Freudenburg, G.
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LOCAL SLICE CONSTRUCTIONS IN $k[X,Y,Z]$

GENE FREUDENBURG

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1. Introduction

Let k be a field of characteristic zero, and let $k^{[n]}$ denote the polynomial ring in n variables over k. Recently, in [5], the first examples of locally nilpotent derivations of $k^{[n]}$ having maximal rank n were constructed. In an attempt to understand and generalize these examples, our present aim is to describe *local slice constructions*, a general procedure by which new locally nilpotent derivations can be constructed from given derivations of a certain type. The examples of [5] can be obtained from familiar derivations via local slice constructions, though they were not originally discovered in this way. In fact, we will see in what follows that every known locally nilpotent derivation of $k^{[3]}$ can be obtained from a partial derivative via a (finite) sequence of local slice constructions. Whether any others exist is the content of *Question 1* below.

The following notation and definitions will be used. Unless otherwise specified, the word *derivation* will mean k-derivation. Let R be an integral k-domain. $R^{[n]}$ will denote the polynomial ring in n variables over R. Let D be a derivation of R, and let $A = \mathrm{Ker}(D)$, the kernel of D. D is *irreducible* if its image is contained in no proper principal ideal of R. D is *locally nilpotent* if, to each $f \in R$, there is an $n \geq 0$ such that $D^n f = 0$.

We say $s \in R$ is a *slice* for D if $Ds \in R^*$. (It is well known that, when D is locally nilpotent, A is factorially closed, and thus $R^* \subset A$.) The main fact concerning slices is the following.

Proposition 1 ([9, Proposition 2.1]). *Suppose R is an integral k-domain, D is a locally nilpotent derivation of R having kernel A, and $s \in R$ is such that $Ds = 1$. Then $R = A[s] \cong A^{[1]}$, and D is given by $D = d/ds$.*

When D is locally nilpotent (and non-zero), we can find $r \in R$ such that $Dr \in A = \mathrm{Ker}(D)$, but $Dr \neq 0$. If $Dr = f$, then D extends to a locally nilpotent derivation D_f on the localization R_f, and r becomes a slice for D_f. Moreover, the above theorem shows $R_f = A_f[r] = A_f^{[1]}$. This gives rise to the following.

DEFINITION 1. Let R be an integral k-domain, and let D be a locally nilpotent derivation of R. An element $r \in R$ is called a *local slice* for D if $Dr \in \mathrm{Ker}(D)/\{0\}$.

Of particular interest is the case $R = k^{[n]}$. In this case, for any locally nilpotent derivation D of R, the *rank* of D is defined to be the least integer $r \geq 0$ for which there exists a system of variables $(X_1, ..., X_n)$ of $k^{[n]}$ satisfying $k[X_{r+1}, ..., X_n] \subset \mathrm{Ker}(D)$.

2. Local Slice Constructions

From now on, B will denote the polynomial ring $k[X, Y, Z] = k^{[3]}$. Given f, $g \in B$, define a derivation $D_{(f,g)}$ of B by

$$D_{(f,g)}(h) = \frac{\partial(f, g, h)}{\partial(X, Y, Z)} \ .$$

The following theorem of Miyanishi is required (c.f. [6]).

Theorem 1. *If D is any non-zero locally nilpotent derivation of $k^{[3]}$, then* $\mathrm{Ker}(D) \cong k^{[2]}$.

Suppose D is a locally nilpotent derivation of B, with kernel $A = k[f, g]$. Then, up to A-multiples, $D = D_{(f,g)}$ (c.f. [3]). Let S denote the set $(k[f] - 0)$, and define

$$\Omega(f, g) = \{r \in B | D(r/g) \in S\},$$

a subset (possibly empty) of the set of local slices of D.

Proposition 2. *Given $r \in \Omega(f, g)$, $\Omega(f, g) = \{r' \in B | S^{-1}A[r'] = S^{-1}A[r]\}$.*

Proof. Given $r' \in \Omega(f, g)$, note that $(Dr')r - (Dr)r' \in A$. Thus, for some non-zero a, $b \in k[f]$, $(g \cdot a(f) \cdot r - g \cdot b(f) \cdot r') \in A$. Since A is factorially closed, it follows that $(a(f) \cdot r - b(f) \cdot r') \in A$. Therefore, $S^{-1}A[r'] = S^{-1}A[r]$, and $\Omega(f, g) \subseteq \{r' \in B | S^{-1}A[r'] = S^{-1}[r]\}$.

Conversely, suppose $S^{-1}A[r'] = S^{-1}A[r]$ for some $r' \in B$. Then $r' = cr + d$ for $c \in (S^{-1}A)^* = k(f)^*$ and $b \in S^{-1}A$. Thus, $D(r'/g) = c \cdot D(r/g) \in k(f)^* \cap (1/g)B = S$, which implies $\{r' \in B | S^{-1}A[r'] = S^{-1}A[r]\} \subseteq \Omega(f, g)$. □

Next, suppose $\Omega(f, g)$ is not empty. Let $\bar{B}$ denote the domain B modulo (g), and let $\bar{D}$ denote the induced locally nilpotent derivation on $\bar{B}$. Then for every $r \in \Omega(f, g)$, $\bar{r} \in \mathrm{Ker}(\bar{D})$. Since $\mathrm{Ker}(\bar{D})$ is the algebraic closure in $\bar{B}$ of $\bar{A} = k[\bar{f}] \cong k^{[1]}$, there exists $\phi \in k[f][r]$ such that $\phi(r) \in (g)$. If we choose ϕ to be of minimal r-degree in $k[f][r]$, and irreducible as an element of $k[f, r] = k^{[2]}$, then ϕ is unique up to non-zero constant multiples.

Define $h = \phi(r)/g$, and let $\Delta(f, g, r)$ (or simplyΔ) denote the derivation on B defined by $D_{(f,h)}$. (Up to non-zero constant multiples, Δ is uniquely determined by f, g, and r.) The crux of the matter is the following fact.

Theorem 2. *If $K = k(f, h)$ and $B_K = K[X, Y, Z]$, then $K[r] = B_K$. Consequently, Δ is locally nilpotent.*

Proof. We have $D_{(f,gh)} = g \cdot D_{(f,h)} + h \cdot D_{(f,g)}$, and therefore

$$g \cdot \Delta = D_{(f,\phi(r))} - h \cdot D = \phi'(r) \cdot D_{(f,r)} - h \cdot D.$$

It follows that $g \cdot \Delta(r) = -h \cdot Dr$, and thus $\Delta(r) = -h \cdot (Dr/g) \in k[f, h]$.

Since $\Delta r \neq 0$, r is transcendental over K, i.e., $K[r] \cong K^{[1]}$. Since $g = \phi(r)/h$, we have $g \in K[r]$; and since $k[f] \cap (g) = \{0\}$, $deg_r g \geq 1$ and $g \notin K[r]^*$.

We claim that g is irreducible in $K[r]$. Since $gh = \phi(f, r)$, it suffices to show that ϕ is irreducible in $K[r]$. However, ϕ was chosen to be irreducible in $k[f, r] \cong k^{[2]}$, hence it is also irreducible in $k[f, h, r] \cong k^{[3]}$. Since ϕ is not in K, it follows easily that ϕ is irreducible in $K[r]$. Consequently, g is also irreducible in $K[r]$.

It follows that $g \cdot K[r]$ is a maximal ideal of $K[r]$, and thus

$$(g \cdot B_K) \cap K[r] = g \cdot K[r].$$

Set $T = \{g^n \cdot a(f) | n \geq 0, a \in (k[f] - 0)\}$. Then $(T^{-1}A)[r] = T^{-1}B$. Given $b \in B$, choose n so that $g^n b \in k(f)[g, r] \subset K[r]$. Then, using the above ideal equality inductively, we obtain $b \in K[r]$. Therefore, $B \subset K[r]$, and we are done. □

We say Δ is the derivation obtained from D via the *local slice construction* using f, g, and r. Note that $r \in \Omega(f, h)$, so we can carry out a further local slice construction. However, since $g = (1/h)\phi(r)$, this simply results in reversing the process: $\Delta(f, h, r) = D$. To continue the process inductively, we may, by *Proposition 2* above, replace r with any r' for which $S^{-1}A[r'] = S^{-1}A[r]$.

It may also happen that the original derivation, D, admits a local slice r such that $Dr = fg$. Then $\Delta r = -fh$, so both $\Omega(f, h)$ and $\Omega(h, f)$ contain r. Thus, to continue the process inductively, we may also use $\Delta(h, f, r)$ instead of $\Delta(f, h, r)$.

In order to determine the kernel and rank of Δ, the following criteria are quite useful.

Proposition 3 (Kernel Criterion). *Suppose a, $b \in B = k[X, Y, Z]$ are such that $\delta := D_{(a,b)}$ is locally nilpotent and non-zero. Then the following are equivalent.*
(i) $k[a, b] = \mathrm{Ker}(\delta)$
(ii) *δ is irreducible, and* $\mathrm{Ker}(\delta) \subset k(a, b)$

Proof. The implication (i) $\Rightarrow$ (ii) follows from [3], Corollary 2.5. Conversely, assume (ii) holds. By Theorem 1 above, there exist $u,\ v \in B$ such that $\mathrm{Ker}(\delta) = k[u,v] \cong k^{[2]}$. It follows that

$$\delta = D_{(a,b)} = \frac{\partial(a,b)}{\partial(u,v)} \cdot D_{(u,v)}.$$

Since δ is irreducible, $\partial(a,b)/\partial(u,v) \in k^*$, i.e., (a,b) is a "Jacobian pair" for $k[u,v]$. Since $k(a,b) = k(u,v)$, the inclusion $k[a,b] \hookrightarrow k[u,v]$ is birational. It is well known that the Jacobian Conjecture is true in the birational case, and we thus conclude $k[a,b] = k[u,v]$.[1] □

Proposition 4 (Rank Criterion [2, Lemma 2.4]). *Let δ be a non-zero locally nilpotent derivation of B, and let $\pi : \mathbf{A}^3 \to \mathbf{A}^2$ be the morphism induced by the inclusion* $\mathrm{Ker}(\delta) \hookrightarrow B$. *Let $\mathcal{R}(\delta)$ denote the set of points of $\mathbf{A}^2$ whose fibers are reducible. If $\mathrm{rank}(\delta) \leq 2$, then $\mathcal{R}(\delta)$ is contained in a univariate set (i.e., a union of parallel lines with respect to some choice of coordinates).*

REMARK. The reader will probably have noticed that, geometerically, passage from $k[f,g,r] \cong k^{[3]}$ to $k[f,h,r] \cong k^{[3]}$ corresponds to a birational transformation of k^3 of a particularly elementary sort. Thus, algebraic passage from D to Δ via a local slice construction may be thought of geometrically as a sequence blow-ups of 3-space, followed by a sequence of blow-downs.

3. The Rank Two Case

We show that, up to change of coordinates on B, every irreducible locally nilpotent derivation of B of rank at most two can be obtained from $(\partial/\partial Z)$ by a sequence of local slice constructions.

Theorem 1. ([4, Corollary 3.2]) *Let D be an irreducible locally nilpotent derivation of $B = k[X,Y,Z]$ having rank at most two, and suppose $DX = 0$. Set $K = k(X)$. Then there exist $P,\ Q \in B$ such that $K[P,Q] = K[Y,Z]$,* $\mathrm{Ker}(D) = k[X,P]$, *and $DQ \in k[X]$. Moreover, the ideal generated by the image of D is (P_Y, P_Z), and if $(P_Y, P_Z) = (1)$, then Q may be chosen so that $k[X,P,Q] = B$ and $D = \partial/\partial Q$.*

Let D be any irreducible locally nilpotent derivation of B having rank at most two, and choose P, Q, and K be as in the theorem. Let $GA_2(K)$ be the group of K-automorphisms of $K[Y,Z]$, and consider the degrees of P and Q as elements of

[1]The same reasoning yields yet another equivalent formulation of the two-dimensional Jacobian Conjecture: Given $a,\ b \in B$, if $D_{(a,b)}$ is irreducible, then $\mathrm{Ker}D_{(a,b)} = k[a,b]$.

$K[Y,Z]$. Consider first the case $\deg_K P = 1$. If $P = aY + bZ$ for a, $b \in k[X]$, then $(P_Y, P_Z) = (a,b)$. Since (a,b) is principal, and since D is irreducible, we conclude that $(a,b) = (1)$, so by the theorem, D is equivalent to $\partial/\partial Z$.

Assume that $\deg_K P > 1$. In this case, the structure theory for $GA_2(K)$ implies that either $\deg_K P > \deg_K Q$ or $\deg_K P < \deg_K Q$ (c.f. [8, Proposition 7]). In the latter case, there exists $Q' \in K[Y,Z]$ such that $K[P,Q'] = K[Y,Z]$ and $\deg_K P > \deg_K Q'$; moreover, since $K[P,Q] = K[P,Q']$, we must have $Q' = \alpha \cdot Q + \beta(P)$ for some $\alpha \in (k[X] - 0)$ and $\beta \in k[X][P]$. Thus, $DQ' = \alpha \cdot DQ \in k[X]$. So it is no loss of generality to assume $\deg_K P > \deg_K Q$. In addition, if (Q) is not a prime ideal of B, there exists $\ell \in k[X]$ dividing Q such that (Q/ℓ) is prime. So it is no loss of generality to further assume Q is irreducible in B.

Observe that $\Omega(X,P) \neq \emptyset$, since $r := PQ \in \Omega(X,P)$. Consider $\Delta = \Delta(X,P,r) = D_{(X,Q)}$. Δ is again locally nilpotent (by Theorem 2), and since $\Delta X = 0$, it is of rank at most two. Also, $\mathrm{Ker}(\Delta)$ is the algebraic closure of of $k[X,Q]$ in B. Since Q is both irreducible and a K-variable, it follows that Δ is irreducible and $\mathrm{Ker}(\Delta) = k[X,Q]$. Finally, since $\deg_K Q < \deg_K P$, we conclude by induction on $\deg_K P$ that, up to a change of coordinates, $(\partial/\partial Z)$ can be obtained from D by a sequence of local slice constructions.

REMARK. Using the results of [4], we see that local slice constructions can be carried out for kernels of locally nilpotent derivations of $k^{[n]}$ having rank at most 2, for all $n \geq 2$. As above, all such kernels can be obtained from a partial derivative via a sequence of local slice constructions. However, for $n \geq 4$, it is not clear how to generalize local slice constructions to derivations of rank greater than 2, owing to the fact that the kernels of such are not in general polynomial rings.

4. Rank Three Examples

The examples discussed in this section are homogeneous in the standard sense (as maps of B). These tend to be easier to work with. It should be noted, however, that local slice constructions can be used to construct rank-3 examples which are weighted-homogeneous, or which are not homogeneous in any (non-trivial) grading of B.

Given any locally nilpotent derivation D of B, D is homogeneous iff there exist homogeneous polynomials f, $g \in B$ such that $\mathrm{Ker}(D) = k[f,g]$ [10]. In this case, we say D is homogeneous of *type* (e_1, e_2), where $e_1 = \min\{\deg f, \deg g\}$, and $e_2 = \max\{\deg f, \deg g\}$. As noted in [5], the rank of D is 3 iff $e_1 > 1$. Following the appearance of the $(2,5)$ example in [5], Daigle gave the following beautiful geometric characterization of the homogeneous locally nilpotent derivations of B. ($\mathbf{P}^2$ denotes the projective plane over k.)

Theorem 1. ([1]) *Let f_1 and f_2 be homogeneous elements of $k^{[3]}$, and let C_1 and C_2 be the corresponding projective plane curves which they define. The following are equivalent.*

- *There exists a locally nilpotent derivation D of $k^{[3]}$ such that* $\mathrm{Ker}(D) = k[f_1, f_2]$.
- $\mathbf{P}^2$ *minus* $(C_1 \cup C_2)$ *is isomorphic to* $\mathbf{P}^2$ *minus 2 lines.*

In other words, f_1 and f_2 define a locally nilpotent derivation precisely when there exists a plane Cremona transformation which is an isomorphism away from C_1 and C_2, and which transforms C_1 and C_2 into a pair of lines.

In order to construct examples, let D be the rank-2 locally nilpotent (linear) derivation on B defined by $DX = 0$, $DY = X$, and $DZ = 2Y$. Then D is homogeneous, and the kernel of D is $k[X, F]$, where $F = XZ - Y^2$. Since $D(FY) = XF$, we see that neither $\Omega(X, F)$ nor $\Omega(F, X)$ is empty.

4.1. Examples of Type (2, 4n+1)

Given $n \in \mathbf{Z}^+$, set $r_n = (X^{2n+1} + F^n Y) \in \Omega(F, X)$, and let $\Delta_n = \Delta(F, X, r_n)$. By Theorem 2, Δ_n is locally nilpotent. Now $F \equiv (-Y^2)$ and $r_n \equiv (-1)^n Y^{2n+1}$ modulo X, so the minimal polynomial we need is $\phi(r_n) = F^{2n+1} + r_n^2$. Therefore, $G_n := \phi(r_n)/X = (ZF^{2n} + 2X^{2n}F^n Y + X^{4n+1})$ is homogeneous of degree $(4n+1)$. It must be shown that Δ_n is irreducible.

The computation in the proof of Theorem 2 above shows that $\Delta_n r_n = -F^n G_n$. Thus,

$$G_n \cdot \Delta_n X = \Delta_n(G_n X) = \Delta_n(\phi(r_n)) = \phi'(r_n) \cdot \Delta_n r_n = (2r_n)(-F^n G_n),$$

which implies $\Delta_n X = -2F^n r_n$. Now if $\Delta_n = f \cdot T$ for some polynomial f and some derivation T, then $f \in \mathrm{Ker}(\Delta_n)$. Since $r_n \notin \mathrm{Ker}(\Delta_n)$ and F is irreducible, we conclude that Δ_n is reducible iff $\Delta_n \equiv 0 (\bmod F)$. Since $F^n Y = r_n - X^{2n+1}$, we have

$$F^n \Delta_n Y = (-F^n G_n) - (2n+1)X^{2n}(-2F^n r_n) ,$$

which implies $\Delta_n Y = 2(2n+1)X^{2n} r_n - G_n$. Modulo F, we have $r_n \equiv X^{2n+1}$ and $G_n \equiv X^{4n+1}$, and it follows that $\Delta_n Y \equiv (4n+1)X^{4n+1} \not\equiv 0$. Therefore Δ_n is irreducible.

Theorem 2 shows $\mathrm{Ker}(\Delta_n) \subset k(F, G_n)$. By the *Kernel Criterion*, it follows that $\mathrm{Ker}(\Delta_n) = k[F, G_n]$. Therefore Δ_n is of type $(2, 4n+1)$, and is consequently of rank 3. Note that when $n = 1$, we obtain the $(2, 5)$ example first discussed in [5].

The set of points fixed by the $\mathbf{G}_a$-action on $\mathbf{A}^3$ induced by Δ_n is precisely the set of points where $\Delta_n X$, $\Delta_n Y$, and $\Delta_n Z$ vanish simultaneously, and it is easy to check that this set is the line defined by $X = Y = 0$. Every other orbit is a line, i.e., isomorphic to $\mathbf{G}_a \cong \mathbf{A}^1$. Let $\pi : \mathbf{A}^3 \to \mathbf{A}^2$ be the morphism induced by the inclusion $A \hookrightarrow B$ ($A = k[F, G_n]$). Then the fiber over the point $(a, b) \in \mathbf{A}^2$ is defined by the ideal $(f - a, g - b)$ in B, and each fiber is a union of orbits. The fiber over the origin $(0, 0)$ is the line of fixed points. If neither a nor b is 0, then the fiber over (a, b) is a single (coordinate) line in $\mathbf{A}^3$. The most interesting fibers lie over points $(0, b)$ and $(a, 0)$ for $a \neq 0$ and $b \neq 0$. Over $(0, b)$, $b \neq 0$, the fiber consists of $4n+1$ (coordinate) lines lying on the surface defined by F. And over $(a, 0)$, $a \neq 0$, the fiber consists of two (coordinate) lines lying on the surface defined by G_n. The situation for $n = 1$ is depicted in Figure 1.

4.2. Examples of Fibonacci Type

We again use D as above, and fix $r = -(X^3 + FY)$ ($F = XZ - Y^2$). Inductively, define functions

$$\begin{aligned} H_0 &= -Y \\ H_2 &= F \\ H_{n+1} &= \frac{1}{H_{n-1}}(H_n^3 + r^{a_n}) \qquad (a_n = \deg H_n). \end{aligned}$$

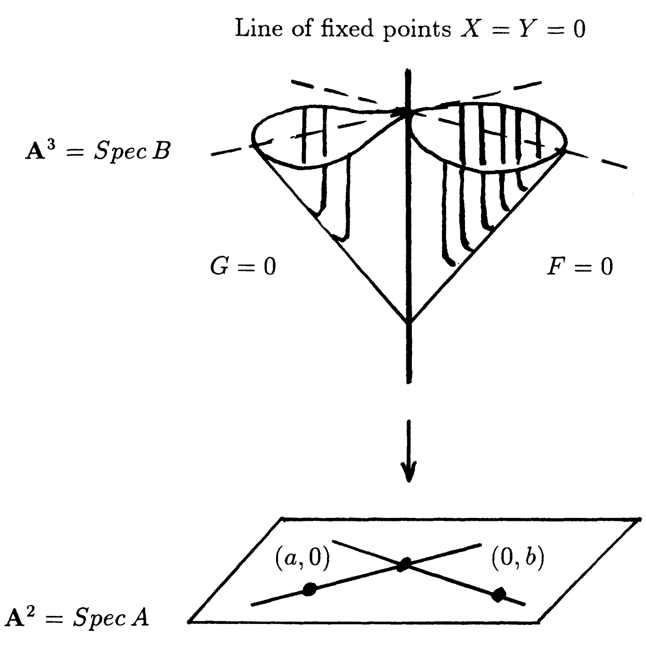

Fig. 1.

(As shown below, each H_n is a polynomial.) Observe that $a_{n+1} = 3a_n - a_{n-1}$, giving every other element of the Fibonacci sequence, and that $a_n \not\equiv 0 (\mathrm{mod} 3)$ for all n. We will show that, for $n \geq 2$, the derivations $D_{(H_n, H_{n+1})}$ are locally nilpotent of rank three and type (a_n, a_{n+1}), having kernel $k[H_n, H_{n+1}]$.

CLAIM 1. $H_n \in B$ and $H_n \notin (r)$ for all $n \geq 1$.

Proof. We show by induction on n that, given $n \geq 2$, the following properties hold for all k such that $2 \leq k \leq n$:

$$H_k \in B; H_k \notin (r); \text{and} (H_k^3 + r^{a_k}) \equiv 0 \pmod{H_{k-1}}.$$

The case $n = 2$ is clear, so assume these properties hold for $k = 2, \ldots, (n-1)$, with $n \geq 3$. Then $H_n \in B$, and

$$\begin{aligned} H_n H_{n-2} &\equiv H_{n-1}^3 + r^{a_{n-1}} \equiv r^{a_{n-1}} \pmod{H_{n-1}} \\ H_{n-1} H_{n-3} &\equiv H_{n-2}^3 + r^{a_{n-2}} \equiv 0 \pmod{H_{n-1}}. \end{aligned}$$

Therefore, modulo H_{n-1}, we have

$$\begin{aligned} (H_n^3 + r^{a_n}) r^{a_{n-2}} &\equiv H_n^3 r^{a_{n-2}} + r^{a_n} r^{a_{n-2}} \\ &\equiv H_n^3 r^{a_{n-2}} + r^{3a_{n-1}} \\ &\equiv H_n^3 r^{a_{n-2}} + (H_n H_{n-2})^3 \\ &\equiv H_n^3 (H_{n-2}^3 + r^{a_{n-2}}) \\ &\equiv 0 \pmod{H_{n-1}}. \end{aligned}$$

Since r is irreducible and $H_{n-1} \notin (r)$, we conclude that $(H_n^3 + r^{a_n}) \equiv 0 \pmod{H_{n-1}}$. Furthermore, $H_n H_{n-2} \equiv H_{n-1}^3 \not\equiv 0 \pmod{r}$, so $H_n \notin (r)$. □

Now define derivations $\delta_n = D_{(H_n, H_{n+1})}$ for $n \geq 1$.

CLAIM 2. Given $n \geq 2$, δ_n is locally nilpotent and $Ker(\delta_n) = k[H_n, H_{n+1}]$. Moreover, $\delta_n = \Delta(H_n, H_{n-1}, r)$, $\delta_n r = -H_n H_{n+1}$, and δ_n is irreducible.

Proof. We proceed by induction on n. Note first that δ_2 equals the $(2, 5)$ example Δ_1 constructed above, and this is obtained from $D = D_{(H_1, H_2)} = \delta_1$. Thus, we also have $\delta_2 r = \Delta_1 r = -FG_1 = -H_2 H_3$, and we see that the claim holds true for $n = 2$.

Consider the case $n \geq 3$. We assume that $\delta_{n-1} = \Delta(H_{n-1}, H_{n-2}, r)$, $\delta_{n-1} r = -H_{n-1} H_n$, δ_{n-1} is irreducible, and $\mathrm{Ker}(\delta_{n-1}) = k[H_{n-1}, H_n]$. It follows that $r \in \Omega(H_n, H_{n-1})$. Claim 1 shows that $\phi_n(r) := (H_n^3 + r^{a_n}) \in (H_{n-1})$. Moreover, since

$\gcd(3, a_n) = 1$ for all n, ϕ_n is irreducible as an element of $k[H_n, r] \cong k^{[2]}$, and thus ϕ_n is the minimal polynomial we need in constructing $\Delta(H_n, H_{n-1}, r)$. Therefore $\delta_n = \Delta(H_n, H_{n-1}, r)$ and $\delta_n r = -H_n H_{n+1}$. By induction, these two equalities hold for all $n \geq 2$. Thus, by Theorem 2, each δ_n is locally nilpotent.

Further, we have:

$$\begin{aligned}\delta_n(H_{n-1}) &= \delta_n\left(\frac{1}{H_{n+1}}(H_n^3 + r^{a_n})\right)\\ &= \frac{1}{H_{n+1}} \cdot a_n r^{a_n - 1} \cdot \delta_n r\\ &= -a_n H_n r^{a_n - 1}.\end{aligned}$$

Now H_n is irreducible, being a generator for $\operatorname{Ker}(\delta_{n-1})$. As in the preceding examples, if $\delta_n = f \cdot T$ for some polynomial f and some derivation T, then f must lie in the kernel of δ_n. Since $r \notin \operatorname{Ker}(\delta_n)$, we conclude that δ_n is reducible iff $\delta_n \equiv 0 \pmod{H_n}$.

Consider:

$$\begin{aligned}\delta_n H_{n-2} &= \delta_n\left(\frac{1}{H_n}(H_{n-1}^3 + r^{a_{n-1}})\right)\\ &= \frac{1}{H_n}\left(3H_{n-1}^2 \delta_n H_{n-1} + a_{n-1} r^{(a_{n-1})-1} \delta_n r\right)\\ &= \frac{1}{H_n}\left(3H_{n-1}^2(-a_n H_n r^{a_n - 1}) + a_{n-1} r^{(a_{n-1})-1}(-H_n H_{n+1})\right)\\ &= -3a_n H_{n-1}^2 r^{a_n - 1} - a_{n-1} r^{(a_{n-1})-1} H_{n+1}.\end{aligned}$$

Since $H_{n-2}H_n = (H_{n-1}^3 + r^{a_{n-1}})$, we see that $H_{n-1}^3 \equiv (-r^{a_{n-1}}) \pmod{H_n}$. Likewise, $H_{n-1}H_{n+1} = (H_n^3 + r^{a_n})$, and thus $H_{n-1}H_{n+1} \equiv r^{a_n} \pmod{H_n}$. It follows that, modulo H_n, we have

$$\begin{aligned}H_{n-1} \cdot \delta_n H_{n-2} &\equiv -3a_n H_{n-1}^3 r^{a_n - 1} - a_{n-1} r^{(a_{n-1})-1} H_{n-1} H_{n+1}\\ &\equiv -3a_n(-r^{a_{n-1}}) r^{a_n - 1} - a_{n-1} r^{(a_{n-1})-1}(r^{a_n})\\ &\equiv (3a_n - a_{n-1}) r^{a_n + (a_{n-1}) - 1}\\ &\equiv a_{n+1} r^{a_n + (a_{n-1}) - 1}\\ &\not\equiv 0 \pmod{H_n}.\end{aligned}$$

Therefore δ_n is irreducible. By Theorem 2 and the *Kernel Criterion*, it follows that $\operatorname{Ker}(\delta_n) = k[H_n, H_{n+1}]$. □

Finally, since δ_n is of type (a_n, a_{n+1}), we conclude that the rank of δ_n is three whenever $n \geq 2$.

4.3. Remarks

The two families of examples described above were also discovered independently by both Daniel Daigle and Peter Russell, following the appearance of the $(2,5)$ example in [5] and Daigle's proof of Theorem 4 above. In constrast to the the author's algebraic approach, their constructions were entirely geometric, yielding a sequence of blow-ups and blow-downs of $\mathbf{P}^2$, rather than explicit polynomials. Daigle used the theory of local trees, which he developed in his thesis. Russell used Hamburger-Noether tableaux. Russell also informed me of the existence of an example of type $(5,38)$. Using local slice constructions, this may be realized as follows.

Consider the $(2,5)$ example, denoted δ_2 (or Δ_1) above. The kernel of δ_2 is $k[F,G]$ for $F = H_2$ and $G = H_3$, and the polynomial $r' := (F^4 - Gr)$ lies in $\Omega(G,F)$ (r as above), since $\delta_2 r' = FG^2$. One checks that $(G^8 + (r')^5) \in (F)$, so that if $H = (1/F)(G^8 + (r')^5)$, then $\Delta(G,F,r')$ is of type $(5,38)$, with kernel $k[G,H]$.

5. The Graph of Kernels

We are actually interested in subrings A of B which occur as the kernel of some locally nilpotent derivation, rather than in any specific derivation of which A is the kernel. With this in mind, we will, in this section, rephrase some of our terms, results and questions in the language of graphs.

Define Γ to be the graph such that $\mathrm{vert}(\Gamma)$ is the set of all kernels of non-zero locally nilpotent k-derivations of B, and such that two vertices A and A' are connected by an edge iff there exist derivations D, D' of B with $\mathrm{Ker}(D) = A$, $\mathrm{Ker}(D') = A'$, and D' is obtained from D by means of a local slice construction. Let A_0 denote the vertex $k[X,Y]$, corresponding to the partial derivative $(\partial/\partial Z)$.

The group $GA_3(k)$ of k-automorphisms of B acts on Γ by conjugation, and we let $\mathcal{G}$ denote the quotient graph. (Observe that, by Rentschler's Theorem [7], the corresponding quotient graph in dimension 2 consists of a single vertex, namely, that corresponding to the partial derivative.) Let $\mathcal{G}_0$ denote the connected component of A_0 in $\mathcal{G}$. The following terminology will be used. (We do not distinguish between $A \in \mathrm{vert}(\Gamma)$ and its equivalence class in $\mathrm{vert}(\mathcal{G})$.)

1. The *rank* of a vertex A is the rank of any derivation D on B with $Ker(D) = A$.
2. A vertex A is *homogeneous* if there exists a homogeneous derivation D of B with $\mathrm{Ker}(D) = A$. The *type* of a homogeneous vertex is the type of the corresponding derivation. (It is possible that more than one vertex could be associated with a given type.)
3. A vertex A is *free* if there exists a locally nilpotent derivation D on B with $Ker(D) = A$ and $(imD) = (1)$.

Note that A_0 is the unique vertex of $\mathcal{G}$ of rank one. Moreover, the results of §3 show the following.

Proposition 1. *$\mathcal{G}_0$ contains every rank-two vertex of $\mathcal{G}$.*

We close with some questions.

Question 1 Is $\mathcal{G}$ connected ?

Question 2 Does every homogeneous vertex of $\mathcal{G}$ lie in $\mathcal{G}_0$?

Question 3 Does $\mathcal{G}_0$ contain a free vertex other than A_0 ?

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Department of Mathematics
University of Southern Indiana
Evans-Ville, Indiana 47712 USA
e-mail: freudenb.ucs@smtp.usi.edu