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SURFACES OF CONSTANT MEAN CURVATURE WITH BOUNDARY IN A SPHERE

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1. Introduction and preliminaries

We consider a compact (always supposed connected) surface Σ embedded in the Euclidean space $\boldsymbol{R}^3$ whose boundary will be represented by $\partial\Sigma$. When the surface has (non zero) constant mean curvature, we abbreviate saying *cmc surface* and *H-cmc surface* when we emphasize the value H of the mean curvature. The mean curvature in a point is defined by the average of the two principal curvatures in this point, so, the sphere of radius $r > 0$ has $1/r$ as mean curvature in anywhere if the Gauss map is chosen to point inside. Remember that a cmc surface is orientable and so, we may choose a globally defined Gauss map N on Σ. A cmc surface in Euclidean three-space can be viewed as a surface where the exterior pressure and the surface tension forces are balanced. For this reason they are thought of as *soap bubbles or films* depending on the considered surface being either closed (that is, compact without boundary) or compact with non-empty boundary. The study of the space of H-cmc surfaces with prescribed non-empty boundary Γ has been the focus of a number of authors. Even in the simplest case, when Γ is a circle of radius 1, it is unknown if the two spherical caps (the large cap and the small one) with radius $1/|H|$ are the only examples. Heinz [4] found that a necessary condition for the existence in this situation is that $|H| \leq 1$. In another hand, the Alexandrov reflection method gives that if the surface is included in one of the two halfspaces determined by the boundary plane, then the surface inherits the symmetries of its boundary and then, is a spherical cap. For that, it is interesting to have hypothesis to assure the surface is over the plane containing the boundary. Partial results have been obtained in [2, 3, 6, 7, 8, 9].

Related with this subject, we pose the following problem: let Γ be a Jordan curve lying in the unit sphere

$$S^2(1) = \{x = (x_1, x_2, x_3) \in \boldsymbol{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

and we ask for the shape of a 1-cmc surface with boundary Γ. It is immediate that Γ determines in S^2 two domains which are 1-cmc surfaces, but it is possible the

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existence of anothers 1-cmc surfaces with boundary Γ. For instance, the intersection of a cylinder of radius $1/2$, whose axis is tangent to $S^2(1)$, with the ball $|x| \leq 1$, has boundary on $S^2(1)$, but is not umbilical.

The basic tool in this paper is the often invoked Maximum Principle due to E. Hopf. To establish it in the context of cmc surfaces, we call that two surfaces *touch at* p if they are tangent at some common interior point p, the orientations in both surfaces agree in p and one of them is above the other one in a neighbourhood of p with respect to the coordinate system given by the tangent plane in p and the unit normal in p. Then the Maximum Principle can be stated as follows:

Maximum Principle [5]. *Let Σ_1, Σ_2 be two cmc surfaces with the same mean curvature. If they touch at a point p, then both surfaces coincide in some neighbourhood of p.*

In relation with the Maximum Principle, it is a basic fact of differential geometry that if two surfaces Σ_1, Σ_2 (not necessarily cmc surfaces) touch at some interior point p and Σ_1 is above Σ_2 in a neighbourhood of p, then the mean curvature of Σ_2 is less or equal than the mean curvature of Σ_1 around p. We will call this fact *Comparison Principle.*

2. The main result

Koiso [6] proved that if Σ is a cmc surface with boundary a Jordan curve Γ included in a plane P and Σ does not intersect the outside of Γ in P, then Σ is included in one of the two halfspaces determined by P. We will obtain a similar result, but in the case that the boundary is included in a sphere. Firstly, set the following notation

$$B = \{x \in \boldsymbol{R}^3; |x| < 1\} \qquad E = \{x \in \boldsymbol{R}^3; |x| > 1\}$$

and the upper hemisphere $S_+ = \{x \in S^2(1); x_3 > 0\}$. If Γ is a Jordan curve included in S_+, it bounds two domains in $S^2(1)$. We call *the bounded domain by Γ in S_+* the only one of both included in S_+. The next theorem is motivated by the paper of Koiso.

Theorem 1. *Let Σ be a 1-cmc surface with $\partial\Sigma$ a Jordan curve included in the hemisphere S_+. Let Ω be the bounded domain by $\partial\Sigma$ in S_+. If Σ does not intersect $S^2(1) - \overline{\Omega}$, then $\Sigma = \overline{\Omega}$ or $\Sigma - \partial\Sigma \subset B$ or $\Sigma - \partial\Sigma \subset E$.*

Proof. We suppose that $\Sigma \neq \overline{\Omega}$. Then to show the Theorem 1 it is sufficient to prove that $\Sigma \cap \Omega = \emptyset$. We will derive a contradiction if we assume that $\Sigma \cap \Omega \neq \emptyset$. In this case, we define the closed embedded surface $F = \Sigma \cup (S^2(1) - \overline{\Omega})$ and let W

be the bounded 3-domain determined by F in $\boldsymbol{R}^3$. Choose in Σ the Gauss map N corresponding with the mean curvature $H = 1$.

Firstly, we prove that N points towards W. For that, we take the function $f(x) = |x|^2$ for $x \in \Sigma$ and let $p \in \Sigma$ be the point where f attains its maximum. Because $\Sigma \cap \Omega \neq \emptyset$, $f(p) \geq 1$ and the point p can be chosen to be an interior point of Σ. Then $N(p) = \pm p/|p|$. If we put a sphere of radius 1 tangent to Σ at p and included in the domain $\{x \in \boldsymbol{R}^3; |x| \geq |p|\}$, the Maximum Principle gets $N(p) = -p/|p|$ (with $H = 1$, the Gauss map of that sphere points inside). Therefore $N(p)$ points towards W and so, N points to W. We have two possibilities about the domain W:

1. near $S^2(1) - \overline{S_+}$, the domain W lies in the side B. From now, we will call this "the property (A)".
2. near $S^2(1) - \overline{S_+}$, the domain W lies in the side E.

We are going to prove that the second case is not possible. If it is so, we take the lower hemisphere $S_- = \{x \in S^2(1); x_3 \leq 0\}$ and we "blow up" it fixing the boundary ∂S_- to obtain spherical caps bounded by ∂S_-, below the plane $\{x_3 = 0\}$ and with radius increasingly. At first, spherical caps are included in W but there is a first time such that Σ and a spherical cap have a common interior point because W is bounded and the curve $\partial\Sigma$ is above the plane $\{x_3 = 0\}$. So, Σ and the spherical cap touch at this point (for positive mean curvature, the Gauss maps of both surfaces point towards W). Then the Comparison Principle gets a contradiction because the mean curvature of the spherical cap is less than 1.

Now we treat the case that $\Sigma \subset \overline{E}$ (in this case, $B \subset W$ by the property (A)). In any point $z \in \Omega \cap \Sigma$, we compare Ω with Σ. If N_Ω is the unit normal field of Ω to have $H = 1$, then $N_\Omega(z) = -z$ and therefore $N_\Omega(z)$ points to W. Thus Ω and Σ touch at z and thanks to the Maximum Principle we get a contradiction.

Therefore, we are going to suppose that $\Sigma \cap B \neq \emptyset$. Now we show that $(\Sigma \cap B) \cap \{x \in \boldsymbol{R}^3; x_3 \leq 0\}$ is empty. In the other case, set $r = (r_1, r_2, r_3) \in \Sigma \cap B$, $r_3 \leq 0$, the point where the third coordinate function x_3 attains its minimum in the subset $\Sigma \cap B$. Then $N(r) = \pm a$, where $a = (0, 0, 1)$. But since N points towards W, $\partial\Sigma \subset S_+$ and by the property (A), we have $N(r) = -a$. Now we compare Σ with a sphere of radius 1 tangent to Σ at the point r and included in the halfspace $\{x \in \boldsymbol{R}^3; x_3 \leq r_3\}$. If we orient this sphere to have 1 as mean curvature, the Gauss map of this sphere points inside and then its value is $-a$, i.e., it agrees with N in the point r. Then the sphere and Σ touch at r, getting a new contradiction by virtue of the Maximum Principle.

Therefore $\Sigma \cap B \subset \{x \in \boldsymbol{R}^3; x_3 > 0\}$. Now we consider the family $\{C_s; s \in (0, 1]\}$ of small spherical caps included in $\{x \in \boldsymbol{R}^3; x_3 \geq 0\}$, with constant mean curvature s and boundary the equator ∂S_+. The Gauss map in C_s points down (respect the direction a). Since Σ is compact and $\Sigma \cap B$ is included in the upper halfspace $x_3 > 0$, there exists $\epsilon > 0$ such that $C_s \cap (\Sigma - \partial\Sigma) = \emptyset$ for any $s \in (0, \epsilon)$.

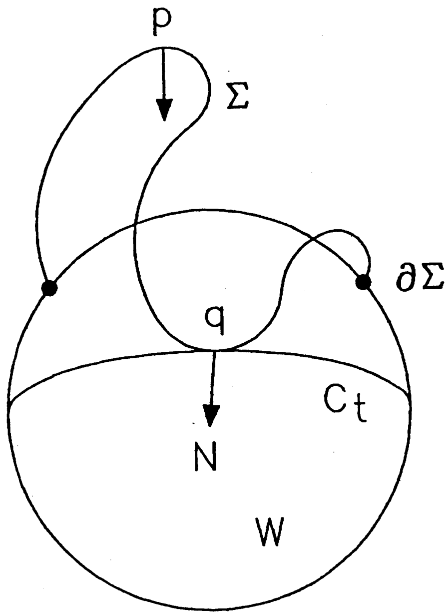

Fig. 1.

We increase $s \to 1$ to intersect C_s with Σ at the first time $t \geq \epsilon$. Notice that $t < 1$ because $\Sigma \cap B \neq \emptyset$. Then C_t and Σ are tangent in some common interior point q. Since the property (A) assures that the lower domain determined by C_t in $S^2(1)$ is included in W, the Gauss map of C_t points towards W and therefore, the Gauss maps of Σ and C_t agree in q (see Figure 1). But it is impossible, since the mean curvature of C_t is $t < 1$, strictly less than the mean curvature of Σ (equal to one) and with the fact that C_t is locally above Σ, in contradiction with the Comparison Principle. □

The proof of Theorem 1 can be generalized immediately to (non zero) constant mean curvature compact, connected hypersurfaces embedded in $\boldsymbol{R}^{n+1}$ ($\equiv$ *cmc hypersurfaces*) with several components in the boundary. For that, let $S^n(1)$ be the Euclidean n-dimensional sphere of radius 1 in $\boldsymbol{R}^{n+1}$, S_+ the upper hemisphere of $S^n(1)$ and B and E the inside and outside of $S^n(1)$. Let Σ be a cmc hypersurface of $\boldsymbol{R}^{n+1}$ and $\Gamma_1 \cup \ldots \cup \Gamma_k$ the decomposition into connected components of $\partial\Sigma$. If $\partial\Sigma$ is included in S_+, we call *the bounded domain Ω by $\partial\Sigma$ in S_+* to the union of the bounded domains by Γ_i in S_+, $i = 1, \ldots, k$. With this notation, we have

Theorem 2. *Let Σ be a 1-cmc hypersurface with boundary $\partial\Sigma = \Gamma_1 \cup \ldots \cup \Gamma_k$ included in S_+. If Σ does not intersect $S^n(1) - \overline{\Omega}$, then $\Sigma - \partial\Sigma \subset B$ or $\Sigma - \partial\Sigma \subset E$ or $k = 1$ and $\Sigma = \overline{\Omega}$.*

Theorem 2 gives the following result on the problem posed in the Introduction to characterize the spherical caps as the only cmc hypersurfaces in $\boldsymbol{R}^{n+1}$ with boundary an $(n-1)$-dimensional Euclidean sphere.

Corollary 2.3. *Let Γ be a round $(n-1)$-dimensional sphere of radius less than one and included in $S^n(1)$ and Σ a 1-cmc hypersurface with boundary Γ. If Σ does not intersect the large spherical cap of $S^n(1)$ determined by Γ, then Σ is the small spherical cap of radius 1 or $\Sigma - \partial\Sigma$ is included completely in E.*

Proof. We can assume that Γ is included in the upper hemisphere S_+. If Ω is the small spherical cap determined by Γ in $S^n(1)$, Theorem 2 gives that Σ is $\overline{\Omega}$ or $\Sigma - \partial\Sigma$ is included in E or included in B. But from a result of Barbosa [1], the last case gives that Σ is the small spherical cap, exactly it is the reflection of Ω with respect to the hyperplane containing Γ. $\square$

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