EXISTENCE OF INVARIANT MEASURES
FOR DIFFUSION PROCESSES ON A WIENER SPACE

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1. Introduction

In this paper, we discuss existence of finite invariant measures for diffusion processes with unbounded drift on an abstract Wiener space (B, H, μ). The processes we treat are the ones which have generators formally expressed as $-(1/2)D^*\sigma D + (b, D\cdot)$, where D stands for the H-derivative in the sense of Malliavin calculus, D^* the adjoint operator of D with respect to the Wiener measure μ, σ a diffusion coefficient, being a function on B taking values in the space of positive symmetric operators on H, and b a drift coefficient, being an H-valued function on B. When $\sigma \equiv$ identity, a given generator has rigorous meaning in L^2 sense under a mild condition on b, and existence of the associated diffusion and invariant measures was proved by Shigekawa [17] in the case that b is bounded. Partially generalized cases were treated by Vintschger [21] and Zhang [22]. We extend these results to the case where σ is not constant and b is not necessarily bounded but has only some kind of exponential integrability. The diffusion process is constructed by using the theory of Dirichlet forms and the Girsanov transformation, so measures charging no set of zero capacity are allowed to be initial measures. In this paper, however, we restrict our attention to absolutely continuous measures with respect to μ, as in the papers mentioned above. That is, our formulation is as follows: Let $\{T_t\}$ be the associated Markovian semigroup on $L^\infty(\mu)$. Is there any non-zero $\rho \in L^1(\mu)$ such that

$$(1.1) \qquad \int_B (T_t f)\rho \, d\mu = \int_B f\rho \, d\mu \quad \text{for every } t > 0 \text{ and } f \in L^\infty(\mu)?$$

If there is, then $\rho \, d\mu$ is an invariant measure of $\{T_t\}$, and of the diffusion process.

Note that, in a different formulation from ours, invariant measures are proved to be absolutely continuous with respect to μ; see [3, 2].

The difficulty in proving existence of invariant measures lies in the point that the resolvent of the generator is no more compact and that spectrums are not necessarily eigenvalues. So, as in [17], first we consider the case when B is finite dimensional, then take approximation in general cases. At this point, the property of hyperbound-

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edness of the associated semigroups plays an important role. We discuss this in L^p category, different from [17, 21, 22] in which they study in L^2 category. This enables us to weaken an assumption (cf. Proposition 3.1).

The organization of this paper is as follows: in Section 2, we determine the Girsanov density corresponding to b and construct a diffusion process. In Section 3, we study conditions so that the associated Markovian semigroup is extended to a hyperbounded as well as strongly continuous one on $L^p(\mu)$. We also study properties of its infinitesimal generator on $L^p(\mu)$, and connection to the associated coercive closed form when it can be defined. In Section 4, the existence of invariant measures is proved.

2. Construction of Diffusions

Let (B, H, μ) be an abstract Wiener space; that is, B is a real separable Banach space, H is a separable Hilbert space which is embedded densely and continuously in B, and μ is the Wiener measure on B, which satisfies

$$\int_B \exp(\sqrt{-1}{}_B(x, \varphi)_{B^*})\mu(dx) = \exp(-|\varphi|^2_{H^*}/2) \quad \text{for } \varphi \in B^* \subset H^*.$$

In the following, B is treated as infinite dimensional unless specifically stated; the situation is simpler when B is finite dimensional. The inner product and the norm of H are denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively. By the Riesz representation theorem, we identify H^* with H. We represent the Borel σ-field of B as $\mathcal{F}$, and the space of all bounded Borel functions on B as $\mathcal{B}_b$. We denote the L^p space with respect to $(B, \mathcal{F}, \mu)$ just by L^p. We also denote by D^p_r the Sobolev space on B with integrability index p and differentiability index r, and by $\mathcal{F}C^\infty_b$ the space of all functions f on B expressed as

$$f(x) = F({}_B(x, \varphi_1)_{B^*}, \ldots, {}_B(x, \varphi_n)_{B^*}), \quad \varphi_1, \ldots, \varphi_n \in B^*, F \in C^\infty_b(\mathbb{R}^n).$$

Let $\mathcal{SL}(H)$ be the space of all bounded and symmetric linear operators on H with the operator norm $\|\cdot\|_{\mathrm{op}}$.

Suppose that we are given an $\mathcal{SL}(H)$-valued strongly measurable function σ on B and an H-valued Borel measurable function b on B which satisfy the following assumptions:

(A1) There exists some $\delta > 0$ such that $\sigma(x) \geq \delta I$ for μ-a.e.x in the form sense, and $\|\sigma\|_{\mathrm{op}} \in L^1$.

(A2) There exists some $\theta > 0$ such that $\exp(\theta|\sigma^{-1/2}b|^2) \in L^1$.

We remark that $|b| \in \bigcap_{1 \leq p < 2} L^p$ also holds under these assumptions. Indeed, for $1 \leq p < 2$,

$$\int_B |b|^p \, d\mu \leq \int_B \|\sigma^{1/2}\|^p_{\mathrm{op}}|\sigma^{-1/2}b|^p \, d\mu$$

$$\leq \left(\int_B \|\sigma\|_{\mathrm{op}} \, d\mu\right)^{p/2} \left(\int_B |\sigma^{-1/2}b|^q \, d\mu\right)^{p/q} < \infty,$$

where $1/2 + 1/q = 1/p$.

We will show that we can construct a diffusion process on B corresponding to the formal generator $-(1/2)D^*\sigma D + (b, D\cdot)$. First consider the following symmetric bilinear form on L^2:

$$\mathcal{E}^\sigma(f,g) = \frac{1}{2}\int_B (\sigma Df, Dg)d\mu, \quad f,g \in \mathcal{F}C_b^\infty.$$

It is known that $(\mathcal{E}^\sigma, \mathcal{F}C_b^\infty)$ is closable, and that its closure $(\mathcal{E}^\sigma, \mathrm{Dom}(\mathcal{E}^\sigma))$ is a quasi-regular, conservative Dirichlet form (cf. [14]). So there exists a conservative symmetric diffusion $\{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P_x, x \in B\}$ properly associated with $\mathcal{E}^\sigma$. Here $\Omega = C([0,\infty) \to B)$, X_t is a coordinate map, $\mathcal{M}_t$ is a minimum completed admissible filtration, and θ_t is a shift operator. The family of probability measures $\{P_x\}_{x \in B}$ on Ω is uniquely determined up to $\mathcal{E}^\sigma$-capacity 0. We omit '$\mathcal{E}^\sigma$-' from now on since capacities and quasi notions are always with respect to $\mathcal{E}^\sigma$ in this paper.

We set $P_\nu(\cdot) = \int_B P_x(\cdot)\nu(dx)$ for a probability measure ν on B, and denote by E_x (resp. E_ν) the expectation with respect to P_x (resp. P_ν). The associated semigroup $\{T_t^\sigma\}$ on L^2 is uniquely extended to L^p for $p \in [1,\infty]$ and is strongly continuous for $p \in [1,\infty)$. Its generator on L^p is denoted by $\mathcal{L}_p^\sigma$. The suffix p is often omitted when it need not be designated.

For a Borel function f in L^2 and $\alpha > 0$, we set

$$R_\alpha^\sigma f = E.\left[\int_0^\infty e^{-\alpha t} f(X_t)dt\right],$$

which is a quasi-continuous modification of the resolvent $G_\alpha^\sigma f = (\alpha - \mathcal{L}^\sigma)^{-1}f$.

The logarithmic Sobolev inequality

$$\int_B f^2 \log(f^2/\|f\|_2^2)d\mu \leq 2\int_B |Df|^2 d\mu, \quad f \in D_1^2$$

and (A1) imply that

$$(2.1) \quad \int_B f^2 \log(f^2/\|f\|_2^2)d\mu \leq (4/\delta)\mathcal{E}^\sigma(f,f), \quad f \in \mathrm{Dom}(\mathcal{E}^\sigma) \subset D_1^2.$$

That is, $\mathcal{E}^\sigma$ also satisfies the logarithmic Sobolev inequality and the logarithmic Sobolev constant does not exceed $\alpha := 4/\delta$. By [5, Theorem 6.1.14], $\{T_t^\sigma\}$ has the following hypercontractive property:

$$(2.2) \quad \|T_t^\sigma\|_{p \to q} \leq 1 \quad \text{for } t > 0, \, 1 < p < q < \infty \text{ with } (q-1)/(p-1) \leq e^{4t/\alpha}.$$

Here $\| \cdot \|_{p \to q}$ stands for the operator norm from L^p to L^q.

To deal with a drift part $(b, D\cdot)$, we review stochastic analysis via additive functionals (AF's for short) in this setting. We say that two AF's $A^{(i)}$ ($i = 1, 2$) are equivalent if for each $t > 0$ and for q.e.x, $A_t^{(1)} = A_t^{(2)}$ P_x-a.e. As usual, we identify equivalent AF's. The energy $e(A)$ of an AF A_t is defined by

$$e(A) = \lim_{t \downarrow 0} \frac{1}{2t} E_\mu[A_t^2]$$

when the limit exists in $[0, \infty]$. The mutual energy $e(A^{(1)}, A^{(2)})$ for AF's $A^{(i)}$ ($i = 1, 2$) of finite energy is also defined by polarization. Let $\overset{\circ}{\mathcal{M}} = \{M \mid M$ is a continuous AF such that for each $t > 0$, $E_x[M_t^2] < \infty$ and $E_x[M_t] = 0$ q.e.x, and $e(M) < \infty\}$. It is not only a Hilbert space under the inner product $e(\cdot, \cdot)$ but also a $\mathcal{B}_b$-module by the stochastic integral $(f \bullet M)_t$ for $f \in \mathcal{B}_b$, $M \in \overset{\circ}{\mathcal{M}}$ as the action of $\mathcal{B}_b$. Each $M \in \overset{\circ}{\mathcal{M}}$ admits a unique positive continuous AF $\langle M \rangle_t$, called quadratic variation, satisfying

$$E_x[\langle M \rangle_t] = E_x[M_t^2] \quad t > 0, \text{q.e.}x.$$

Its Revuz measure $\mu_{\langle M \rangle}$ is called the energy measure of M. There is a connection between $e(M)$ and $\mu_{\langle M \rangle}$: $e(M) = (1/2)\mu_{\langle M \rangle}(B)$. Quadratic variations and energy measures are also polarized and denoted by $\langle \cdot, \cdot \rangle_t$ and $\mu_{\langle \cdot, \cdot \rangle}$, respectively. For $f \in \mathcal{B}_b$ and $M^{(1)}, M^{(2)} \in \overset{\circ}{\mathcal{M}}$, it holds that $d\mu_{\langle f \bullet M^{(1)}, M^{(2)} \rangle} = f \, d\mu_{\langle M^{(1)}, M^{(2)} \rangle}$.

Every $f \in \mathrm{Dom}(\mathcal{E}^\sigma)$ has a quasi-continuous modification $\tilde{f}$. According to the Fukushima decomposition theorem in the non-locally-compact version, it holds that

$$\tilde{f}(X_t) = \tilde{f}(X_0) + M_t^{[f]} + N_t^{[f]} \quad P_x\text{-a.e., q.e.}x,$$

where $M^{[f]} \in \overset{\circ}{\mathcal{M}}$ and $N^{[f]}$ is a continuous AF of zero energy. $M^{[f]}$ and $N^{[f]}$ are uniquely determined as AF's. Moreover, the quadratic variation of $M^{[f]}$ and $M^{[g]}$ for $f, g \in \mathrm{Dom}(\mathcal{E}^\sigma)$, and its associated signed measure are expressed as

$$\langle M^{[f]}, M^{[g]} \rangle_t = \int_0^t (\sigma Df, Dg)(X_s)ds, \quad d\mu_{\langle M^{[f]}, M^{[g]} \rangle} = (\sigma Df, Dg)d\mu,$$

respectively (see [1, Proposition 4.5]).

Set

$$V = \{d \mid d \text{ is an } H\text{-valued Borel measurable function on } B$$

$$\text{such that } |\sigma^{1/2}d| \in L^2\} \Big/ \sim .$$

Here $d^{(1)} \sim d^{(2)}$ means that $d^{(1)} = d^{(2)}$ μ-a.e. Then V is a Hilbert space under the inner product $(\cdot|\cdot)_V = (1/2) \int_B (\sigma\cdot, \cdot)d\mu$ as well as $\mathcal{B}_b$-module by the pointwise product as the action of $\mathcal{B}_b$. We construct an isometry I between V and $\overset{\circ}{\mathcal{M}}$.

Proposition 2.1. *For $u \in \mathrm{Dom}(\mathcal{E}^\sigma)$, define $I(Du) = M^{[u]}$. Then I is uniquely extended to an operator from V to $\overset{\circ}{\mathcal{M}}$ which is isometric as well as $\mathcal{B}_b$-homomorphic. Furthermore, It holds that*

$$(2.3) \qquad \langle I(d^{(1)}), I(d^{(2)}) \rangle_t = \int_0^t (\sigma d^{(1)}, d^{(2)})(X_s) ds$$

for $d^{(i)} \in V$, $i = 1, 2$.

Proof. Define a subspace V_0 of V by

$$V_0 = \left\{ \sum_{i=1}^n f_i Du_i \, \middle| \, n \in \mathbb{N}, f_i \in \mathcal{B}_b, u_i \in \mathrm{Dom}(\mathcal{E}^\sigma), i = 1, \ldots, n \right\}.$$

For $d \in V_0$ represented as $d = \sum_{i=1}^n f_i Du_i$, we set $I(d) = \sum_{i=1}^n f_i \bullet M^{[u_i]}$. Then for $d^{(k)} = \sum_{i=1}^{n_k} f_i^{(k)} Du_i^{(k)} \in V_0$ $(k = 1, 2)$,

$$\begin{aligned}
e(I(d^{(1)}), I(d^{(2)})) &= \frac{1}{2} \int_B \sum_{i,j} f_i^{(1)} f_j^{(2)} d\mu_{\langle u_i^{(1)}, u_j^{(2)} \rangle} \\
&= \frac{1}{2} \int_B \sum_{i,j} f_i^{(1)} f_j^{(2)} (\sigma Du_i^{(1)}, Du_j^{(2)}) d\mu \\
&= (d^{(1)} | d^{(2)})_V.
\end{aligned}$$

This implies that $I|_{V_0}$ is an isometry into $\overset{\circ}{\mathcal{M}}$ as well as well-defined. If V_0 is dense in V and $I(V_0)$ is dense in $\overset{\circ}{\mathcal{M}}$, then I can be extended to an isometry between V and $\overset{\circ}{\mathcal{M}}$. The denseness of $I(V_0)$ follows from a variant of [6, Lemma 5.6.3]. We prove the denseness of V_0. Let $\{e_i\} \subset B^* \subset H$ be a c.o.n.s. of H. For each n, a projection operator π_n on H is defined by $\pi_n h = \sum_{i=1}^n (h, e_i) e_i$, $h \in H$. Let $F(s) = \arctan s$, $s \in \mathbb{R}$. For any $d \in V$, define Borel subsets $\{A_n\}_{n \in \mathbb{N}}$ of B by

$$A_n = \{ |\sigma^{1/2} \pi_n d|^2 \leq |\sigma^{1/2} d|^2 + 1, |(d, e_i)| \leq n F'(_B(\cdot, e_i)_{B^*}), \ i = 1, \ldots, n \}.$$

Then $\lim_{n \to \infty} A_n = B$. Take

$$d_n := \sum_{i=1}^n \frac{(d, e_i)}{F'(_B(\cdot, e_i)_{B^*})} \cdot 1_{A_n} D\Big(F(_B(\cdot, e_i)_{B^*}) \Big) \in V_0.$$

By noting that $d_n = \sum_{i=1}^n (d, e_i) \cdot 1_{A_n} e_i = 1_{A_n} \pi_n d$, it holds that

$$(d_n - d | d_n - d)_V = \int_B |1_{A_n} \sigma^{1/2} \pi_n d - \sigma^{1/2} d|^2 d\mu.$$

Since the integrand is dominated by $4|\sigma^{1/2}d|^2 + 2$ and converges to 0 μ-a.e., we have that $d_n \to d$ in V as $n \to \infty$ by the dominated convergence theorem. Therefore, V_0 is dense in V.

It is clear that $I|_{V_0}$ is $\mathcal{B}_b$-homomorphic. So is the extended I due to the fact that the actions of $\mathcal{B}_b$ on V and $\overset{\circ}{\mathcal{M}}$ are continuous.

Uniqueness of the extension is evident from the way of construction.

Lastly, we prove (2.3). For any Borel set $A \subset B$, we have

$$\frac{1}{2}\int_A (\sigma d^{(1)}, d^{(2)})d\mu = (d^{(1)}|1_A d^{(2)})_V = e(I(d^{(1)}), I(1_A d^{(2)}))$$

$$= e(I(d^{(1)}), 1_A \bullet I(d^{(2)})) = \frac{1}{2}\mu_{\langle I(d^{(1)}), I(d^{(2)})\rangle}(A).$$

Hence we obtain that $d\mu_{\langle I(d^{(1)}), I(d^{(2)})\rangle} = (\sigma d^{(1)}, d^{(2)})d\mu$. This concludes (2.3). □

In particular, for $d \in V$ and $u \in \text{Dom}(\mathcal{E}^\sigma)$,

$$(2.4) \qquad\qquad \langle I(d), Du\rangle_t = \int_0^t (\sigma d, Du)(X_s)ds.$$

We set $Y := I(\sigma^{-1}b)$ and $Z'_t := \exp(Y_t - (1/2)\langle Y\rangle_t)$, the exponential local martingale of Y.

Proposition 2.2. (1) $\{Z'_t\}_{t \geq 0}$ *is a continuous martingale under* P_x *for q.e.x.*

(2) *For any* $p \geq 1$, *there exists* $T > 0$ *such that* $\{Z'_t\}_{0 \leq t \leq T}$ *is an* L^p-*martingale under* P_μ, *and under* P_x *for q.e.x.*

(3) *For* $p > 0$ *and* $r > 1$, $E_x[Z'^p_t] \leq E_x[\exp\{(pr(pr - r + 1)/(2(r - 1)))$ $\int_0^t |(\sigma^{-1/2}b)(X_s)|^2 ds\}]^{1/r}$ *for q.e.x.*

Proof.   First we prove (2) and (3). Let r' be the conjugate exponent of $r > 1$. For any stopping time τ, Hölder's inequality and (2.3) imply that for q.e.x,

$$E_x[Z'^p_{t\wedge\tau}] \leq E_x\left[\exp\left(pr'Y_{t\wedge\tau} - \frac{1}{2}p^2 r'^2\langle Y\rangle_{t\wedge\tau}\right)\right]^{1/r'} E_x\left[\exp\left\{\frac{pr(pr' - 1)}{2}\langle Y\rangle_{t\wedge\tau}\right\}\right]^{1/r}$$

$$\leq E_x\left[\exp\left\{\frac{pr(pr - r + 1)}{2(r - 1)}\int_0^t |(\sigma^{-1/2}b)(X_s)|^2 ds\right\}\right]^{1/r}.$$

This includes (3). Furthermore, by Jensen's inequality,

$$E_x[Z'^p_{T\wedge\tau}] \leq \left[\frac{e^T}{T}R_1^\sigma\left(\exp\left\{\frac{pr(pr - r + 1)}{2(r - 1)}T|\sigma^{-1/2}b|^2\right\}\right)(x)\right]^{1/r},$$

which is finite q.e. for $T = (r-1)\theta/(pr(pr-r+1))$, $p \geq 1$. Moreover,

$$(2.5) \quad E_\mu[Z'^p_{T \wedge \tau}] \leq \left[\frac{e^T}{T} \int_B \exp\left\{\frac{pr(pr-r+1)}{2(r-1)} T|\sigma^{-1/2}b|^2\right\} d\mu\right]^{1/r} < \infty.$$

Therefore, (2) follows. Lastly, we prove (1). For any smooth probability measure ν, we have

$$E_\nu[Z'_{2T}|\mathcal{F}_T] = E_\nu[Z'_T \circ \theta_T \cdot Z'_T|\mathcal{F}_T] = E_\nu[Z'_T \circ \theta_T|\mathcal{F}_T]Z'_T$$

$$= E_{X_T}[Z'_T]Z'_T = Z'_T \qquad P_\nu\text{-a.e.}$$

Therefore, $E_\nu[Z'_{2T}] = 1$. On the other hand, $\{Z'_t\}_{t \geq 0}$ is a supermartingale under P_x q.e., so in particular, $E_x[Z'_{2T}] \leq 1$. Combining these, we have that $E_x[Z'_{2T}] = 1$, ν-a.e.x. Thus $E_x[Z'_{2T}] = 1$ q.e. and $\{Z'_t\}_{0 \leq t \leq 2T}$ is a martingale under P_x q.e. Repeating this argument, we obtain (1). □

Take an exceptional Borel set N_0 of B such that $\{Z'_t\}$ is a continuous martingale under P_x for $x \in B \backslash N_0$. Put $Z_t = \exp\{(Y_t - (1/2)\langle Y\rangle_t) \cdot 1_{\{X_0 \in B \backslash N_0\}}\}$, and define probability measures $\{Q_x\}$, $x \in B$ on Ω by $Q_x|_{\mathcal{M}_t} = Z_t P_x|_{\mathcal{M}_t}$ for all $t > 0$. Then $\{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, Q_x, x \in B\}$ is a conservative diffusion process on B. The choice of N_0 does not have influence on the process in the following sense.

Proposition 2.3. *If $N_1 \supset N_0$ is an exceptional Borel set such that $B \backslash N_1$ is (X_t, P_x)-invariant, then $B \backslash N_1$ is also (X_t, Q_x)-invariant.*

Proof. Let $\sigma_{N_1} = \inf\{t > 0; X_t \in N_1\}$ be the first hitting time. For $x \in B \backslash N_1$ and $t > 0$,

$$Q_x(\sigma_{N_1} < t) = E_x[1_{\{\sigma_{N_1} < t\}}Z_t] = 0.$$

Hence $Q_x(\sigma_{N_1} < \infty) = 0$. □

We remark that there always exists such N_1 as above.

Now, for $t > 0$ and a bounded Borel function f on B, we define $T_t f$ by

$$T_t f(x) := \int_B f(X_t) dQ_x = E_x[f(X_t)Z_t].$$

$\{T_t\}$ is uniquely extended to a conservative Markovian semigroup on L^∞. In the next section, we will show that under an additional condition, $\{T_t\}$ is extended to a strongly continuous semigroup on L^p for some $p > 1$, and its infinitesimal generator is consistent with an operator $\mathcal{L}^\sigma + (b, D\cdot)$.

3. Analysis on L^p space

From this section, we impose the following additional assumption.
(A3) $\theta > 2/\delta$.
Then there exists $p > 1$ such that $\theta > (2/\delta)(p/(p-1))^2$. We fix this p.

Proposition 3.1. $\{T_t\}$ can be uniquely extended to a strongly continuous semigroup on L^p. Moreover, $\{T_t\}$ has a hyperbounded property in the following sense: for $t > 0$, $p < q < \infty$, $\alpha' > \alpha = 4/\delta$ with $(\alpha/2)((p/(p-1)) \cdot (\alpha'/(\alpha'-\alpha)))^2 < \theta$ and $(q-1)/(p-1) \leq e^{4t/\alpha'}$,

$$(3.1) \quad \|T_t\|_{p \to q} \leq \left\| \exp\left\{ \frac{\alpha}{2} \left(\frac{p}{p-1} \cdot \frac{\alpha'}{\alpha'-\alpha} \right)^2 \left| \sigma^{-1/2} b \right|^2 \right\} \right\|_1^{(2/(p-1))((q-1)/q)^2(1/\alpha-1/\alpha')t} .$$

In particular,

$$\|T_t\|_{p \to p} \leq \left\| \exp\left\{ \frac{\alpha}{2} \left(\frac{p}{p-1} \right)^2 \left| \sigma^{-1/2} b \right|^2 \right\} \right\|_1^{2(p-1)t/(p^2\alpha)} .$$

Proof.   We take $\alpha'' \geq \alpha'$ so that $(q-1)/(p-1) = e^{4t/\alpha''}$. For fixed large $n \in \mathbb{N}$, we define successively $p = p_0 < p_1 < \cdots < p_n = q$ by $(p_{k+1} - 1)/(p_k - 1) = e^{4t/n\alpha''}$, $k = 0, 1, \ldots, n-1$. For fixed $\lambda \in (0, 1 - \alpha/\alpha'')$, we take sequences $\{a_k\}$, $\{r_k\}$, $\{s_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ such that

$$\frac{a_k - 1}{p_k - 1} = \lambda, \quad r_k = \frac{p_k}{a_k} > 1, \quad \frac{s_k - 1}{r_k - 1} = e^{4t/n\alpha},$$

$$\frac{1}{a_k s_k} + \frac{1}{\beta_k} = \frac{1}{p_{k+1}}, \quad \frac{1}{a_k} + \frac{1}{\beta_k} + \frac{1}{\gamma_k} = 1,$$

$k = 0, 1, \ldots, n-1$. We remark that $\beta_k > 0$ when n is sufficiently large, and that

$$(3.2) \qquad\qquad \lim_{n \to \infty} \max_{0 \leq k \leq n-1} \frac{1}{\beta_k} = 0.$$

Indeed, these follow from

$$\frac{1}{\beta_k} = \frac{1}{p_{k+1}} - \frac{1}{a_k s_k}$$

$$= \frac{1}{p_k + (p_k - 1)(e^{4t/n\alpha''} - 1)} - \frac{1}{p_k + (p_k - 1)(1 - \lambda)(e^{4t/n\alpha} - 1)}$$

and

$$\frac{(1-\lambda)(e^{4t/n\alpha} - 1)}{e^{4t/n\alpha''} - 1} \xrightarrow{n \to \infty} \frac{(1-\lambda) \cdot 4t/\alpha}{4t/\alpha''} > 1.$$

We also note that when n is large enough,

(3.3) $1/\gamma_k$ is greater than a positive constant independent of n and k.

For $f \in L^\infty$, we have

$$\|T_{t/n}f\|_{p_{k+1}} = \left(\int_B |E_x[f(X_{t/n})Z_{t/n}]|^{p_{k+1}} \mu(dx) \right)^{1/p_{k+1}}$$

$$\leq \left(\int_B E_x[|f(X_{t/n})|^{a_k}]^{p_{k+1}/a_k} E_x[Z_{t/n}^{a_k/(a_k-1)}]^{p_{k+1}(a_k-1)/a_k} \mu(dx) \right)^{1/p_{k+1}}.$$

By applying Proposition 2.2(3) with $p = a_k/(a_k - 1)$ and $r = (a_k - 1)\beta_k/a_k$, this is dominated by

$$\left(\int_B E_x[|f(X_{t/n})|^{a_k}]^{p_{k+1}/a_k} \right.$$

$$\times E_x \left[\exp \left\{ \int_0^{t/n} \frac{\beta_k(\gamma_k - 1)}{2} |(\sigma^{-1/2}b)(X_s)|^2 ds \right\} \right]^{p_{k+1}/\beta_k} \mu(dx) \right)^{1/p_{k+1}}$$

$$\leq \left(\int_B E_x[|f(X_{t/n})|^{a_k}]^{s_k} \mu(dx) \right)^{1/a_k s_k}$$

$$\times \left(\int_B E_x \left[\exp \left\{ \int_0^{t/n} \frac{\beta_k(\gamma_k - 1)}{2} |(\sigma^{-1/2}b)(X_s)|^2 ds \right\} \right] \mu(dx) \right)^{1/\beta_k}$$

$$=: I_1 \times I_2.$$

Then

$$I_1 = \left\| T_{t/n}^\sigma(|f|^{a_k}) \right\|_{s_k}^{1/a_k} \leq \||f|^{a_k}\|_{r_k}^{1/a_k} \quad \text{(by (2.2))}$$

$$= \|f\|_{a_k r_k} = \|f\|_{p_k},$$

$$I_2 \leq \left\| \exp \left\{ \frac{\beta_k(\gamma_k - 1)t}{2n} |\sigma^{-1/2}b|^2 \right\} \right\|_1^{1/\beta_k}.$$

Therefore, if

(3.4) $$\frac{\beta_k(\gamma_k - 1)t}{2n} \leq \theta \quad \text{for every } k = 0, 1, \ldots, n - 1,$$

then $T_{t/n}$ can be extended to a bounded operator from L^{p_k} to $L^{p_{k+1}}$, and it holds that

(3.5) $$\|T_t\|_{p \to q} \leq \prod_{k=0}^{n-1} \|T_{t/n}\|_{p_k \to p_{k+1}} \leq \prod_{k=0}^{n-1} \left\| \exp \left\{ \frac{\beta_k(\gamma_k - 1)t}{2n} |\sigma^{-1/2}b|^2 \right\} \right\|_1^{1/\beta_k}$$

$$\leq \left\| \exp \left[\left\{ \max_{0 \leq k \leq n-1} \frac{\beta_k(\gamma_k - 1)t}{2n} \right\} |\sigma^{-1/2}b|^2 \right] \right\|_1^{\sum_{k=0}^{n-1} 1/\beta_k}.$$

We will show that (3.4) holds for suitable λ and for sufficiently large n, and estimate the last term in (3.5). First we estimate

$$\frac{\beta_k(\gamma_k - 1)t}{2n} = \frac{1}{2}(a_k - 1)(\gamma_k - 1)\left(\frac{n}{t} \cdot \frac{a_k - 1}{\beta_k}\right)^{-1}.$$

The equation $(a_k - 1)(\gamma_k - 1) = 1 + a_k\gamma_k/\beta_k$, (3.2) and (3.3) imply that

$$\lim_{n \to \infty} \max_{0 \le k \le n-1} (a_k - 1)(\gamma_k - 1) = 1.$$

We also have

$$\frac{n}{t} \cdot \frac{a_k - 1}{\beta_k} = \frac{n}{t} \cdot \frac{\lambda(p_k - 1)}{\beta_k}$$

$$= \frac{\lambda n}{t} \cdot \frac{(1 - \lambda)(e^{4t/n\alpha} - 1) - (e^{4t/n\alpha''} - 1)}{\{p_k/(p_k - 1) + (e^{4t/n\alpha''} - 1)\}\{p_k/(p_k - 1) + (1 - \lambda)(e^{4t/n\alpha} - 1)\}}$$

$$\ge \frac{\lambda n}{t} \cdot \frac{(1 - \lambda)(e^{4t/n\alpha} - 1) - (e^{4t/n\alpha''} - 1)}{\{p/(p - 1) + (e^{4t/n\alpha''} - 1)\}\{p/(p - 1) + (1 - \lambda)(e^{4t/n\alpha} - 1)\}}.$$

Hence

$$\varliminf_{n \to \infty} \min_{0 \le k \le n-1} \left(\frac{n}{t} \cdot \frac{a_k - 1}{\beta_k}\right) \ge \frac{\lambda\{(1 - \lambda) \cdot (4/\alpha) - (4/\alpha'')\}}{(p/(p - 1))^2}.$$

The right-hand side, regarded as a function of λ, takes its maximum $(1/\alpha) \times (((p - 1)/p) \cdot ((\alpha'' - \alpha)/\alpha''))^2$ at $\lambda = (1 - (\alpha/\alpha''))/2$. Then we have

$$\varlimsup_{n \to \infty} \max_{0 \le k \le n-1} \frac{\beta_k(\gamma_k - 1)t}{2n} \le \frac{\alpha}{2}\left(\frac{p}{p - 1} \cdot \frac{\alpha''}{\alpha'' - \alpha}\right)^2 < \theta,$$

which implies (3.4) for large enough n. Also, when $\lambda = (1 - (\alpha/\alpha''))/2$,

$$\sum_{k=0}^{n-1} \frac{1}{\beta_k}$$

$$= \sum_{k=0}^{n-1} \frac{1}{p_k - 1} \cdot \frac{(1 - \lambda)(e^{4t/n\alpha} - 1) - (e^{4t/n\alpha''} - 1)}{\{p_k/(p_k - 1) + (e^{4t/n\alpha''} - 1)\}\{p_k/(p_k - 1) + (1 - \lambda)(e^{4t/n\alpha} - 1)\}}$$

$$\le \frac{n}{p - 1} \cdot \frac{(1 - \lambda)(e^{4t/n\alpha} - 1) - (e^{4t/n\alpha''} - 1)}{\{q/(q - 1) + (e^{4t/n\alpha''} - 1)\}\{q/(q - 1) + (1 - \lambda)(e^{4t/n\alpha} - 1)\}}$$

$$\xrightarrow{n \to \infty} \frac{1}{p - 1} \cdot \frac{(1 - \lambda) \cdot (4t/\alpha) - 4t/\alpha''}{(q/(q - 1))^2}$$

$$= \frac{2}{p - 1}\left(\frac{q - 1}{q}\right)^2\left(\frac{1}{\alpha} - \frac{1}{\alpha''}\right)t.$$

Letting $n \to \infty$ in (3.5), we get that

$$\|T_t\|_{p \to q} \leq \left\| \exp\left\{ \frac{\alpha}{2} \left(\frac{p}{p-1} \cdot \frac{\alpha''}{\alpha'' - \alpha} \right)^2 |\sigma^{-1/2}b|^2 \right\} \right\|_1^{(2/(p-1))((q-1)/q)^2(1/\alpha - 1/\alpha'')t}$$

Jensen's inequality yields (3.1).

Lastly we show that $\{T_t\}$ is strongly continuous on L^p. Take $T > 0$ and a bounded continuous function f on B. Then for $0 < t < T$,

$$|T_t f(x) - f(x)| = |E_x[\{f(X_t) - f(X_0)\}Z_T]|,$$

which converges to 0 pointwise as $t \downarrow 0$ by the dominated convergence theorem. Therefore, $T_t f \to f$ in L^p as $t \downarrow 0$ by using the dominated convergence theorem again. $\qquad \Box$

The infinitesimal generator of $\{T_t\}$ on L^p is denoted by A_p. It has consistency with respect to p: $A_p \supset A_{p'}$ if $p < p'$. We will give the explicit expression of A_p. We prepare an operator $\tilde{A}$ defined by

$$\tilde{A}f = \mathcal{L}^\sigma f + (b, Df), \quad \text{Dom}(\tilde{A}) = \{f \mid f \in \text{Dom}(\mathcal{L}_1^\sigma), Df \in L^1\}.$$

Proposition 3.2. *If* $f \in \left(\bigcup_{r>1} \text{Dom}(\mathcal{L}_r^\sigma) \right) \cap \text{Dom}(\mathcal{E}^\sigma) \cap L^p$ *satisfies* $\tilde{A}f \in L^p$, *then* f *also belongs to* $\text{Dom}(A_p)$ *and* $A_p f = \tilde{A}f$.

Proof. Since $nG_n^\sigma f \in \text{Dom}(A_2)$, the Fukushima decomposition theorem induces that

$$n\widetilde{G_n^\sigma f}(X_t) - n\widetilde{G_n^\sigma f}(X_0)$$
$$= M_t^{[nG_n^\sigma f]} + \int_0^t (\mathcal{L}^\sigma nG_n^\sigma f)(X_s)ds \quad \text{for every } t \geq 0, \ P_x\text{-a.e. for q.e.} x,$$

where $\tilde{}$ means a quasi-continuous version. Noticing that $nG_n^\sigma f \to f$ in $\mathcal{E}_1^\sigma$-sense and $\mathcal{L}^\sigma G_n^\sigma f = G_n^\sigma \mathcal{L}^\sigma f$ μ-a.e., we have

$$\tilde{f}(X_t) - \tilde{f}(X_0) = M_t^{[f]} + \int_0^t (\mathcal{L}^\sigma f)(X_s)ds \quad \text{for every } t \geq 0, \ P_x\text{-a.e. for } \mu\text{-a.e.} x.$$

We set a stopping time T_a, $a > 0$ by

$$T_a = \inf\{t > 0; |Z_t| \geq a \text{ or } \langle Z \rangle_t \geq a \text{ or } |\tilde{f}(X_t)| \geq a\}.$$

Using the equation $Z_t = 1 + \int_0^t Z_s dY_s$, Itô's formula, and (2.4), we have, for μ-a.e.x, and any $t > 0$, the following identities; P_x-a.e.,

$$\tilde{f}(X_t)Z_{t \wedge T_a} = \tilde{f}(X_0) + \int_0^t Z_{s \wedge T_a} dM_s^{[f]} + \int_0^t Z_{s \wedge T_a}(\mathcal{L}^\sigma f)(X_s) ds$$
$$+ \int_0^t \tilde{f}(X_s) dZ_{s \wedge T_a} + \int_0^{t \wedge T_a} Z_s d\langle M^{[f]}, Y \rangle_s$$
$$= \tilde{f}(X_0) + \int_0^t Z_{s \wedge T_a} dM_s^{[f]} + \int_0^t Z_{s \wedge T_a}(\mathcal{L}^\sigma f)(X_s) ds$$
$$+ \int_0^{t \wedge T_a} \tilde{f}(X_s) dZ_s + \int_0^{t \wedge T_a} Z_s (b, Df)(X_s) ds.$$

Therefore,

$$E_x[\tilde{f}(X_t)Z_{t \wedge T_a}] = E_x[\tilde{f}(X_0)] + E_x[\int_0^t Z_{s \wedge T_a}(\mathcal{L}^\sigma f)(X_s) ds]$$
$$+ E_x[\int_0^t Z_s (b, Df)(X_s) 1_{\{s \le T_a\}} ds].$$

From Proposition 2.2(2), for some $T > 0$, the integrand of each term is uniformly integrable with respect to a for $t \le T$. Letting $a \to \infty$, we have

$$T_t f(x) = f(x) + \int_0^t T_s(\tilde{A}f)(x) ds \quad \text{if } t \le T.$$

Hence

$$\frac{1}{t}(T_t f(x) - f(x)) = \frac{1}{t} \int_0^t T_s(\tilde{A}f)(x) ds.$$

Since $\tilde{A}f \in L^p$ and $\{T_t\}$ is strongly continuous on L^p, we conclude that $f \in \text{Dom}(A_p)$ and $A_p f = \tilde{A}f$. $\square$

This proposition formally justifies to say that $\{T_t\}$ is a corresponding semigroup of $\tilde{A}$. We do not know in general, however, whether the class of functions satisfying the assumption is large. The next proposition shows that smooth functions belong to the domain if σ has some smoothness.

Proposition 3.3. *Suppose that* $|b| \in L^p$ *and that* $\sigma\varphi \in D_1^{p'}(H)$, $p' = p \vee 2$ *for every* $\varphi \in B^* \subset H$. *Then* $\mathcal{F}C_b^\infty \subset \text{Dom}(A_p)$.

Proof.   Take $f \in \mathcal{F}C_b^\infty$. By the assumption, $\sigma Df \in D_1^{p'}(H)$. Thus $D^*\sigma Df$ can be defined and belongs to $L^{p'}$. Let q' be the conjugate exponent of p'. For any $g \in$

$\mathrm{Dom}(\mathcal{L}_{q'}^{\sigma}) \cap \mathrm{Dom}(\mathcal{E}^{\sigma})$, we have

$$\int_B f(-\mathcal{L}_{q'}^{\sigma} g) \, d\mu = \lim_{t \downarrow 0} \int_B f \cdot \frac{g - T_t^{\sigma} g}{t} \, d\mu$$
$$= \frac{1}{2} \int_B (\sigma Df, Dg) \, d\mu = \frac{1}{2} \int_B (D^* \sigma Df) g \, d\mu.$$

Therefore, $f \in \mathrm{Dom}((\mathcal{L}_{q'}^{\sigma})^*) = \mathrm{Dom}(\mathcal{L}_{p'}^{\sigma}) \subset \mathrm{Dom}(\mathcal{L}_p^{\sigma})$ and $\mathcal{L}_p^{\sigma} f = -(1/2)D^* \sigma Df$. Proposition 3.2 completes the proof. $\qquad\square$

We say that a bilinear form $\mathcal{E}$ on a Hilbert space $\mathcal{H}$ with a domain $D(\mathcal{E})$ is a coercive closed form with bound constant M if $\mathcal{E}_M(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) + M(\cdot, \cdot)_{\mathcal{H}}$ is a coercive closed form in the usual sense (cf. [14, Chapter I]). Then there exists a corresponding resolvent, a semigroup, and an infinitesimal generator.

When θ is large enough for $\{T_t\}$ to be a semigroup on L^2, we will see that there exists an associated closed form.

Proposition 3.4. *Suppose that $\theta > 8/\delta$. Then the bilinear form*

$$\mathcal{E}(f,g) = \int_B \left\{ \frac{1}{2}(\sigma Df, Dg) - (bg, Df) \right\} \, d\mu, \quad \mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(\mathcal{E}^{\sigma})$$

is a coercive closed form on L^2 with some bound constant $M \geq 0$. The corresponding generator is equal to A_2.

Proof.   The proof of the first part is standard; by Hausdorff-Young's inequality $st \leq e^s + t \log t - t$, $s \in \mathbb{R}$, $t > 0$, and the logarithmic Sobolev inequality (2.1), we have

$$\left| \int_B (bg, Df) d\mu \right| \leq \|\sigma^{1/2} Df\|_2 \|\sigma^{-1/2} bg\|_2$$

$$\leq \sqrt{2} \mathcal{E}^{\sigma}(f,f)^{1/2} \left\{ \int_B \theta^{-1} \left(e^{\theta |\sigma^{-1/2}b|^2} + \frac{g^2}{\|g\|_2^2} \log \frac{g^2}{\|g\|_2^2} \right) d\mu \right\}^{1/2} \|g\|_2$$

$$\leq (2\theta^{-1})^{1/2} \mathcal{E}^{\sigma}(f,f)^{1/2} \left\{ \|e^{\theta |\sigma^{-1/2}b|^2}\|_1^{1/2} \|g\|_2 + (4\delta^{-1})^{1/2} \mathcal{E}^{\sigma}(g,g)^{1/2} \right\}.$$

Therefore we see easily that

$$c \mathcal{E}_1^{\sigma}(f,f) \leq \mathcal{E}_M(f,f) \leq c^{-1} \mathcal{E}_1^{\sigma}(f,f),$$

$$\mathcal{E}_{M+1}(f,g) \leq K \mathcal{E}_{M+1}(f,f)^{1/2} \mathcal{E}_{M+1}(g,g)^{1/2}$$

for some $M \geq 0$, $c > 0$ and $K \geq 1$.

Let A' be the corresponding generator to $\mathcal{E}$. Take $f \in \mathrm{Dom}(A') \subset \mathrm{Dom}(\mathcal{E}^\sigma)$. Then for $g \in \mathrm{Dom}(\mathcal{L}_q^\sigma)$, $q > 2$,

$$\int_B f(-\mathcal{L}_q^\sigma g) \, d\mu = \mathcal{E}^\sigma(f, g) = \mathcal{E}(f, g) + \int_B (bg, Df) \, d\mu = \int_B \{-A'f + (b, Df)\}g \, d\mu.$$

Thus $f \in \mathrm{Dom}(\mathcal{L}_p^\sigma)$ for every $1 < p < 2$ and $\mathcal{L}^\sigma f = A'f - (b, Df)$. From Proposition 3.2, we have that $f \in \mathrm{Dom}(A_2)$ and $A_2 f = A'f$. Hence $A' \subset A_2$. Since $(\kappa - A')^{-1} \supset (\kappa - A_2)^{-1}$ for sufficiently large κ and both operators are everywhere defined, we conclude that $A' = A_2$. $\qquad\square$

4. Existence of invariant measures

The purpose of this section is to prove the following theorem.

Theorem 4.1. *Under* (A1), (A2), *and* (A3), *there exists a unique function* $\rho \in L^1$ *such that* $\rho \geq 0$, $\|\rho\|_1 = 1$, *and* (1.1) *holds. In this case,* $\rho > 0$ μ-a.e. *Moreover, if* $\theta > 2q^2/\delta$ *for* $q > 1$, *then* $\rho \in L^q$.

The uniqueness and the strict positivity are shown in the same way as [17, Proposition 3.1]. So we concentrate on the existence part. The following proposition assures that we can take an approximation method.

Proposition 4.2. *Let* $p > 1$ *and let* $\{T_t\}$ *be a semigroup on* L^p. *Suppose that we are given a sequence of sub σ-fields* $\{\mathcal{F}_n\}$ *of* $\mathcal{F}$ *and semigroups* $\{T_t^n\}$ *on* $L^p(\mathcal{F}_n) = L^p(B, \mathcal{F}_n, \mu)$. *Define the projection* P_n *from* L^p *to* $L^p(\mathcal{F}_n)$ *by* $P_n f = E[f|\mathcal{F}_n]$, *the conditional expectation with respect to* μ *given* $\mathcal{F}_n$. *We also assume the following*:
(1)   *For every* $f \in L^p$, $P_n f$ *converges to* f *in* L^p.
(2)   *For every* $f \in L^p$ *and* $t > 0$, $T_t^n P_n f$ *converges to* $T_t f$ *in* L^p.
(3)   *For every* n, *there exists an invariant measure* $\rho_n \, d\mu$ *of* $\{T_t^n\}$ *with* $\rho_n \in L^q(\mathcal{F}_n)$, $\rho_n \geq 0$, $\rho_n \not\equiv 0$. *Here* q *is the conjugate exponent of* p.
(4)   *For some* $t_1 > 0$ *and some* $p' > p$,

$$(4.1) \qquad\qquad \sup_n \|T_{t_1}^n\|_{L^p(\mathcal{F}_n) \to L^{p'}(\mathcal{F}_n)} < \infty.$$

Then there exists an invariant measure $\rho \, d\mu$ *of* $\{T_t\}$ *with* $\rho \in L^q$, $\rho \geq 0$, $\rho \not\equiv 0$.

Proof.   We may assume that $\|\rho_n\|_q = 1$ by normalization. Then we can take a sequence $\{n_k\}$ such that ρ_{n_k} converges weakly to some ρ in L^q. Letting $k \to \infty$ in the equation

$$\int_B (T_t^{n_k} P_{n_k} f) \rho_{n_k} \, d\mu = \int_B (P_{n_k} f) \rho_{n_k} \, d\mu, \quad f \in L^p,$$

we obtain that ρ satisfies (1.1). Thus the rest to be proved is that ρ is not identically zero. Suppose that $\rho \equiv 0$. Then $\rho_{n_k} \to 0$ in L^1 since $\rho_n \geq 0$. This implies that $\rho_{n_k} \to 0$ in L^r for any $r \in [1, q)$ by Hölder's inequality. For any k,

$$1 = \int_B \rho_{n_k}^q \, d\mu = \int_B T_{t_1}^{n_k}(\rho_{n_k}^{q-1})\rho_{n_k} \, d\mu \leq \|T_{t_1}^{n_k}(\rho_{n_k}^{q-1})\|_{p'}\|\rho_{n_k}\|_{q'} \quad (1/p' + 1/q' = 1)$$

$$\leq \|T_{t_1}^{n_k}\|_{L^p(\mathcal{F}_{n_k}) \to L^{p'}(\mathcal{F}_{n_k})}\|\rho_{n_k}^{q-1}\|_p\|\rho_{n_k}\|_{q'}.$$

The right-hand side converges to 0 as $k \to \infty$, by virtue of (4.1) and the facts that $\|\rho_{n_k}^{q-1}\|_p = 1$, and that $\|\rho_{n_k}\|_{q'} \to 0$ as $k \to \infty$. This is a contradiction. $\qquad \square$

The following characterization of invariant measures is proved in the same way as Proposition 3.2 in [17].

Lemma 4.3. *Let $p > 1$ and let q be the conjugate exponent of p. For a strongly continuous semigroup $\{T_t\}$ on L^p and $\rho \in L^q$, the following are equivalent:*
(i) *ρ satisfies (1.1).*
(ii) *$\rho \in \mathrm{Ker}(A_p^*)$, where A_p^* is an operator on L^q which is adjoint of A_p.*

Proof of Theorem 4.1.   We divide the proof into three steps.
(Step 1–the case when B is finite dimensional and b is bounded)   It is enough to prove that $\mathrm{Ker}(A_p^*) \neq \{0\}$ for any $p > 1$, since if ρ satisfies (1.1) then the positive part of ρ also satisfies (1.1) (see [17, Lemma 3.1]). From Proposition 3.4, there is a coercive closed form $\mathcal{E}$ with some bound constant $M \geq 0$, associated with A_2. Then we have the following sequence of continuous embeddings:

$$(\mathrm{Dom}(A_2), \|(M - A) \cdot \|_2 + \| \cdot \|_2) \subset (\mathrm{Dom}(\mathcal{E}^\sigma), \mathcal{E}_{M+1}(\cdot, \cdot)^{1/2}) \subset D_1^2 \subset L^2.$$

Since B is finite dimensional, the last embedding is a compact one. Hence, the inclusion $(\mathrm{Dom}(A_2), \|(M - A) \cdot \|_2 + \| \cdot \|_2) \subset L^2$ is compact, that is, A_2 is a compact resolvent operator. On the other hand, for any $p > 1$, $(\kappa - A_p)^{-1}$ is defined for sufficiently large $\kappa > M$. We can apply the argument in [4, Theorem 1.6.1] to show that it is also compact on L^p. From Riesz-Schauder's theorem, we obtain that $(\kappa - A_p^*)^{-1}$ is compact on L^q, $1/p + 1/q = 1$, its spectrum is the same as that of $(\kappa - A_p)^{-1}$, and every spectrum except 0 is an eigenvalue. Since 1 is an eigenfunction of the eigenvalue $1/\kappa$ of $(\kappa - A_p)^{-1}$, $(\kappa - A_p^*)^{-1}$ also has $1/\kappa$ as an eigenvalue. Therefore we conclude that $\mathrm{Ker}(A_p^*) \neq \{0\}$.
(Step 2–the case when B is infinite dimensional and b is bounded)   We apply Proposition 4.2. Let $\{e_i\} \subset B^* \subset H$ be a c.o.n.s. of H and B_n a linear span of $\{e_1, \ldots, e_n\}$. For each n, let $\mathcal{F}_n$ be a sub σ-field of $\mathcal{F}$ generated by $\{e_1, \ldots, e_n\}$. The projection P_n from L^p to $L^p(\mathcal{F}_n) = L^p(B, \mathcal{F}_n, \mu)$, $p \in [1, \infty]$ is defined by $P_n f = E[f|\mathcal{F}_n]$, the conditional expectation with respect to μ given $\mathcal{F}_n$. Let a

bilinear form $(\mathcal{E}^n, \mathrm{Dom}(\mathcal{E}^n))$ on L^2 be the smallest closed extension of $(\mathcal{E}, \{f \in \mathcal{FC}_b^\infty; f$ is $\mathcal{F}_n$-measurable$\})$. We can regard $(\mathcal{E}^n, \mathrm{Dom}(\mathcal{E}^n))$ as a form on $L^2(\mathcal{F}_n)$, in which case we write $(\mathcal{E}'^n, \mathrm{Dom}(\mathcal{E}'^n))$ to avoid confusion. Since $\mathrm{Dom}(\mathcal{E}'^n)$ is dense in $L^2(\mathcal{F}_n)$, there exists an associated strongly continuous semigroup $T_t'^n$ on $L^2(\mathcal{F}_n)$. Noting that Df is B_n-valued $\mathcal{F}_n$-measurable function for $f \in \mathrm{Dom}(\mathcal{E}'^n)$, we have the following expression:

$$\mathcal{E}'^n(f, g) = \int_B \left\{ \frac{1}{2}(\sigma_n Df, Dg) - (b_n g, Df) \right\} d\mu, \quad f, g \in \mathrm{Dom}(\mathcal{E}'^n)$$

where $\sigma_n : B \to \mathcal{SL}(B_n)$ and $b_n : B \to B_n$ are defined by

$$\sigma_n(\cdot)(h) = \sum_{i,j=1}^n E[(\sigma e_i, e_j)|\mathcal{F}_n](h, e_i)e_j, \quad h \in B_n,$$

$$b_n = \sum_{i=1}^n E[(b, e_i)|\mathcal{F}_n]e_i.$$

Since $(B, \mathcal{F}_n, \mu)$ and $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n),$ the standard Gaussian measure) are isomorphic as measure spaces and have the same differential structures, we can apply (Step 1) to prove the existence of an invariant measure $\rho_n \, d\mu$ of $\{T_t'^n\}$ with $\rho_n \in \bigcap_{p>1} L^p(\mathcal{F}_n)$, $\rho_n \geq 0$, $\rho_n \not\equiv 0$. We also remark that $T_t'^n$ has hyperbounded properties by Proposition 3.1. From [9, Example 4.1], $T_t'^n P_n$ converges strongly to T_t in L^2 for any $t > 0$. Besides, since the operator norms of $\{T_t'^n P_n\}_{n \in \mathbb{N}}$ on L^p is uniformly bounded for any $p \in (1, \infty)$ from Proposition 3.1, the convergence holds in L^p sense by virtue of Hölder's inequality. Hence we can apply Proposition 4.2 to get the assertion.

(Step 3–general cases) Let $p > 1$ satisfy that $\theta > (2/\delta)(p/(p-1))^2$. We take an approximate sequence $\{b_n\}$ of b by $b_n = b \cdot 1_{\{|b| \leq n\}}$. Set $Y^n = I(\sigma^{-1}b_n) \in \overset{\circ}{\mathcal{M}}$, and denote the corresponding Girsanov density and the semigroup on L^p by Z_t^n and $\{T_t^n\}$, respectively. Since $Y^n \to Y$ in $\overset{\circ}{\mathcal{M}}$ and $\langle Y^n \rangle_t \to \langle Y \rangle_t$ in $L^1(\Omega, P_\mu)$ from Proposition 2.1, there exists a sequence $\{n_k\} \uparrow \infty$ such that $Y_t^{n_k} \to Y_t$ and $\langle Y^{n_k} \rangle_t \to \langle Y \rangle_t$ P_μ-a.e. for every $t > 0$. Hence, $Z_t^{n_k} \to Z_t$ P_μ-a.e. for every $t > 0$. On the other hand, from the same type of estimates as (2.5), there is some $T > 0$ such that for $0 < t \leq T$, $\{E_\mu[(Z_t^n)^{p+1}]\}_{n \in \mathbb{N}}$ is uniformly bounded. Therefore, $Z_t^{n_k} \to Z_t$ in $L^p(\Omega, P_\mu)$ when $0 < t \leq T$. Now, for $f \in \mathcal{B}_b$ and $0 < t \leq T$,

$$\|T_t^{n_k}f - T_t f\|_p = \|E_\cdot[f(X_t)Z_t^{n_k}] - E_\cdot[f(X_t)Z_t]\|_p$$

$$\leq \|f\|_\infty \|E_\cdot[|Z_t^{n_k} - Z_t|]\|_p$$

$$\leq \|f\|_\infty E_\mu[|Z_t^{n_k} - Z_t|^p]^{1/p}$$

$$\to 0 \quad \text{as } k \to \infty.$$

Since $\{\|T_t^n\|_{p \to p}\}_{n \in \mathbb{N}}$ are uniformly bounded by Proposition 3.1, $T_t^{n_k}$ converges

strongly to T_t in L^p for any $t > 0$. By using (Step 2) and Proposition 3.1 again, an application of Proposition 4.2 yields the assertion.   □

Lemma 4.3 and Proposition 3.3 induce an additional property of the density function ρ of an invariant measure.

Corollary 4.4. *Suppose that there exists some $p > 1$ such that $\theta > (2/\delta) \times (p/(p-1))^2$, $|b| \in L^p$ and $\sigma\varphi \in D_1^{p'}(H)$, $p' = 2 \vee p$ for every $\varphi \in B^*$. Then ρ satisfies the following equation*:

$$\int_B \left\{ -\frac{1}{2}D^*\sigma Df + (b, Df) \right\} \rho \, d\mu = 0 \quad \text{for every } f \in \mathcal{FC}_b^\infty.$$

Finally, we remark about regularity of ρ. Suppose that $\theta > 8/\delta$. From Proposition 3.4, A_2 is defined and has an associated form the domain of which is equal to $\text{Dom}(\mathcal{E}^\sigma)$. So the same is true for A_2^*, and we conclude that $\rho \in \text{Dom}(A_2^*) \subset \text{Dom}(\mathcal{E}^\sigma) \subset D_1^2$.

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