and $eS({}_RRf) \leq eIf = 0$, a contradiction. Hence $MK = MReS({}_RRf) \neq 0$. Thus M satisfies the property in (2). □

Theorem 3.5 (Baba [2, Theorem 2]). *Let $E = E(T(fR_R))$ and let $({}_{P_i}e_iR, Rf_Q)$ be a right (or left) finite colocal pair for any $i = 1, \dots, n$, where $e_i, f \in \mathrm{Pi}(R), P_i = e_iRe_i$ and $Q = fRf$. Put $P = \mathrm{End}E_R$. Then the following conditions are equivalent.*

(1) $S({}_RRf) \cong T({}_RRe_1) \oplus \cdots \oplus T({}_RRe_n)$.

(2) $T(E_R) \cong T(e_1R_R) \oplus \cdots \oplus T(e_nR_R)$.

Moreover in case the conditions are satisfied, $S({}_RRf)$ (or equivalently $T(E_R)$) is square-free and the pair $({}_PE, Rf_Q)$ is finite.

Proof. Note that for any $e \in \mathrm{Pi}(R)$, the following property (P) holds.

(P) $eS({}_RRf) \neq 0$ implies $T(E_R)e \neq 0$.

If $eS({}_RRf) \neq 0$, then by Lemma 2.1 $EeS({}_RRf) \neq 0$ holds and we have $EJS({}_RRf) = 0$ clearly, which shows that (P) holds.

(1) $\Rightarrow$ (2). Assume (1). Then $S({}_RRf)$ is square-free by Lemma 3.4 (1). Hence by the property (P), $T(E_R)$ has a direct summand isomorphic to $T(e_1R_R) \oplus \cdots \oplus T(e_nR_R)$. By (1) we have $S({}_RRf) = K_1 \oplus \cdots \oplus K_n$ for some $K_i \leq {}_RRf$ with $K_i \cong T({}_RRe_i); i = 1, \dots, n$. By Lemma 3.4, for each $i = 1, \dots, n$, there exists a quasi-injective submodule M_i of E_R such that $({}_PM_i, Rf_Q)$ is a finite left faithful colocal pair with $T(M_i) \cong T(e_iR)$ and $M_iK_i \neq 0$. Putting $M = M_1 + \cdots + M_n$, then M_R is a quasi-injective module with $|{}_PM| < \infty$. Since $({}_PE, Rf_Q)$ is a left faithful colocal pair, so is $({}_PM, Rf_Q)$. If $0 \neq a \in Rf$, then $Ra \geq K_i$ for some i and

consequently $Ma \geq MK_i \neq 0$. Hence $({}_PM, Rf_Q)$ is a finite faithful colocal pair. Moreover we have the surjective ring homomorphism $\rho_M : P \to \mathrm{End}(M)$. Therefore M_R is injective by Theorem 2.8, which implies $E = M$. Thus $T(E_R)$ $(= T(M_R))$ is isomorphic to a direct summand of $T(e_1R_R) \oplus \cdots \oplus T(e_nR_R)$ and consequently we have (2). Moreover (E, Rf) is finite because of $E = M$.

(2) $\Rightarrow$ (1). Assume (2). Then by the property (P) and Lemma 3.4 (1), we may assume that $S({}_RRf) \cong T({}_RRe_1) \oplus \cdots \oplus T({}_RRe_m)$ for some $m; 1 \leq m \leq n$. Therefore from the implication (1) $\Rightarrow$ (2), $T(E_R) \cong T(e_1R_R) \oplus \cdots \oplus T(e_mR_R)$ is obtained. Thus $m = n$ holds and consequently we have (1). □

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