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UNIQUENESS OF SOLUTIONS OF AN ELLIPTIC SINGULAR BOUNDARY VALUE PROBLEM

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1. Introduction

Let $(\check{M},\check{g})$ be a Riemannian manifold ($n=\dim\check{M}\geq 2$), M a relatively compact domain in $\check{M}$, and $g:=\check{g}|_M$. In this paper, we study the following elliptic singular boundary value problem:

$$(*)_\infty \qquad \begin{cases} \Delta_g u = F(x,u) & \text{on} \quad M \\ u>0 & \\ u\to+\infty & \text{as} \quad r_{\partial M}\to 0, \end{cases}$$

where Δ_g is the Laplacian of g (i.e. $\Delta_g:=g^{ij}\nabla_{ij}$), $F(x,t)$ a nonnegative locally Hölder continuous function on $M\times[0,+\infty)$ satisfying certain conditions, and $r_{\partial M}$ is the distance function to the boundary ∂M.

In this decade, many authors investigated existence, behavior, uniqueness and nonexistence of solutions of this problem from the viewpoints of differential equation, conformal geometry, and probability theory. In this paper, we consider mainly the uniqueness of solutions of the problem $(*)_\infty$. For instance, in the most typical case when $F(x,t)=h(x)t^q$ ($q>1$), we show the following:

Theorem 1. *Let M be a relatively compact domain in $\check{M}$ with $(n-1)$-dimensional C^2-boundary, $g:=\check{g}|_M$, h a nonnegative locally Hölder continuous function on M, and q a number larger than* 1. *If h satisfies $h\sim {r_{\partial M}}^\ell$ near ∂M for some number $\ell>-2$, then the problem*

$$(**)_\infty \qquad \begin{cases} \Delta_g u = hu^q & \text{on} \quad M \\ u>0 & \\ u\to+\infty & \text{as} \quad r_{\partial M}\to 0, \end{cases}$$

possesses a unique solution.

Here and throughout this paper, we use the notation "$h\sim\tilde{h}$" to mean that $h/\tilde{h}$ is bounded between two positive constants (i.e. $C\tilde{h}\leq h\leq C'\tilde{h}$ for some $C>0$ and $C'>0$).

The assertion of Theorem 1 was known in the special cases when (1) $h \equiv 1$ on M in $\mathbf{R}^n$ ([10, Theorem 4], [4, Theorem 1]); (2) $h \sim 1$ on M in $\mathbf{R}^n$, and h is continuous on $\overline{M}$ ([9, Corollary], [2, Theorem 2.7], [14, Theorem 1.1]); (3) $h(x) \sim r_{\partial M}{}^{\ell}$ on $M = B_1(0)$ the unit ball in $\mathbf{R}^n$, and $h/r_{\partial M}{}^{\ell}$ is continuous on $\overline{M}$ for some number $\ell > -2$ ([13, Theorem 7.1 (II)]). Also in a certain general case including $F(x,t) = h(x)t^q$, the uniqueness was shown under the assumption of continuity similar to the above ones ([2, Theorems 2.4 and 2.7]). In these works, the uniqueness was given as a consequence of discribing the exact behaviors of the solutions by using the assumption on continuity on the boundary essentially. However, we emphasize here that we need only rough estimates for the solutions of the problem $(*)_\infty$ to show the uniqueness if $F(x,t)$ has polynomial order with respect to t. To see this, we improve, in Lemma 2.1, the uniqueness result of the previous paper [6]. This lemma plays a crucial role in this paper.

When $M = B_1(0)$ in $(\mathbf{R}^n, g_0)$, Ratto-Rigoli-Veron [13, Theorem 7.1 (I)] showed that, if $F(x,t) = h(x)t^{(n+2)/(n-2)}$ and h is radially nonincreasing with respect to a point in M, then the problem $(*)_\infty$ has at most one solution. For the proof, they used a method similar to Iscoe [4, Proposition 3.15] which is valid so long as (M,g) is a star-shaped domain in $(\mathbf{R}^n, g_0)$. By means of Lemma 2.1, we can show that, *even if* h *is not radially nonincreasing, the uniqueness holds when* $h \sim h_1$ *near the boundary for some radially nonincreasing* h_1*, and moreover, it is also true for some more general non-star-shaped domains, e.g. annulus domains* (see Theorem 3.6).

As we observed in [6], the equation

$$(*) \qquad \begin{cases} \Delta_g u = F(x,u) \\ u > 0 \end{cases} \quad \text{on} \quad M$$

often possesses a solution with mixed behavior, i.e. which is neither maximal nor asymptotic to any harmonic function. In some cases, such a solution can be regarded as a maximal solution of a Dirichlet problem with partially singular boundary value. Lemma 2.1 can be modified to be applicable to such solutions (see Lemma 4.1). As an application of this, we give a structure theorem of the scalar curvature equation on a compact Riemannian manifold punctured by a finite number of points.

Theorem 2. *Let* $(\overline{M}, \overline{g})$ *be a compact Riemannian manifold* ($n = \dim \overline{M} \geq 3$) *with positive scalar curvature,* $p_1, \ldots, p_k$ *points in* $\overline{M}$*, and* $(M,g) := (\overline{M} \setminus \{p_1, \ldots, p_k\}, \overline{g}|_{\overline{M}\setminus\{p_1,\ldots,p_k\}})$. *Set* $G_{p_i}(x) := G(p_i, x)$ $(\sim r_{p_i}(x)^{2-n})$*, where* G *is the Green function of the conformal Laplacian* $L_{\overline{g}} := -4\{(n-1)/(n-2)\}\Delta_{\overline{g}} + S_{\overline{g}}$. *Let* f *be a nonpositive smooth function on* M. *If* f *satisfies* $f \sim -r_{p_i}{}^{\ell_i}$ *near* p_i *for a number* $\ell_i > 2$ $(i = 1, \ldots, k)$*, then, for any* $\gamma = (\gamma_1, \ldots, \gamma_k) \in (0, +\infty]^k$*, the scalar curvature equation*

$$(f,M) \qquad \begin{cases} L_g u := -4\dfrac{n-1}{n-2}\Delta_g u + S_g u = f u^{(n+2)/(n-2)} & \text{on} \quad M \\ u > 0 \end{cases}$$

possesses a unique solution u_γ *such that* $u_\gamma(x)/G_{p_i}(x) \to \gamma_i$ *as* $x \to p_i$ $(i = 1, \dots, k)$ *and the metric* ${u_\gamma}^{4/(n-2)}g$ *is complete. Conversely, any solution* u *of the equation* (f,M) *such that* $u^{4/(n-2)}g$ *is complete coincides with* u_γ *for some* γ*. Namely, the space of complete conformal metrics on* M *with scalar curvature* f *is parametrized by* $(0,+\infty]^k$.

First, in Section 2, we prove our main lemma. In Section 3, we give a proof of Theorem 1 with more general style, and also consider various applications.

In Section 4, we modify Lemma 2.1 and study its application, and we prove Theorem 2 in Section 5.

2. A proof of main lemma

In this section, we first prove our main lemma. Here we call a solution U of the equation $(*)$ is maximal if and only if $U \geq u$ holds for any solution u of $(*)$.

Lemma 2.1. *Let* (M,g) *be an open Riemannian manifold, and* $F(x,t)$ *a nonnegative locally Hölder continuous function on* $M \times [0,+\infty)$ *satisfying the following conditions:*

(F.1) *For any* $x \in M$*,* $F(x,t)$ *is differentiable with respect to* t*,* $\partial F/\partial t(x,t)$ *is continuous on* $M \times [0,+\infty)$ *and nondecreasing with respect to* t*;*

(F.2) *There exists a number* $q > 1$ *such that*

$$\delta F(x,t) \leq F(x,\delta t) \leq \delta^q F(x,t) \qquad \text{for} \quad \delta \geq 1, (x,t) \in M \times [0,+\infty).$$

Suppose the equation $(*)$ *possesses a maximal solution* U*. Then, for any solution* u *of the equation* $(*)$*, if* $u \sim U$ *on* M*, then* $u \equiv U$ *on* M*.*

Proof. First, note that the assumptions (F.1) and (F.2) imply

$$F(x,t) \leq t\frac{\partial F}{\partial t}(x,t) \leq qF(x,t) \qquad \text{for} \quad (x,t) \in M \times [0,+\infty).$$

In particular, for any $x \in M$, "$F(x,t) > 0, \partial F/\partial t(x,t) > 0$ for any $t > 0$ and $F(x,0) = 0$" or "$F(x,t) = 0$ for any $t \geq 0$".

Set $\beta := \sup_M(U/u)$, and $\delta_0 := \beta/(\beta-1)$. Then, by the assumption $u \sim U$, we have $\beta < +\infty$. Suppose $u \not\equiv U$. Then $\beta > 1$ and, by the strong maximum principle, it holds that $u < U$. Set $v_+ := \delta U$ and $v_- := \delta(U-u)$, where δ is a positive number larger than δ_0 and chosen later. Then clearly $v_\pm > 0$ and $v_+ - v_- = \delta u > 0$.

Moreover, we get

$$\Delta_g v_+ = \delta\Delta_g U = \delta F(x,U) \le F(x,\delta U) = F(x,v_+),$$

namely, v_+ is a supersolution of the equation $(*)$. On the other hand,

$$\Delta_g v_- = \delta(\Delta_g U - \Delta_g u) = \delta(F(x,U) - F(x,u)).$$

Now, we claim that there is a positive number $\delta > \delta_0$ such that

$$\Delta_g v_- = \delta(F(x,U) - F(x,u)) \ge F(x,\delta(U-u)) = F(x,v_-),$$

namely, v_- is a subsolution of the equation $(*)$.

For any $\delta > \delta_0$, $x \in M$ and $u > 0$, set

$$F_1(s) := \delta(F(x,s) - F(x,u)) - F(x,\delta(s-u)).$$

Then

$$F_1{}'(s) = \delta\left(\frac{\partial F}{\partial t}(x,s) - \frac{\partial F}{\partial t}(x,\delta(s-u))\right).$$

Since we assume that $\partial F/\partial t(x,\cdot)$ is monotonically nondecreasing,

$$\begin{array}{ll} \dfrac{\partial F}{\partial t}(x,s) \ge \dfrac{\partial F}{\partial t}(x,\delta(s-u)) & \text{for} \quad s \le \dfrac{\delta}{\delta-1}u \ \ (<\beta u). \\ \text{(resp. } \le \text{)} & \text{(resp. } \ge \text{)} \end{array}$$

Therefore

$$F_1(U) \ge \min\{F_1(u), F_1(\beta u)\} = \min\{0, F_1(\beta u)\}.$$

Set

$$F_2(\delta) := F_1(\beta u) = \delta(F(x,\beta u) - F(x,u)) - F(x,\delta(\beta-1)u).$$

Then, by the convexity of $F(x,\cdot)$, we have

$$\begin{aligned} F_2(\delta_0) &= \delta_0(F(x,\beta u) - F(x,u)) - F(x,\beta u) \ge \delta_0 u \frac{F(x,\beta u) - F(x,u)}{\beta u - u} \\ &\ge \delta_0 u \frac{\partial F}{\partial t}(x,u) \ge \delta_0 F(x,u), \end{aligned}$$

and

$$\begin{aligned} F_2(\delta) - F_2(\delta_0) &= (\delta - \delta_0)(F(x, \beta u) - F(x, u)) - (F(x, \delta(\beta - 1)u) - F(x, \beta u)) \\ &\geq 0 - \frac{\partial F}{\partial t}(x, \delta(\beta - 1)u)\{\delta(\beta - 1)u - \beta u\} \\ &\geq -q\frac{F(x, \delta(\beta - 1)u)}{\delta(\beta - 1)u}\{\delta(\beta - 1) - \beta\}u \\ &\geq -q\frac{\{\delta(\beta - 1)\}^q F(x, u)}{\delta(\beta - 1)u}(\beta - 1)(\delta - \delta_0)u \\ &= -q\delta^{q-1}(\beta - 1)^q(\delta - \delta_0)F(x, u) \end{aligned}$$

from which it follows that

$$F_2(\delta) \geq \{\delta_0 - q\delta^{q-1}(\beta - 1)^q(\delta - \delta_0)\}F(x, u).$$

Remark here that

$$\delta_0 - q\delta^{q-1}(\beta - 1)^q(\delta - \delta_0) \to \delta_0(> 0) \qquad \text{as} \quad \delta \to \delta_0.$$

Then, for any δ enough close to δ_0, we have $F_2(\delta) \geq 0$ and v_- is a subsolution of the equation $(*)$.

By the method of supersolutions and subsolutions, the equation $(*)$ possesses a solution v satisfying $v_+ \geq v \geq v_-$.

Now, by the definition of β, there exists a sequence $\{x_i\}_{i \in \mathbf{N}}$ of points in M such that $\lim_{i \to +\infty}(U(x_i)/u(x_i)) = \beta$. Hence we get

$$\frac{v(x_i)}{u(x_i)} \geq \frac{v_-(x_i)}{u(x_i)} = \delta\left(\frac{U(x_i)}{u(x_i)} - 1\right) \to \delta(\beta - 1) > \beta \qquad \text{as} \quad i \to +\infty$$

from which it follows that, for any i large enough,

$$\frac{v(x_i)}{u(x_i)} > \beta \geq \frac{U(x_i)}{u(x_i)},$$

namely, $v(x_i) > U(x_i)$. This contradicts the assumption that U is maximal. Therefore we have $u \equiv U$. □

A typical example we can apply our lemma is of the form $F(x, t) = \sum_{i=1}^{k} h_i(x)t^{q_i} + h_0(x)t$ $(q_i > 1)$ with nonnegative h_i's. When $k = 1$, this assertion was proved in [6, Theorem 1].

In Lemma 2.1, we assumed the existence of the maximal solution of the equation $(*)$. Indeed we may assume this in many cases. We state it and give a proof here under the assumption which is suitable for our cases.

Proposition 2.2. *Let $(\check{M}, \check{g})$ be a Riemannian manifold, M a relatively compact domain in $\check{M}$ with $(n-1)$-dimensional C^2-boundary, and $g := \check{g}|_M$.*

Let $h_+(x)$ be a nonnegative C^1-function on M which is positive near ∂M, $h_-(x)$ a positive C^1-function on M, and $F(t)$ a nonnegative nondecreasing C^1-function on $[0,+\infty)$ such that

$$\text{(F.3)} \qquad \int_t^{+\infty} \left(\int_0^s F(r)dr \right)^{-1/2} ds < +\infty \qquad \text{for} \quad t > 0.$$

Suppose $F(x,t)$ is a nonnegative locally Hölder continuous function on $M \times [0,+\infty)$ satisfying (F.1), (F.2) *and*

$$h_+(x)F(t) \le F(x,t) \le h_-(x)F(t) \qquad \text{for} \quad (x,t) \in M \times [0,+\infty).$$

If there exists a positive subsolution u_- of the equation $(*)$, then the equation $(*)$ possesses a solution U which is maximal in the sense that $U \ge u$ for any solution u of the equation $(*)$.

Proof. Let Ω_1 be a relatively compact domain of M such that $f_+ > 0$ on $M \setminus \Omega_1$, $\{\Omega_i\}_{i \in \mathbf{N}}$ a sequence of relatively compact domains of M with smooth boundaries which satisfies $\Omega_i \subset\subset \Omega_{i+1}$ and $\cup_{i\in\mathbf{N}}\Omega_i = M$. Set $C_{i+} := \min_{\overline{\Omega}_i \setminus \Omega_1} h_+$ and $C_{i-} := \max_{\overline{\Omega}_i} h_-$. By the same way as Keller [8, Theorem III] (cf. [12]), for any $i > 1$, the equation

$$\begin{cases} \Delta_g u = C_{i\pm}F(u) \\ u > 0 \end{cases} \qquad \text{on} \quad \Omega_i$$

possesses a solution $v_{i\pm}$ such that $v_{i\pm} \to +\infty$ as $r_{\partial\Omega_i} \to 0$ and $v_{i+} \ge v_{i-}$. Set $u_{i-} := v_{i-}$. Then we have

$$\Delta_g u_{i-}(x) = C_{i-}F(u_{i-}(x)) \ge h_-(x)F(u_{i-}(x)) \ge F(x,u_{i-}(x)) \qquad \text{for} \quad x \in \Omega_i,$$

namely, u_{i-} is a subsolution of the equation $(*)$ on Ω_i. On the other hand, let η be a positive smooth function on $\overline{M}$ such that $-\Delta_g\eta$ is bounded below by a positive number on M (e.g. choose the first Dirichlet eigenfunction of $-\Delta_g$ on a domain including $\overline{M}$). Set $u_{i+} := v_{i+} + \beta_i\eta$, where $\beta_i := \max_{\overline{\Omega}_1}(C_{i+}F(v_{i+})/|\Delta_g\eta|)$. Then we get

$$\Delta_g u_{i+}(x) = C_{i+}F(v_{i+}(x)) + \beta_i\Delta_g\eta \le 0 \le F(x,u_{i+}) \qquad \text{for} \quad x \in \Omega_1,$$

and

$$\begin{aligned} \Delta_g u_{i+}(x) &= C_{i+}F(v_{i+}(x)) + \beta_i\Delta_g\eta \\ &\le h_+(x)F(v_{i+}(x)) + 0 \le F(x,v_{i+}) \le F(x,u_{i+}) \qquad \text{for} \quad x \in \Omega_i \setminus \Omega_1, \end{aligned}$$

namely, u_{i+} is a supersolution of the equation $(*)$ on Ω_i. Since $u_{i+} \geq u_{i-} > 0$ on Ω_i, by the method of supersolutions and subsolutions, the equation $(*)$ possesses a solution u_i on Ω_i satisfying $u_{i+} \geq u_i \geq u_{i-}$.

Since $u_i \to +\infty$ as $r_{\partial\Omega_i} \to 0$, and both u_{i+1} and u_- are bounded above in Ω_i, by the maximum principle, we get $u_i \geq u_{i+1}$ and $u_i \geq u_-$ in Ω_i, that is, $\{u_i\}_{i>1}$ is monotonically decreasing and bounded below by u_-. Therefore, if we set $U := \lim_{i\to+\infty} u_i$, then U is a solution of the equation $(*)$. By the maximum principle again, it is clear that $u_i \geq u$ for any solution u of the equation $(*)$. Hence we see that $U \geq u$ or U is the maximal solution. □

REMARK 2.3. The existence of the maximal solution U of the equation $(*)$ is shown by Proposition 2.2 under the assumption of each theorem in Section 3. In particular, $U \to +\infty$ as $r_{\partial M} \to 0$. Hence, to prove the uniqueness of solutions of the problem $(*)_\infty$, we have only to show that any two solutions u and $\tilde{u}$ satisfy $u \sim \tilde{u}$ near ∂M. Indeed, if we can show this, then any solution u of the problem $(*)_\infty$ satisfies $u \sim U$ near ∂M. Since u is bounded below by a positive constant, we get $u \sim U$ on M. Now, by Lemma 2.1, we have $u \equiv U$.

3. An elliptic singular boundary value problem

In this section, we apply our Lemma 2.1 to an elliptic singular boundary value problem $(*)_\infty$. First, we show the following result which includes the most typical case as in Theorem 1.

Theorem 3.1. *Let $(\check{M}, \check{g})$ be a Riemannian manifold, M a relatively compact domain in $\check{M}$ with $(n-1)$-dimensional C^2-boundary, and $g := \check{g}|_M$. Suppose $F(x,t)$ is a nonnegative locally Hölder continuous function on $M \times [0,+\infty)$ satisfying* (F.1), (F.2) *and*

$$F(x,t) \begin{cases} \geq C_+ \cdot r_{\partial M}(x)^\ell t^q & \text{for} \quad (x,t) \in (B_R(\partial M) \cap M) \times [0,+\infty) \\ \leq C_- \cdot r_{\partial M}(x)^\ell t^q & \text{for} \quad (x,t) \in M \times [0,+\infty) \end{cases}$$

for positive constants C_+, C_-, R, $q > 1$ and a number $\ell > -2$, where $B_R(\partial M)$ denotes R-neighborhood of ∂M. Then the problem $(*)_\infty$ possesses a unique solution u. Moreover u satisfies the estimate $u \sim {r_{\partial M}}^{-(\ell+2)/(q-1)}$.

Proof. It is clear that $F(t) = t^q$ $(q > 1)$ satisfies the condition (F.3). Therefore, by Proposition 2.2 and Lemma 3.2 below, the equation $(*)$ possesses a maximal solution U such that $U \to +\infty$ as $r_{\partial M} \to 0$.

Let u be a solution of the problem $(*)_\infty$. Standard calculations give a priori upper estimate $u \leq C_1 {r_{\partial M}}^{-(\ell+2)/(q-1)}$ near ∂M for a positive constant C_1 (see e.g. [13, Proposition 2.3]). We may assume $|\Delta_g r_{\partial M}| \leq C_2$ on $\overline{B}_R(\partial M)$ for a positive constant C_2.

LOWER ESTIMATE IN THE CASE $\ell \geq 0$. For this estimate, the subsolutions in Loewner-Nirenberg [10, Theorem 4] can be generalized to be adaptable. Indeed, let R' be a positive number given by $R' := \min\{R, (\alpha+1)/2C_2\}$, where $\alpha := (\ell+2)/(q-1)$. For any $0 \leq \epsilon \leq R'$, set

$$u_{\epsilon -} := \gamma_-\{(r_{\partial M}+\epsilon)^{-\alpha} - (R'+\epsilon)^{-\alpha}\} \qquad \text{on} \quad B_{R'}(\partial M) \cap M,$$

where $\gamma_- := (C_3/C_-)^{1/(q-1)}$ and $C_3 := \alpha(\alpha+1-2R'C_2)$. By direct computation, we have

$$\begin{aligned} &{u_{\epsilon -}}^{-q} \Delta_g u_{\epsilon -} \\ &\quad = {\gamma_-}^{1-q} \alpha (r_{\partial M}+\epsilon)^{-\alpha-2} \\ &\qquad \times \{\alpha+1-(r_{\partial M}+\epsilon)\Delta_g r_{\partial M}\}\{(r_{\partial M}+\epsilon)^{-\alpha} - (R'+\epsilon)^{-\alpha}\}^{-q} \\ &\quad \geq {\gamma_-}^{1-q} \alpha {r_{\partial M}}^{\ell} (r_{\partial M}+\epsilon)^{-\alpha-2+\alpha q - l} (\alpha+1-2R'C_2) \left\{1 - \left(\frac{r_{\partial M}+\epsilon}{R'+\epsilon}\right)^{\alpha}\right\}^{-q} \\ &\quad \geq {\gamma_-}^{1-q} C_3 {r_{\partial M}}^{\ell}. \end{aligned}$$

Hence, for any $0 < \epsilon \leq R'$,

$$\Delta_g u_{\epsilon -}(x) \geq {\gamma_-}^{1-q} C_3 {r_{\partial M}}^{\ell} u_{\epsilon -}(x)^q \geq {\gamma_-}^{1-q} C_3 {C_-}^{-1} F(x, u_{\epsilon -}(x)) = F(x, u_{\epsilon -}(x)),$$

namely $u_{\epsilon -}$ is a subsolution of the equation $(*)$ on $B_{R'}(\partial M) \cap M$ which is finite on ∂M and equal to 0 on $\partial B_{R'}(\partial M) \cap M$. Since ϵ can be chosen arbitrarily small, we get

$$u \geq \lim_{\epsilon \to +0} u_{\epsilon -} = \gamma_-({r_{\partial M}}^{-\alpha} - {R'}^{-\alpha}) \geq C_4 {r_{\partial M}}^{-(\ell+2)/(q-1)} \qquad \text{near} \quad \partial M$$

for a positive constant C_4.

LOWER ESTIMATE IN THE CASE $-2 < \ell < 0$. Let R' be a positive number given by $R' := \min\{R, |\ell|/2C_2\}$. For any $\epsilon \geq 0$, set

$$u_{\epsilon -} := \gamma_-({r_{\partial M}}^{\alpha(q-1)/2} + \epsilon)^{-2/(q-1)} \qquad \text{on} \quad B_{R'}(\partial M) \cap M,$$

where $\gamma_- := \min\{(C_5/C_-)^{1/(q-1)}, m_0 {R'}^{\alpha}\}$, $C_5 := \alpha(\alpha+1-R'C_2)$ $(\geq \alpha\{\alpha+(\ell+2)/2\} > 0)$, $m_0 := \min_M u$ and α is as before. By direct computation, we have

$$\begin{aligned} &{u_{\epsilon -}}^{-q} \Delta_g u_{\epsilon -} \\ &\quad = {\gamma_-}^{1-q} \alpha {r_{\partial M}}^{\alpha(q-1)/2-2} \\ &\qquad \times \left\{\epsilon\left(\frac{|\ell|}{2} - r_{\partial M}\Delta_g r_{\partial M}\right) + {r_{\partial M}}^{\alpha(q-1)/2}(\alpha+1-r_{\partial M}\Delta_g r_{\partial M})\right\} \end{aligned}$$

$$
\begin{aligned}
&\geq \gamma_-{}^{1-q}\alpha r_{\partial M}{}^{(l-2)/2}\left\{\epsilon\left(\frac{|\ell|}{2}-R'C_2\right)+r_{\partial M}{}^{(\ell+2)/2}(\alpha+1-R'C_2)\right\}\\
&\geq \gamma_-{}^{1-q}\alpha r_{\partial M}{}^{(\ell-2)/2+(\ell+2)/2}(\alpha+1-R'C_2)\\
&\geq \gamma_-{}^{1-q}C_5 r_{\partial M}{}^{\ell}.
\end{aligned}
$$

Hence, for any $\epsilon > 0$,

$$
\begin{aligned}
\Delta_g u_{\epsilon-}(x) &\geq \gamma_-{}^{1-q}C_5 r_{\partial M}(x)^{\ell} u_{\epsilon-}(x)^q\\
&\geq \gamma_-{}^{1-q}C_5 C_-{}^{-1}F(x,u_{\epsilon-}(x)) \geq F(x,u_{\epsilon-}(x)),
\end{aligned}
$$

namely $u_{\epsilon-}$ is a subsolution of the equation $(*)$ on $B_{R'}(\partial M)\cap M$ which is finite on ∂M and satisfies $u_{\epsilon-} \leq \gamma_- R'^{-\alpha} \leq m_0 \leq u$ on $\partial B_{R'}(\partial M)\cap M$. Since ϵ can be chosen arbitrarily small, we get

$$
u \geq \lim_{\epsilon\to+0} u_{\epsilon-} = \gamma_-\cdot r_{\partial M}{}^{-(\ell+2)/(q-1)} \qquad \text{on} \quad B_{R'}(\partial M)\cap M.
$$

Therefore, any solution of the problem $(*)_\infty$ satisfies $u \sim r_{\partial M}{}^{-(\ell+2)/(q-1)}$. Now, by Remark 2.3, we get the assertion of the theorem. □

Lemma 3.2. *Let (M,g) and $F(x,t)$ be as in Theorem 3.1. Then there exists a positive subsolution u_- of the equation $(*)$ which tends to $+\infty$ as $r_{\partial M}\to 0$.*

Proof. It is enough to show the case when $-2 < \ell < 0$. Let α be as before, $R' := \min\{R, (\alpha+1)/2C_2\}$, and χ a nonnegative smooth function on M such that $\chi\equiv 1$ on $B_{R'}(\partial M)\cap M$ and $\chi \equiv 0$ on $M\setminus B_{2R'}(\partial M)$. By direct computation, we have

$$
\begin{aligned}
\Delta_g(\chi\cdot r_{\partial M}{}^{-\alpha}) &= \Delta_g(r_{\partial M}{}^{-\alpha}) = \alpha(\alpha+1-r_{\partial M}\Delta_g r_{\partial M})r_{\partial M}{}^{-\alpha-2}\\
&\geq C_6 r_{\partial M}{}^{-\alpha-2} \qquad \text{on} \quad B_{R'}(\partial M)\cap M,
\end{aligned}
$$

where $C_6 := \alpha(\alpha+1-R'C_2)$. On the other hand, it is clear that $\Delta_g(\chi\cdot r_{\partial M}{}^{-\alpha})\equiv 0$ on $M\setminus B_{2R'}(\partial M)$. Hence there is a positive constant C_7 such that $\Delta_g(\chi\cdot r_{\partial M}{}^{-\alpha}) \geq -C_7$ on M. Let η be as in the proof of Proposition 2.2, and set

$$
u_- := \gamma\{\chi\cdot r_{\partial M}{}^{-\alpha} + \beta(\delta-\eta)\},
$$

where $\gamma := \{C_8/(1+\beta\delta R'^{\alpha})^q C_-\}^{1/(q-1)}$, $C_8 := \min\{C_6, C_7 R'^{\alpha q - l}\}$, $\beta := 2C_7/\min_{x\in\overline{M}}|\Delta_g\eta|$ and $\delta := 2\max_{x\in\overline{M}}\eta$. Clearly $u_- > 0$ on M. By direct computation, we have

$$
\begin{aligned}
\Delta_g u_- &= \gamma\{\Delta_g(r_{\partial M}{}^{-\alpha}) + \beta(-\Delta_g\eta)\} \geq \gamma\Delta_g(r_{\partial M}{}^{-\alpha}) \geq \gamma C_6 r_{\partial M}{}^{-\alpha-2},\\
u_- &\leq \gamma(r_{\partial M}{}^{-\alpha}+\beta\delta) \leq \gamma(1+\beta\delta R'^{\alpha})r_{\partial M}{}^{-\alpha},
\end{aligned}
$$

and hence

$$\begin{aligned} u_-^{-q}\Delta_g u_- &\geq \gamma^{1-q}C_6(1+\beta\delta R'^{\alpha})^{-q}r_{\partial M}{}^{-\alpha-2+q\alpha} \\ &\geq \gamma^{1-q}C_8(1+\beta\delta R'^{\alpha})^{-q}r_{\partial M}{}^{\ell} = C_- \cdot r_{\partial M}{}^{\ell} \qquad \text{on} \quad B_{R'}(\partial M)\cap M. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \Delta_g u_- &= \gamma\{\Delta_g(\chi\cdot r_{\partial M}{}^{-\alpha}) + \beta(-\Delta_g\eta)\} \geq \gamma(-C_7+2C_7) \geq \gamma C_7, \\ u_- &\leq \gamma(R'^{-\alpha}+\beta\delta) \leq \gamma R'^{-\alpha}(1+\beta\delta R'^{\alpha}), \end{aligned}$$

and hence

$$\begin{aligned} u_-^{-q}\Delta_g u_- &\geq \gamma^{1-q}C_7R'^{\alpha q}(1+\beta\delta R'^{\alpha})^{-q} \\ &\geq \gamma^{1-q}C_8R'^{\ell}(1+\beta\delta R'^{\alpha})^{-q} \\ &= C_-R'^{\ell} \geq C_-\cdot r_{\partial M}{}^{\ell} \qquad \text{on} \quad M\setminus B_{R'}(\partial M). \end{aligned}$$

Therefore

$$\Delta_g u_-(x) \geq C_-\cdot r_{\partial M}(x)^{\ell}u_-(x)^q \geq F(x,u_-(x)) \qquad \text{for} \quad x\in M,$$

namely u_- is a subsolution of the equation $(*)$. In particular, $u_- \longrightarrow +\infty$ as $r_{\partial M} \to 0$. □

We note here that the assumption $\ell > -2$ is essential in Lemma 3.2. Indeed, if $F(x,t) \geq C'_+ \cdot r_{\partial M}{}^{-2}t^q$ for some $C'_+ > 0$, then not only the problem $(*)_\infty$ but also the equation $(*)$ possesses no positive (sub-)solutions even when ∂M is not regular (cf. [7]).

Now, suppose $n = \dim \check{M} \geq 3$. Denote the scalar curvature of g $(= \check{g}|_M)$ by S_g. It is well known that a smooth function f on M can be realized as the scalar curvature of some metric $\tilde{g}$ conformal to g, if and only if there exists a smooth solution u of the scalar curvature equation

$$(f,M) \qquad \begin{cases} L_g u := -4\dfrac{n-1}{n-2}\Delta_g u + S_g u = f u^{(n+2)/(n-2)} \\ u > 0 \end{cases} \qquad \text{on} \quad M.$$

Indeed, the metric $\tilde{g} = u^{4/(n-2)}g$ has the scalar curvature $S_{\tilde{g}} = f$.

On the other hand, if $(\check{M},\check{g})$ has nonnegative scalar curvature $S_{\check{g}}$, then there exists a metric g' on M $(\subset\subset \check{M})$ which is pointwise conformal and uniformly equivalent to g and has vanishing scalar curvature. Therefore, by the transformation rule of the conformal Laplacian L_g (cf. [11]), the scalar curvature equation (f,M)

is equivalent to

$$\begin{cases} L_{g'}v := -4\dfrac{n-1}{n-2}\Delta_{g'}v = fv^{(n+2)/(n-2)} \\ v > 0 \end{cases} \quad \text{on} \quad M,$$

and the condition $u \to +\infty$ as $r_{\partial M} \to 0$ is equivalent to $v \to +\infty$ as $r_{\partial M} \to 0$. Hence Theorem 3.1 yields the following

Corollary 3.3. *Let $(\check{M}, \check{g})$ be a Riemannian manifold ($n = \dim \check{M} \geq 3$) with nonnegative scalar curvature, M a relatively compact domain in $\check{M}$ with $(n-1)$-dimensional C^2-boundary, and $g := \check{g}|_M$. Let f be a nonpositive smooth function on M. If f satisfies $f \sim -r_{\partial M}{}^{\ell}$ near ∂M for a nonnegative number ℓ, then any solution u of the equation (f, M) satisfies the estimate $u \geq C_9 r_{\partial M}{}^{-(\ell+2)(n-2)/4}$ for some positive constant C_9, and the metric $u^{4/(n-2)}g$ $(= v^{4/(n-2)}g')$ is complete. In particular, u coincides with the maximal solution of (f, M).*

In [6, Theorem 2], we gave a uniqueness result for the case when $\dim \partial M < n-2$. The result above can be regarded as a correspondent for $\dim \partial M = n-1$. In the case $f \equiv -1$, this result was shown by Andresson-Chrusciel-Friedrich [1, Theorem 1.2].

Next, we consider $F(x,t)$ of another type.

Theorem 3.4. *Let (M, g) be as in Theorem 3.1, $F(t)$ a nonnegative C^1-function on $[0, +\infty)$ satisfying both of the following conditions:*
(F.4) *$F(t) > 0$ and $F'(t) > 0$ for any $t > 0$, $F(0) = 0$, and*

$$\delta F(t) \leq F(\delta t) \qquad \text{for} \quad \delta \geq 1, t \geq 0;$$

(F.5) *There exist numbers $q' > 1$ and $T \geq 0$ such that*

$$\delta^{q'} F(t) \leq F(\delta t) \qquad \text{for} \quad \delta \geq 1, t \geq T.$$

Suppose $F(x,t)$ is a nonnegative locally Hölder continuous function on $M \times [0, +\infty)$ satisfying (F.1), (F.2) *and*

$$F(x,t) \begin{cases} \geq C_+ F(t) & \text{for} \quad (x,t) \in (B_R(\partial M) \cap M) \times [0, +\infty) \\ \leq C_- F(t) & \text{for} \quad (x,t) \in M \times [0, +\infty) \end{cases}$$

for positive constants C_+, C_- and R. Then the problem $(*)_\infty$ possesses a unique solution.

Proof. The conditions (F.4) and (F.5) imply the condition (F.3) (cf. [2, Lemma 2.1]). By the same way as Keller [8, Theorem III], the equation

$$\begin{cases} \Delta_g u = C_- F(u) \\ u > 0 \end{cases} \quad \text{on} \quad M$$

possesses a solution u_- such that $u_- \to +\infty$ as $r_{\partial M} \to 0$. u_- satisfies

$$\Delta_g u_-(x) \geq C_- F(u_-(x)) \geq F(x, u_-(x)) \qquad \text{for} \quad x \in M,$$

namely u_- is a subsolution of the equation $(*)$. Hence, by Proposition 2.2, the equation $(*)$ possesses a maximal solution U such that $U \to +\infty$ as $r_{\partial M} \to 0$.

Define a function $\phi(r)$ by

$$\phi^{-1}(t) := \int_t^{+\infty} \left(2 \int_0^s F(r) dr \right)^{-1/2} ds.$$

Apply the proofs of Bandle-Marcus [2, Lemma 2.2 and Theorem 2.3] to an arbitrary solution u of the problem $(*)_\infty$, but replace the radially symmetric supersolution $V(|x - z_y|)$ in the proof of [2, Theorem 2.3] by $\phi(\beta(R^2 - r_{z_y}(x)^2))$ with a positive number β small enough. Then, by the same way as in the proof of Theorem 3.1 and using Lemma 3.5 below, we can show that, $u/\phi(r_{\partial M})$ is bounded between two positive constants, and we have $u \equiv U$. □

Lemma 3.5. *Let ϕ be as above. For any constant $a > 1$, there exist positive constants $C_{10} < 1$ and $R > 0$ such that $C_{10}\phi(r) \leq \phi(a^{1/2} r) \leq \phi(r)$ for $r \leq R$.*

Proof. By the assumption (F.4) and (F.5), we have

$$\begin{aligned} \int_0^s a^{2q'/(q'-1)} F(r) dr &\leq \int_T^s F(a^{2/(q'-1)} r) dr + \int_0^T F(a^{2q'/(q'-1)} r) dr \\ &= a^{-2/(q'-1)} \int_{a^{2/(q'-1)} T}^{a^{2/(q'-1)} s} F(r) dr \\ &\quad + a^{-2/(q'-1)} \int_0^{a^{2/(q'-1)} T} F(a^2 r) dr \\ &= a^{-2/(q'-1)} \left(\int_0^{a^{2/(q'-1)} s} F(r) dr + C_{11} \right) \qquad \text{for} \quad s \geq T, \end{aligned}$$

where

$$C_{11} := \int_0^{a^{2/(q'-1)} T} \left(F(a^2 r) - F(r) \right) dr.$$

Set $T_1 := \max\{C_{11}/(a-1)F(1) + 1, T\}$. Then, for any $s \geq T_1$,

$$\int_0^{a^{2/(q'-1)}s} F(r)dr \geq \int_1^{a^{2/(q'-1)}s} r \cdot F(1)dr \geq F(1)(T_1 - 1) \geq \frac{C_{11}}{a-1},$$

and hence

$$\int_0^{a^{2/(q'-1)}s} F(r)dr + C_{11} \leq a \int_0^{a^{2/(q'-1)}s} F(r)dr.$$

Therefore, for any $t \geq T_1$,

$$\begin{aligned}
&\int_t^{+\infty} \left(2 \int_0^s a^{2q'/(q'-1)} F(r)dr\right)^{-1/2} ds \\
&\geq \int_t^{+\infty} \left(2a^{(q'-3)/(q'-1)} \int_0^{a^{2/(q'-1)}s} F(r)dr\right)^{-1/2} ds \\
&= a^{-(q'+1)/2(q'-1)} \int_{a^{2/(q'-1)}t}^{+\infty} \left(2 \int_0^s F(r)dr\right)^{-1/2} ds,
\end{aligned}$$

from which it holds that $a^{-1/2}\phi^{-1}(t) \geq \phi^{-1}(a^{2/(q'-1)}t)$. Set $R := a^{-1/2}\phi^{-1}(T_1)$. Note here that ϕ is monotonically decreasing. Then, for any $r \leq R$, since $\phi(a^{1/2}r) \geq T_1$, we have

$$\begin{aligned}
\phi(r) &\geq \phi(a^{1/2}r) = a^{-2/(q'-1)}\phi(\phi^{-1}(a^{2/(q'-1)}\phi(a^{1/2}r))) \\
&\geq a^{-2/(q'-1)}\phi(a^{-1/2}\phi^{-1}(\phi(a^{1/2}r))) = a^{-2/(q'-1)}\phi(r).
\end{aligned}$$

This completes the proof. □

Now, we consider the case of star-shaped domains or domains satisfying a certain weaker geometric condition. Here we say a domain M in $\mathbf{R}^n$ is star-shaped with respect to a point $p \in M$ if and only if it holds that $\{p + t(x-p) \in \mathbf{R}^n | 0 \leq t < 1\} \subset M$ for any $x \in \partial M$. We prove the following theorem by combining our Lemma 2.1 with the idea of Iscoe [4, Proposition 3.15]. It is applicable to not only star-shaped domains but also to a wider class of domains e.g. annulus domains as we mentioned in Introduction.

Theorem 3.6. *Let M be a relatively compact domain in $(\mathbf{R}^n, g_0)$ with C^2-boundary such that $M = M_1$ or $M = M_1 \setminus \cup_{i=2}^k \overline{M}_i$, where M_1 is a star-shaped domain with respect to a point $p_1 \in M_1$, M_i is a relatively compact subdomain of M_1 which is star-shaped with respect to a point $p_i \in M_i$ $(i = 2, \ldots, k)$ and $\overline{M}_i \cap \overline{M}_{i'} = \emptyset$ for any $i \neq i'$ $(i, i' = 2, \ldots, k)$. Let $h_1(x)$ be a positive continuous function on M_1 such that $h_1(p_1 + s(x - p_1))$ is nonincreasing with respect to $s \in [0, 1)$ for any*

$x \in \partial M_1$, $h_i(x)$ a positive continuous function on $M_1 \setminus \overline{M}_i$ such that $h_i(p_i + s(x - p_i))$ is nondecreasing with respect to $s \in (1, +\infty)$ for any $x \in \partial M_i$ $(i = 2, \ldots, k)$, and $F_i(t)$ a nonnegative C^1-function on $[0, +\infty)$ satisfying the following condition with $F(t) = F_i(t)$ $(i = 1, \ldots, k)$:

(F.6) *$F(t) > 0$ and $F'(t) > 0$ for any $t > 0$, $F(0) = 0$, and there exists a number $q' > 1$ such that*

$$\delta^{q'} F(t) \leq F(\delta t) \qquad \text{for} \quad \delta \geq 1, t \geq 0.$$

Suppose $F(x,t)$ is a nonnegative locally Hölder continuous function on $M \times [0, +\infty)$ satisfying (F.1), (F.2) *and*

$$F(x,t) \begin{cases} \geq C_{i+} h_i(x) F_i(t) & \text{for} \quad (x,t) \in (B_R(\partial M_i) \cap M) \times [0, +\infty) \\ \leq C_{i-} h_i(x) F_i(t) & \text{for} \quad (x,t) \in M \times [0, +\infty) \end{cases}$$

for positive constants C_{i+}, C_{i-} and R. Then the problem $(*)_\infty$ possesses a unique solution.

Proof. We give a proof for the case when $M = M_1 \setminus \cup_{i=2}^k \overline{M}_i$. When $M = M_1$, namely M is star-shaped, if we regard $\cup_{i=2}^k \overline{M}_i = \emptyset$, then our proof below is valid.

Define a domain $M_{i,r}$ by

$$M_{i,r} := \{x \in \mathbf{R}^n | p_i + r^{-1}(x - p_i) \in M_i\}$$

for any positive number r $(i = 1, \ldots, k)$. There is a number $R' > 1$ such that $M \setminus \overline{M}_{1,1/R'} \subset B_R(\partial M_1) \cap M$ and $M_{i,R'} \setminus \overline{M} \subset B_R(\partial M_i) \cap M$ $(i = 2, \ldots, k)$. Since the condition (F.6) implies the condition (F.3), and h_i is bounded on M $(i = 1, \ldots, k)$, by the same way as in the proof of Theorem 3.4, the equation $(*)$ possesses a maximal solution U such that $U \to +\infty$ as $r_{\partial M} \to 0$. Set $m_1 := \max_{\partial M_{1,1/R'}} U$ and $m_i := \max_{\partial M_{i,R'}} U$ $(i = 2, \ldots, k)$.

Let u be a solution of the problem $(*)_\infty$, and set $m_0 := \min_M u (> 0)$. For any number $1 < \epsilon < R'$, define

$$u_{1,\epsilon}(x) := \gamma_1 \epsilon^{-2/(q'-1)} U(p_1 + \epsilon^{-1}(x - p_1)) \qquad \text{on} \quad \overline{M}_1 \setminus M_{1,\epsilon/R'},$$

where $\gamma_1 := \min\{(C_{1+}/C_{1-})^{1/(q'-1)}, m_0/m_1\}$. Then, by direct computation, we have

$$\begin{aligned} \Delta u_{1,\epsilon}(x) &= \gamma_1 \epsilon^{-2/(q'-1)} \epsilon^{-2} F(p_1 + \epsilon^{-1}(x - p_1), U(p_1 + \epsilon^{-1}(x - p_1))) \\ &\geq \gamma_1 \epsilon^{-2q'/(q'-1)} C_{1+} h_1(p_1 + \epsilon^{-1}(x - p_1)) F_1(U(p_1 + \epsilon^{-1}(x - p_1))) \\ &\geq \gamma_1 \epsilon^{-2q'/(q'-1)} C_{1+} h_1(x) F_1(\gamma_1{}^{-1} \epsilon^{2/(q'-1)} u_{1,\epsilon}(x)) \\ &\geq \gamma_1 \epsilon^{-2q'/(q'-1)} C_{1+} h_1(x) \gamma_1{}^{-q'} \epsilon^{2q'/(q'-1)} F_1(u_{1,\epsilon}(x)) \end{aligned}$$

$$
\begin{aligned}
&= {\gamma_1}^{1-q'} C_{1+} h_1(x) F_1(u_{1,\epsilon}(x)) \\
&\geq {\gamma_1}^{1-q'} C_{1+} {C_{1-}}^{-1} F(x, u_{1,\epsilon}(x)) \\
&= F(x, u_{1,\epsilon}(x)),
\end{aligned}
$$

namely $u_{1,\epsilon}$ is a subsolution of the equation $(*)$ on $M_1 \setminus M_{1,\epsilon/R'}$ which is finite on ∂M_1 and satisfies $u_{1,\epsilon} \leq \gamma_1 m_1 \leq m_0 \leq u$ on $\partial M_{1,\epsilon/R'}$. Since ϵ can be chosen arbitrarily close to 1, we get $u \geq \lim_{\epsilon\to 1} u_{1,\epsilon} = \gamma_1 U$ on $M_1 \setminus M_{1,1/R'}$.

On the other hand, for any $2 \leq i \leq k$ and any number $1 < \epsilon < R'$, define

$$
u_{i,\epsilon}(x) := \gamma_i \epsilon^{2/(q'-1)} U(p_i + \epsilon(x - p_i)) \qquad \text{on} \quad \overline{M}_{i,R'/\epsilon} \setminus M_i,
$$

where $\gamma_i := \min\{(C_{i+}/C_{i-})^{1/(q'-1)}, m_0/m_i R'^{2/(q'-1)}\}$. Then, by direct computation, we have

$$
\begin{aligned}
\Delta u_{i,\epsilon}(x) &= \gamma_i \epsilon^{2/(q'-1)} \epsilon^2 F(p_i + \epsilon(x - p_i), U(p_i + \epsilon(x - p_i))) \\
&\geq \gamma_i \epsilon^{2q'/(q'-1)} C_{i+} h_i(p_i + \epsilon(x - p_i)) F_i(U(p_i + \epsilon(x - p_i))) \\
&\geq \gamma_i \epsilon^{2q'/(q'-1)} C_{i+} h_i(x) F_i({\gamma_i}^{-1} \epsilon^{-2/(q'-1)} u_{i,\epsilon}(x)) \\
&\geq \gamma_i \epsilon^{2q'/(q'-1)} C_{i+} h_i(x) {\gamma_i}^{q'} \epsilon^{-2q'/(q'-1)} F_i(u_{i,\epsilon}(x)) \\
&= {\gamma_i}^{1-q'} C_{i+} h_i(x) F_i(u_{i,\epsilon}(x)) \\
&\geq {\gamma_i}^{1-q'} C_{i+} {C_{i-}}^{-1} F(x, u_{i,\epsilon}(x)) \\
&= F(x, u_{i,\epsilon}(x)),
\end{aligned}
$$

namely $u_{i,\epsilon}$ is a subsolution of the equation $(*)$ on $\overline{M}_{i,R'/\epsilon} \setminus \overline{M}_i$ which is finite on ∂M_i and satisfies $u_{i,\epsilon} \leq \gamma_i m_i \leq m_0 \leq u$ on $\partial M_{i,R'/\epsilon}$. Since ϵ can be chosen arbitrarily close to 1, we get $u \geq \lim_{\epsilon\to 1} u_{i,\epsilon} = \gamma_i U$ on $\overline{M}_{i,R'} \setminus \overline{M}_i$.

Since U is maximal, we have $u \leq U$, and hence we get $u \sim U$. Now, by Remark 2.3, we have $u \equiv U$. □

We remark here that, even when M is the unit ball and $F(x,t) = h(x)t^{(n+2)/(n-2)}$, our assumption above allows a far larger class of h than that in Ratto-Rigoli-Veron [13, Theorem 7.1(I)].

4. The partially singular case

In this section, we generalize our uniqueness result to non-maximal solutions. As we mentioned in Introduction, the equation $(*)$ often possesses solutions with mixed behavior. For such solutions, we can modify Lemma 2.1 as follows:

Lemma 4.1. *Let (M,g) and $F(x,t)$ as in Lemma* 2.1 *in Introduction. Let K be a compact hypersurface of M such that $M \setminus K$ is not connected, M' a union* *of some connected components of* $M \setminus K$, $M'' := M \setminus (K \cup M')$, *and* $\{\Omega_i\}_{i \in \mathbf{N}}$ *a sequence of relatively compact domains of* M *with smooth boundaries which satisfies* $K \subset \Omega_i$, $\Omega_i \subset\subset \Omega_{i+1}$ *and* $\cup_{i \in \mathbf{N}} \Omega_i = M$. *Suppose the equation* $(*)$ *possesses a solution* U' *satisfying* $U' \geq u_-$ *for any positive subsolution* u_- *of the equation* $(*)$ *such that* $\lim_{i \to +\infty}(\sup_{M' \setminus \Omega_i}(u_-/U')) \leq 1$. *Then, for any solution* u *of the equation* $(*)$, $u \equiv U'$ *on* M *provided that* $\lim_{i \to +\infty} (\sup_{M' \setminus \Omega_i} |u/U' - 1|) = 0$ *and* $u \sim U'$ *on* M.

Proof. Clearly, u satisfies $\lim_{i \to +\infty}(\sup_{M' \setminus \Omega_i}(u/U')) = 1$. Hence, by the assumption on U', we have $u \leq U'$. Suppose $u \not\equiv U'$. Then, by the strong maximum principle, it holds that $u < U'$. Set $\beta := \sup_M(U'/u)(> 1)$, $\delta_0 := \beta/(\beta - 1)$, and $v_- := \delta(U' - u)$. Then, by the same calculation as in the proof of Lemma 2.1, we can choose a number $\delta > \delta_0$ such that v_- is a positive subsolution of the equation $(*)$. Since $\lim_{i \to +\infty}(\sup_{M' \setminus \Omega_i}(v_-/U')) = 0 \leq 1$, by the assumption on U' again, we get $v_- \leq U'$. Hence we get the following contradiction:

$$\beta = \sup_M \left(\frac{U'}{u}\right) \geq \sup_M \left(\frac{v_-}{u}\right) = \sup_M \left\{\delta \left(\frac{U'}{u} - 1\right)\right\} = \delta(\beta - 1) > \delta_0(\beta - 1) = \beta.$$

Now, we conclude that $u \equiv U'$. □

REMARK 4.2. The assumption "$U' \geq u_-$ for any positive subsolution u_- ..." in the theorem above seems to be stronger than that in Lemma 2.1. Indeed, in Lemma 2.1, "maximal" means only "$U \geq u$ for any solution u". Among the known existence theorems of the maximal solution, since the technical condition "h is positive outside a compact subset" (or a slightly weaker one) is assumed, we can easily see that the maximal solution U satisfies "$U \geq u_-$ for any subsolution u_-". However, it is not clear in general whether we can expect it or not.

We can observe a typical application in the following

Theorem 4.3. *Let* $(\check{M}, \check{g})$ *be a Riemannian manifold,* M *a relatively compact domain in* $\check{M}$, *whose boundary* ∂M *is* C^2 *and disconnected,* Σ_1 *a union of some (not all) connected components of* ∂M, $\Sigma_2 := \partial M \setminus \Sigma_1$, *and* $g := \check{g}|_M$. *Suppose* $F(x,t)$ *is a nonnegative locally Hölder continuous function on* $M \times [0, +\infty)$ *satisfying* (F.1), (F.2) *and*

$$C_+ t^q \leq F(x,t) \leq C_- t^q \qquad \textit{for} \quad (x,t) \in M \times [0, +\infty)$$

for positive constants C_+, C_- *and* $q > 1$. *Then, for any positive continuous function* ψ *on* Σ_1, *the equation* $(*)$ *possesses a unique solution* u *which is continuous on* $M \cup \Sigma_1$, *and satisfies* $u = \psi$ *on* Σ_1 *and* $u \to +\infty$ *as* $r_{\Sigma_2} \to 0$.

To prove the theorem above, we need to show the following

Lemma 4.4. *Let (M,g) and $F(x,t)$ be as in Theorem 4.3. Then, for any positive continuous function ψ on Σ_1, the equation $(*)$ possesses at least one solution u which is continuous on $M \cup \Sigma_1$, and satisfies $u = \psi$ on Σ_1 and $u \to +\infty$ as $r_{\Sigma_2} \to 0$.*

Proof. Let α be a positive number smaller than 1. Clearly, there is a positive number R such that $B_R(\Sigma_1) \cap B_R(\Sigma_2) = \emptyset$, and r_{Σ_1} satisfies $|\Delta_g r_{\Sigma_1}| \leq C_2 < (1-\alpha)/R$ on $\overline{B}_R(\Sigma_1)$.

Set

$$u_+ := \gamma\{r_{\Sigma_1}{}^{\alpha} + (R - r_{\Sigma_1})^{-2/(q-1)} - R^{-2/(q-1)}\} \qquad \text{on} \quad B_R(\Sigma_1) \cap M,$$

where γ is a positive number chosen later. By direct computation, we have

$$\begin{aligned}\Delta_g u_+ &= \gamma\Bigg[-\alpha(1-\alpha - r_{\Sigma_1}\Delta_g r_{\Sigma_1}) r_{\Sigma_1}{}^{\alpha-2} \\ &\qquad + \frac{2}{q-1}(R - r_{\Sigma_1})^{-2/(q-1)-2}\left\{\frac{2}{q-1} + 1 + (R - r_{\Sigma_1})\Delta_g r_{\Sigma_1}\right\}\Bigg] \\ &\leq \gamma\{-C_{12} r_{\Sigma_1}{}^{\alpha-2} + C_{13}(R - r_{\Sigma_1})^{-2/(q-1)-2}\},\end{aligned}$$

where $C_{12} := \alpha(1-\alpha-RC_2)$, $C_{13} := \{2/(q-1)\}\{2/(q-1)+1+RC_2\}$. Hence, if we set $R' := \min\{R/2, (R^{2/(q-1)+2}C_{12}/2^{2/(q-1)+2}C_{13})^{1/(2-\alpha)}\}$, then we get

$$\Delta_g u_+ \leq \gamma\left\{-C_{12} r_{\Sigma_1}{}^{\alpha-2} + C_{13}\left(\frac{R}{2}\right)^{-2/(q-1)-2}\right\} \leq 0 \qquad \text{on} \quad \overline{B}_{R'}(\Sigma_1) \cap M.$$

On the other hand,

$$\begin{aligned} u_+ &\geq \gamma\{(R - r_{\Sigma_1})^{-2/(q-1)} - R^{-2/(q-1)}\} \\ &\geq \gamma C_{14}(R - r_{\Sigma_1})^{-2/(q-1)} \qquad \text{on} \quad (B_R(\Sigma_1) \setminus \overline{B}_{R'}(\Sigma_1)) \cap M,\end{aligned}$$

where $C_{14} := 1 - (1 - R'/R)^{2/(q-1)}$. Therefore, we get

$$u_+{}^{-q}\Delta_g u_+ \leq \begin{cases} 0 & \text{on} \quad \overline{B}_{R'}(\Sigma_1) \cap M \\ \dfrac{C_{13}}{\gamma^{q-1} C_{14}{}^{q}} & \text{on} \quad (B_R(\Sigma_1) \setminus \overline{B}_{R'}(\Sigma_1)) \cap M. \end{cases}$$

Now, if we choose $\gamma := (C_{13}/C_{14}{}^{q}C_+)^{1/(q-1)}$, then

$$\Delta_g u_+ \leq C_+ u_+{}^{q} \leq F(x, u_+(x)) \qquad \text{for} \quad (x,t) \in (B_R(\Sigma_1) \cap M) \times [0, +\infty),$$

namely u_+ is a supersolution of the equation $(*)$ on $B_R(\Sigma_1) \cap M$. Moreover, since

$$\Delta_g(u_+ + \varphi) = \Delta_g u_+ \leq F(x, u_+) \leq F(x, u_+ + \varphi),$$

$u_+ + \varphi$ is also a supersolution for any positive harmonic function φ on M satisfying the boundary condition $\varphi = \psi$ on Σ_1.

For any positive number m, let u_m be a solution of the equation $(*)$ satisfying the boundary conditions $u_m = \psi$ on Σ_1 and $u_m = m$ on Σ_2.

Each u_m is bounded above by the maximal solution U of the equation $(*)$. Since $u_+ + \varphi = \psi$ on Σ_1 and $u_+ + \varphi \to +\infty$ as $r_{\partial B_R(\Sigma_1) \cap M} \to 0$, each u_m is bounded also by $u_+ + \varphi$ on $B_R(\Sigma_1) \cap M$. By the maximum principle, we see $u_m \geq u_{m'}$ for any $m \geq m'$. Therefore, if we set $u := \lim_{m \to +\infty} u_m$, then u is a solution of the equation $(*)$ with the desired properties. □

Lemma 4.5. *Let (M, g) and $F(x, t)$ be as in Theorem* 4.3. *Then, for any positive continuous function ψ on Σ_1, the equation $(*)$ possesses a solution U_ψ which is continuous on $M \cup \Sigma_1$ and has the following properties:*

(1) *$U_\psi = \psi$ on Σ_1, and $U_\psi \to +\infty$ as $r_{\Sigma_2} \to 0$;*

(2) *If a positive subsolution u_- of the equation $(*)$ satisfies $\limsup_{x \to p} u_-(x) \leq \psi(p)$ for any $p \in \Sigma_1$, then it holds that $u_- \leq U_\psi$ on M. Moreover, if $u_- \not\equiv U_\psi$, then $u_- < U_\psi$ on M;*

(3) *If a solution u of the equation $(*)$ which is continuous on $M \cup \Sigma_1$ satisfies $u = \psi$ on Σ_1 and $u \sim U_\psi$ on M, then it holds that $u \equiv U_\psi$ on M.*

Proof. Let R be as in the proof of Lemma 4.4. By Lemma 4.4, for any positive number $r \leq R$, there exists a solution u_r of the equation $(*)$ on $M \setminus \overline{B}_r(\Sigma_2)$ which is continuous on $(M \cup \Sigma_1) \setminus \overline{B}_r(\Sigma_2)$ and satisfies $u_r = \psi$ on Σ_1 and $u_r \to +\infty$ as $r_{\partial B_r(\Sigma_2) \cap M} \to 0$. Let u_ψ be the solution of the equation $(*)$ on M given by Lemma 4.4.

Since $u_r = \psi = u_{r'} = u_\psi$ on Σ_1 and both $u_{r'}$ and u_ψ are bounded above near $\partial B_r(\Sigma_2) \cap M$ for any $r' < r$, we get $u_r \geq u_{r'}$ and $u_r \geq u_\psi$ on $M \setminus \overline{B}_r(\Sigma_2)$, that is, $\{u_r\}_{0 < r \leq R}$ is monotonically increasing and bounded below by u_ψ. Therefore, if we set $U_\psi := \lim_{r \to 0} u_r$, then U_ψ is a solution of the equation $(*)$. Since $u_\psi \leq U_\psi \leq u_r$, U_ψ can be extended on Σ_1 continuously and satisfies $U_\psi = \psi$ on Σ_1 and $U_\psi \to +\infty$ as $r_{\Sigma_2} \to 0$, namely U_ψ satisfies the condition (1).

Moreover, it is clear that $u_r \geq u_-$ for any subsolution u_- of the equation $(*)$ satisfying the assumption in (2). Hence we see that $U_\psi \geq u_-$. By the strong maximum principle, if $u_- \not\equiv U_\psi$, then $u_- < U_\psi$ on M.

Now, the assertion of (3) follows from (2) and Lemma 4.1. □

Proof of Theorem 4.3. The existence follows from Lemma 4.4. Let u be a solution of the equation $(*)$ satisfying $u = \psi$ on Σ_1 and $u \to +\infty$ as $r_{\Sigma_2} \to 0$.

Then, by the same way as in the proof of Theorem 3.1, we have $u \sim r_{\Sigma_2}{}^{-2/(q-1)}$ near Σ_2, from which it follows that $u \sim U_\psi$ on M, where U_ψ is the solution given in Lemma 4.5. Now, by Lemma 4.5 (3), we have $u \equiv U_\psi$. □

5. An application to the scalar curvature equation

We can apply Lemma 4.1 also for many other cases. One of them can be found in the scalar curvature equation on a compact Riemannian manifold punctured by a finite number of points.

Cheng-Ni [3, Theorem II] proved the uniqueness of solutions of the maximal order when $M = \mathbf{R}^n$ and $F(x,t) = h(x)t^q$ and classify all the solutions under a certain typical assumption. It is well known that the scalar curvature equation on $(\mathbf{R}^n, g_0)$ is equivalent to that on the standard sphere $(\mathbf{S}^n, g_1)$ punctured by a point. In [6, Theorem 3], we generalized [3, Theorem II] to the case of any compact Riemannian manifold $(\overline{M}, \overline{g})$ $(n = \dim \overline{M} \geq 3)$ of positive scalar curvature punctured by a point $\{p_1\}$.

By the same way as the proofs of Theorem 4.3 and Lemmas 4.4-5, we can combine the proof of [6, Theorem 3] and Lemma 4.1, and generalize [6, Theorem 3] to Theorem 2 of this work which treats the case of $(\overline{M}, \overline{g})$ $(n = \dim \overline{M} \geq 3)$ punctured by a finite number of points $\{p_1, \dots, p_k\}$.

The existence part of Theorem 2 has already been proven in [6, Proposition 5.2]. The proof runs along a similar line as that for $\#\Sigma = 1$. However, we have to analize the solutions more minutely to apply Theorem 4.1. Throughout this section, we refer the scalar curvature equation as "the equation (f, M)" with various f and M.

Lemma 5.1. *Let (M, g) and f be as in Theorem 2. Then the equation (f, M) possesses a maximal solution U which satisfies $U \sim r_{p_i}{}^{-(\ell_i+2)(n-2)/4}$ near p_i $(i = 1, \dots, k)$.*

Proof. Clearly, there is a positive number R such that $B_{2R}(p_i) \cap B_{2R}(p_{i'}) = \emptyset$ $(i \neq i')$, r_{p_i} is smooth and $f < 0$ on $\overline{B}_R(p_i)$ $(i = 1, \dots, k)$.

For each i, let f_{i-} be a nonpositive smooth function on $\overline{M} \setminus \{p_i\}$ such that $f_{i-} \leq f$ on M and $f_{i-} \equiv f$ on $B_R(p_i)$. By [5, Theorem IV], the equation $(k^{4/(n-2)} f_{i-}, \overline{M} \setminus \{p_i\})$ possesses a maximal solution U_{i-} which satisfies $U_{i-} \sim r_{p_i}{}^{-(\ell_i+2)(n-2)/4}$. Set $U_- := \sum_{i=1}^k U_{i-}$. Then we have

$$
\begin{aligned}
L_g U_- &= \sum_{i=1}^k L_g U_{i-} = \sum_{i=1}^k k^{4/(n-2)} f_{i-} U_{i-}{}^{(n+2)/(n-2)} \\
&\leq \sum_{i=1}^k k^{4/(n-2)} f U_{i-}{}^{(n+2)/(n-2)}
\end{aligned}
$$

$$\leq f\left(\sum_{i=1}^{k} U_{i-}\right)^{(n+2)/(n-2)} = fU_{-}{}^{(n+2)/(n-2)} \qquad \text{on} \quad M,$$

namely U_- is a positive subsolution of the equation (f, M). Therefore, by [5, Theorem I], the equation (f, M) possesses a maximal solution U which satisfies

$$U \geq U_- > U_{i-} \geq C_{15} r_{p_i}{}^{-(\ell_i+2)(n-2)/4} \qquad \text{near} \quad p_i \qquad (i = 1, \dots, k)$$

for a positive constant C_{15}.

On the other hand, by the proof of [5, Theorem IV], we have an a priori upper estimate $U \leq C_{16} r_{p_i}{}^{-(\ell_i+2)(n-2)/4}$ near p_i $(i = 1, \dots, k)$ for a positive constant C_{16}, and hence we get $U \sim r_{p_i}{}^{-(\ell_i+2)(n-2)/4}$. □

The following lemma corresponds to Lemma 4.5.

Lemma 5.2. *Let (M, g) and f be as in Theorem 2. Then, for any $\gamma = (\gamma_1, \dots, \gamma_k) \in (0, +\infty]^k$, the equation (f, M) possesses a solution U_γ which has the following properties:*

(1) *$U_\gamma(x)/G_{p_i}(x) \to \gamma_i$ as $x \to p_i$ $(i = 1, \dots, k)$;*

(2) *If a subsolution u_- of the equation (f, M) satisfies $\limsup_{x\to p_i}(u_-(x)/G_{p_i}(x)) \leq \gamma_i$ for any i, then it holds that $u_- \leq U_\gamma$ on M. Moreover, if $u_- \not\equiv U_\gamma$, then $u_- < U_\gamma$;*

(3) *If a solution u of the equation (f, M) satisfies $\lim_{x\to p_i}(u(x)/G_{p_i}(x)) = \gamma_i$ for any i and $u \sim U_\gamma$, then it holds that $u \equiv U_\gamma$ on M.*

Proof. For $\gamma = (+\infty, \dots, +\infty)$, the existence of the maximal solution U of the equation (f, M) which satisfies the property (1) follows from Lemma 5.1. By the strong maximum principle, U satisfies (2), and (3) is given by Lemma 2.1 (or [6, Theorem 1]).

For any $\gamma \in (0, +\infty)^k$, we also know the existence of solutions u_γ of the equation (f, M) satisfying (1) (see [5, Theorem V]). In this case, (2) and (3) are consequences of the strong maximum principle.

When some γ_i's are finite and the others are $+\infty$, we may assume $\gamma_i < +\infty$ for $i \leq k'$ and $\gamma_i = +\infty$ for $i \geq k'+1$ without loss of generality. Set $\Sigma_1 := \{p_1, \dots, p_{k'}\}$ and $\Sigma_2 := \{p_{k'+1}, \dots, p_k\}$. By [6, Proposition 5.2], the equation (f, M) possesses a solution u_γ satisfying $u_\gamma(x)/G_{p_i}(x) \to \gamma_i$ as $x \to p_i$ $(i = 1, \dots, k)$.

Let R be as in the proof of Lemma 5.1. Let $f_{1\pm}$ be nonpositive smooth functions on $\overline{M}\backslash\Sigma_1$ such that $f_{1+} \geq f \geq f_{1-}$ on M and $f_{1\pm} \equiv f$ near Σ_1, and $f_{2\pm}$ nonpositive smooth functions on $\overline{M} \setminus \Sigma_2$ such that $f_{2+} \geq f \geq f_{2-}$ on M and $f_{2\pm} \equiv f$ near Σ_2. Now, let u_{1+} (resp. u_{1-}) be the solution of the equation $(f_{1+}, \overline{M} \setminus \Sigma_1)$ (resp. $(2^{4/(n-2)} f_{1-}, \overline{M}\setminus\Sigma_1)$) satisfying $\lim_{x\to p_i}(u_{1\pm}(x)/G_{p_i}(x)) = \gamma_i$ for any $i = 1, \dots, k'$, and, for any positive number $r \le R$, $U_{2+,r}$ (resp. $U_{2-,r}$) the maximal solution of the equation $(f_{2+}, \overline{M} \setminus \overline{B}_r(\Sigma_2))$ (resp. $(2^{4/(n-2)} f_{2-}, \overline{M} \setminus \overline{B}_r(\Sigma_2))$) which satisfies $U_{2\pm,r} \to +\infty$ as $r_{\partial B_r(\Sigma_2)} \to 0$. Set $u_{\pm,r} := u_{1\pm} + U_{2\pm,r}$. Then $u_{+,r}$ (resp. $u_{-,r}$) is a supersolution (resp. subsolution) of the equation $(f, M \setminus \overline{B}_r(\Sigma_2))$, and the equation $(f, M \setminus \overline{B}_r(\Sigma_2))$ possesses a solution u_r satisfying $u_{+,r} \ge u_r \ge u_{-,r}$ from which it follows that $\lim_{x\to p_i}(u_r(x)/G_{p_i}(x)) = \gamma_i$ for any $i = 1, \dots, k'$ and $u_r \to +\infty$ as $r_{\partial B_r(\Sigma_2)} \to 0$.

Define a scalar flat conformal metric by $\tilde{g} := (\sum_{i=1}^{k'} G_{p_i})^{4/(n-2)} g$. Then both $\tilde{u}_r := u_r / \sum_{i=1}^{k'} G_{p_i}$ and $\tilde{u}_\gamma := u_\gamma / \sum_{i=1}^{k'} G_{p_i}$ are solutions of the equation $(f, M \setminus \overline{B}_r(\Sigma_2))$ with the metric $\tilde{g}$. Note here that $u_r \to +\infty$ as $r_{\partial B_r(\Sigma_2)} \to 0$, both $u_{r'}$ ($r' < r$) and u_γ are bounded above near $\partial B_r(\Sigma_2)$, $\lim_{x\to p_i}\{(u_r(x) - u_{r'}(x))/G_{p_i}(x)\} = 0$ and $\lim_{x\to p_i}\{(u_r(x) - u_\gamma(x))/G_{p_i}(x)\} = 0$ ($i = 1, \dots, k'$). Then, by the maximum principle, we get $u_r \ge u_{r'}$ and $u_r \ge u_\gamma$ on $M \setminus \overline{B}_r(\Sigma_2)$, that is, $\{u_r\}_{0<r\le R}$ is monotonically increasing and bounded below by u_γ. Therefore, if we set $U_\gamma := \lim_{r\to 0} u_r$, then U_γ is a smooth solution of the equation (f, M). Since $u_\gamma \le U_\gamma \le u_r$, we have $\lim_{x\to p_i}(U_\gamma(x)/G_{p_i}(x)) = \gamma_i$ for any i, namely U_γ satisfies the condition (1).

The assertions (2) and (3) follow by the same way as the proof of Lemma 4.5. □

We can show the following inequality by the same way as the proof of Cheng-Ni [3, Proposition 5.2] or [6, Lemma 4.1].

Lemma 5.3. *Let (M, g) and f be as in Theorem 2, and suppose u is a solution of the equation (f, M). Then there is a positive constant C_{17} which is independent of both u and r and satisfies*

$$\max_{\partial B_r(p_i)} u \le C_{17} \min_{\partial B_r(p_i)} u$$

for any positive number r small enough.

Proof of Theorem 2. The existence follows from [6, Proposition 5.2].

Suppose u is a solution of the equation (f, M) such that $u^{4/(n-2)} g$ is complete. If u/G_{p_i} is bounded near p_i for some i, let χ_i be a nonnegative smooth function on $\overline{M}$ such that $\chi_i \equiv 1$ on $B_R(p_i)$ and $\chi_i \equiv 0$ on $\overline{M} \setminus B_{2R}(p_i)$. Then $\chi_i u$ is a nonnegative smooth function on $\overline{M} \setminus \{p_i\}$. Note here that $-L_{\overline{g}}(\chi_i u) \equiv |f| u^{(n+2)/(n-2)} \le C_{18} {r_{p_i}}^{\ell_i - n - 2}$ on $B_R(p_i) \setminus \{p_i\}$ for a positive constant C_{18}, and that $-L_{\overline{g}}(\chi_i u)$ is bounded on $\overline{M} \setminus B_R(p_i)$ since it vanishes on $\overline{M} \setminus B_{2R}(p_i)$. It is clear that

$$\Phi_i(x) := -\int_{\overline{M}\setminus\{p_i\}} G(x, y) L_{\overline{g}}(\chi_i(y) u(y)) dy$$

is a smooth function on $\overline{M} \setminus \{p_i\}$ and satisfies $L_{\overline{g}}\Phi_i = -L_{\overline{g}}(\chi_i u)$, and we get, by standard calculation, that $-C_{19} \leq \Phi_i \leq C_{20}{r_{p_i}}^{\ell-n}$ near p_i for positive constants $\ell \in (2, \min\{\ell_i, n\})$, C_{19} and C_{20}. In particular, it holds that $\Phi_i(x)/G_{p_i}(x) \to 0$ as $x \to p_i$. Set $\varphi_i := \chi_i u + \Phi_i + \gamma_i' G_{p_i}$, where γ_i' is a real number chosen to satisfy $\varphi_i > 0$ on $\overline{M} \setminus \{p_i\}$. Then φ_i is a solution of the equation $(0, \overline{M} \setminus \{p_i\})$, and hence there is a positive number γ_i'' such that $\varphi_i \equiv \gamma_i'' G_{p_i}$ (see [5, Section 4]). We get

$$\frac{\chi_i(x)u(x)}{G_{p_i}(x)} = (\gamma_i'' - \gamma_i') - \frac{\Phi_i(x)}{G_{p_i}(x)} \to \gamma_i'' - \gamma_i' \qquad \text{as} \quad x \to p_i,$$

and hence $u(x)/G_{p_i}(x) \to \gamma_i'' - \gamma_i'$ as $x \to p_i$. Since u/G_{p_i} is positive, we have $\gamma_i'' - \gamma_i' \geq 0$.

Set $\gamma_i := \gamma_i'' - \gamma_i'$. Here we claim $\gamma_i > 0$. Indeed, suppose $\gamma_i = 0$. Then $u(x)/G_{p_i}(x) \to 0$ as $x \to p_i$. For each $i' \neq i$, let $f_{i'+}$ be a nonpositive smooth function on $\overline{M} \setminus \{p_{i'}\}$ such that $f_{i'+} \geq f$ on M and $f_{i'+} \equiv f$ on $B_R(p_{i'})$. Let $U_{i',R}$ be a solution of the equation $(f_{i'+}, \overline{M} \setminus \overline{B}_R(p_{i'}))$ such that $U_{i',R} \to +\infty$ as $r_{\partial B_R(p_{i'})} \to 0$. For any positive number t, set $u_{i,t} := tG_{p_i} + \sum_{i' \neq i} U_{i',R}$. Then

$$\begin{aligned} L_{\overline{g}} u_{i,t} &= tL_{\overline{g}} G_{p_i} + \sum_{i' \neq i} L_{\overline{g}} U_{i',R} = 0 + \sum_{i' \neq i} f_{i'+} {U_{i',R}}^{(n+2)/(n-2)} \\ &\geq \sum_{i' \neq i} f {U_{i',R}}^{(n+2)/(n-2)} \geq f \left\{ (tG_{p_i})^{(n+2)/(n-2)} + \sum_{i' \neq i} {U_{i',R}}^{(n+2)/(n-2)} \right\} \\ &\geq f \left(tG_{p_i} + \sum_{i' \neq i} U_{i',R} \right)^{(n+2)/(n-2)} \\ &= f{u_{i,t}}^{(n+2)/(n-2)} \qquad \text{on} \quad \overline{M} \setminus (\{p_i\} \textstyle\bigcup \cup_{i' \neq i} \overline{B}_R(p_{i'})), \end{aligned}$$

namely $u_{i,t}$ is a positive supersolution of the equation $(f, \overline{M} \setminus (\{p_i\} \bigcup \cup_{i' \neq i} \overline{B}_R(p_{i'})))$. Since $u_{i,t}(x)/G_{p_i}(x) \to t > 0$ as $x \to p_i$ and $u_{i,t}(x) \to +\infty$ as $r_{\cup_{i' \neq i} \partial B_R(p_{i'})} \to 0$, u satisfies $u \leq u_{i,t}$ on $\overline{M} \setminus (\{p_i\} \bigcup \cup_{i' \neq i} \overline{B}_R(p_{i'})))$. Hence we have $u \leq \lim_{t \to 0} u_{i,t} = \sum_{i' \neq i} U_{i',R}$. In particular, u is bounded near p_i. This contradicts the completeness of the metric $u^{4/(n-2)}g$. Therefore we get $\gamma_i > 0$. When $\gamma_i > 0$, clearly $u^{4/(n-2)}g$ is complete near p_i.

On the other hand, if u/G_{p_i} is unbounded near p_i, then $\limsup_{x \to p_i}(u(x)/G_{p_i}(x)) = +\infty$. Hence, there is a sequence $\{x_{i,j}\}_{j \in \mathbf{N}}$ of points in M such that $\lim_{j \to +\infty} x_{i,j} = p_i$ and $u(x_{i,j})/G_{p_i}(x_{i,j}) \geq C_{17} j$ for any j, where C_{17} is the constant given in Lemma 5.3. By Lemma 5.3, it holds that $u/G_{p_i} \geq j$ on $\partial B_{r_{p_i}(x_{i,j})}(p_i)$ for any j. Let f_{i-} be as in the proof of Lemma 5.1, U_{i-} the maximal solution of the equation $(f_{i-}, \overline{M} \setminus \{p_i\})$ which satisfies $U_{i-} \sim {r_{p_i}}^{-(\ell_i+2)(n-2)/4}$, and $u_{i-,j}$ a unique solution of the equation

$(f_{i-}, \overline{M} \setminus \{p_i\})$ satisfying $u_{i-,j} \le jG_{p_i}$ and $\lim_{x\to p_i}(u_{i-,j}(x)/G_{p_i}(x)) = j$. By [6, Corollary 3.2], we have $\lim_{j\to+\infty} u_{i-,j} \equiv U_{i-}$.

Now, $u \ge u_{i-,j}$ on $\partial B_{r_{p_i}(x_{i,j})}(p_i)$. On the other hand, for any $i' \ne i$, if $u/G_{p_{i'}}$ is bounded near $p_{i'}$, then $u(x)/G_{p_{i'}}(x) \to \gamma_{i'}$ as $x \to p_{i'}$ for some $\gamma_{i'} > 0$. Since $u_{i-,j}$ is bounded near $p_{i'}$, we have $u > u_{i-,j}$ near $p_{i'}$. If $u/G_{p_{i'}}$ is also unbounded near $p_{i'}$, then, by the consideration above, $u/G_{p_{i'}} \ge \max_{x\in \overline{B}_R(p_{i'})}(u_{i-,j}(x)/G_{p_{i'}}(x))$ on $\partial B_{R'}(p_{i'})$ for some positive number $R' \le R$. Therefore, by [5, Lemma 2.2], $u \ge u_{i-,j}$ on $\overline{M} \setminus (B_{r_{p_i}(x_{i,j})}(p_i) \bigcup \cup_{i'\ne i} B_R(p_{i'}))$ for any j, from which it follows that

$$u \ge \lim_{j\to+\infty} u_{i-,j} = U_{i-} \ge C_{21} r_{p_i}{}^{-(\ell_i+2)(n-2)/4} \qquad \text{on} \quad \overline{M} \setminus (\{p_i\} \bigcup \cup_{i'\ne i} B_R(p_{i'}))$$

for a positive constant C_{21}. Combining this with the upper estimate for the maximal solution, we have $u \sim r_{p_i}{}^{-(\ell_i+2)(n-2)/4}$ near p_i.

Now, by Lemmas 5.1-2, we see that the solution u coincides with one of the solutions given in [6, Proposition 5.2] (or Lemma 5.2). This completes the proof. □

Taking account of the fact that u/G_{p_i} must be bounded near p_i when $f \equiv 0$ near p_i (see the proof of [6, Theorem 6.1]), we can also prove the following theorem by the same way as the proof of Theorem 2.

Theorem 5.4. *Let (M, g) be as in Theorem 2, and f a nonpositive smooth function on M. If f satisfies $f \sim -r_{p_i}{}^{\ell_i}$ near p_i for a number $\ell_i > 2$ $(i = 1, \dots, k')$ and $f \equiv 0$ near p_i $(i = k'+1, \dots, k)$, then, for any $\gamma = (\gamma_1, \dots, \gamma_k) \in (0, +\infty]^{k'} \times (0, +\infty)^{k-k'}$, the scalar curvature equation (f, M) possesses a unique solution u_γ such that $u_\gamma(x)/G_{p_i}(x) \to \gamma_i$ as $x \to p_i$ $(i = 1, \dots, k)$ and the metric $u_\gamma{}^{4/(n-2)} g$ is complete. Conversely, any solution u of the equation (f, M) such that $u^{4/(n-2)} g$ is complete coincides with u_γ for some γ. Namely, the space of complete conformal metrics on M with scalar curvature f is parametrized by $(0, +\infty]^{k'} \times (0, +\infty)^{k-k'}$.*

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