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ERROR ESTIMATE IN OPERATOR NORM OF EXPONENTIAL PRODUCT FORMULAS FOR PROPAGATORS OF PARABOLIC EVOLUTION EQUATIONS

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1. Introduction

In the present work we study an error estimate in the operator norm of exponential product approximation for propagators of parabolic evolution equations. The obtained result applies to Schrödinger operators $-\Delta + V(t,x)$ with a certain class of time dependent singular potentials. One of typical examples is a positive Coulomb potential like $V(t,x) = c/|x - g(t)|$, $c > 0$, which has a singularity moving with time t.

Let X be a Hilbert space and let $\|\ \|$ denote the operator norm of bounded operators acting on X. We are now given positve self-adjoint operators A, $B(t) \geq c > 0$, t being in a compact interval $[0, T]$. We note that the assumption of positivity is not essential. In the discussion below, we have only to assume that these operators are semibounded uniformly in t. If the domain of $B(t)$ fulfills the inclusion relation

$$\mathcal{D}(A^{\alpha}) \subset \mathcal{D}(B(t)) \tag{1.1}$$

for some α, $0 \leq \alpha < 1$, independent of t, then the sum $C(t) = A + B(t)$ also becomes a positive self-adjoint operator with domain $\mathcal{D}(C(t)) = \mathcal{D}(A)$ independent of t. If, in addition, $B(t)$ satisfies a suitable continuity condition (see assumption (A) below), then $C(t)$ generates the propagator $U(t,s)$, $0 \leq s \leq t \leq T$, to the evolution equation

$$\partial_t U(t,s) = -C(t)U(t,s), \quad U(s,s) = \mathrm{Id},$$

where Id is the identity operator. As is easily seen, $U(t,s) : X \to X$ is a contraction operator.

We now consider the exponential product approximation for the propagator $U(t,0)$. Let

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = t, \quad t_j = j\tau, \quad \tau = t/N,$$

for t, $0 < t \leq T$, fixed. Then we define the following operators:

$$\begin{aligned}
&K_j(\tau) = \exp(-\tau A/2)\exp(-\tau B(t_{j-1}))\exp(-\tau A/2),\\
(1.2)\qquad &F_j(\tau) = K_j(\tau)K_{j-1}(\tau) \times \cdots \times K_2(\tau)K_1(\tau),\\
&\tilde{F}_{N-j}(\tau) = K_N(\tau)K_{N-1}(\tau) \times \cdots \times K_{j+2}(\tau)K_{j+1}(\tau)
\end{aligned}$$

for $1 \leq j \leq N$. By definition, these operators are all contraction. It is known ([2]) that $U(t,0)$ is approximated by the product formula

$$(1.3)\qquad U(t,0) = s - \lim_{N\to\infty} F_N(\tau)$$

in the strong topology. If $B(t) = B$ is time independent, then the product formula above

$$(1.4)\qquad \exp(-tC) = s - \lim_{N\to\infty} [\exp(-tA/2N)\exp(-tB)\exp(-tA/2N)]^N$$

with $C = A + B$ is called the Trotter-Kato product formula ([1, 3, 5]). For this formula (1.4), the error bound $O(N^{-1/2}\log N)$ in the operator norm has been established by Rogava [4] under the assumption that $C = A + B$ is self-adjoit with domain $\mathcal{D}(C) = \mathcal{D}(A) \subset \mathcal{D}(B)$. We here extend this result to the time dependent product formula (1.3) and prove the improved error bound $O(N^{-1}\log N)$ under the slightly restrictive assumption (1.1). It should be noted that the result obtained by [4] includes the case $\alpha = 1$. However the method developed there does not seem to apply to the time dependent case directly.

We shall formulate the obtained result more precisely. Let $A \geq c$ and $B(t) \geq c$, $c > 0$, be as above. We make the following assumption for these operators.

(A) There exists α, $0 \leq \alpha < 1$, independent of t, $t \in [0,T]$, such that : $\mathcal{D}(A^\alpha) \subset \mathcal{D}(B(t))$, $B(t)A^{-\alpha} : X \to X$ is uniformly bounded and

$$\|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\| \leq d|t-s|, \quad d > 0.$$

Throughout the entire discussion, the constant α is used with the meaning ascribed in the above assumption (A). The main theorem is fomulated as follows.

Theorem 1.1. *Let the notations be as above. Assume that assumption* (A) *is fulfilled. Then*

$$\|U(t,0) - F_N(\tau)\| = O(N^{-1}\log N), \quad N \to \infty,$$

uniformly in t, $0 \leq t \leq T$.

As stated above, the theorem immediately extends to the case that A and $B(t)$ are semi-bounded. We here discuss the application to the Schrödinger operators $-\Delta +$

$V(t,x)$ acting on $L^2(R^3)$. For example, we consider the positive Coulomb potential $V(t,x) = |x - g(t)|^{-1}$, which has a singularity moving with time t. If $g(t) : [0,\infty) \to R^3$ is of C^1-class, then it follows from the Hardy inequality that assumption (A) is fulfilled with $\alpha = 1/2$. As is easily seen from the Sobolev imbedding theorem, the theorem also applies to a certain class of potentials with $|x - g(t)|^{-p}$, $0 < p < 3/2$, as a local singularity.

In the final section (section 7), we further make two comments on the main theorem. First the same error bound as above is shown to remain true for the different product formula

$$\|U(t,0) - G_N(\tau) \times \cdots \times G_1(\tau)\| = O(N^{-1} \log N)$$

with $G_j(\tau) = \exp(-\tau A)\exp(-\tau B(t_{j-1}))$. The second comment is concerned with the case that $A(t)$ is also time dependent in the sum $C(t) = A(t) + B(t)$. If the domain $\mathcal{D}(A(t))$ is independent of t and if $A(t)$ satisfies a continuity condition similar to assumption (A), the method in the present work extends to such a case without any essential change.

2. Propagators of evolution equations

We here summarize several basic properties of the propagator generated by $C(t) = A + B(t)$. We first discuss the existence of such a propagator. Throughout the section, we again assume that $0 \le s \le t \le T$. By assumption (A), $C(t)A^{-1}$ and $C(t)^{-1}A$ are uniformly bounded and also it follows that

$$\|C(t)^\rho(C(t)^{-1} - C(s)^{-1})C(s)^\rho\| = O(|t-s|), \quad \rho = 1 - \alpha > 0. \tag{2.1}$$

According to [6], this guarantees the existence of propagator $U(t,s)$ to the evolution equations

$$\partial_t U(t,s) = -C(t)U(t,s), \quad U(s,s) = \mathrm{Id},$$
$$\partial_s U(t,s) = U(t,s)C(s), \quad U(t,t) = \mathrm{Id}.$$

We now mention some important properties of the propagator $U(t,s)$, which are required to prove the main theorem.

Lemma 2.1. *Let $0 \le \gamma \le 1$. Then the propagator $U(t,s)$ has the following properties*:

(1) $\|U(t,s)A^\gamma\| = O((t-s)^{-\gamma})$, $\|A^\gamma U(t,s)\| = O((t-s)^{-\gamma})$.
(2) $\|A^{-\gamma}U(t,s)A^\gamma\| = O(1)$, $\|A^\gamma U(t,s)A^{-\gamma}\| = O(1)$.
(3) $\|A^{-\gamma}U(t,s)A\| = O((t-s)^{\gamma-1})$, $\|AU(t,s)A^{-\gamma}\| = O((t-s)^{\gamma-1})$.

Proof. The lemma is verified in almost the same way as in [6] (Proposition 3.1).

We give only a sketch for a proof. We also prove only the first relations in the statements (1), (2) and (3). The second ones can be obtained in a similar way.

(1) By interpolation, it is enough to show the statement for the case $\gamma = 1$. Let

$$C_n(t) = C(t)(\mathrm{Id} + n^{-1}C(t))^{-1}, \quad n \gg 1,$$

be the Yosida approximation of $C(t)$. Then $C_n(t)$ is bounded for each n and it satisfies

$$\|C_n(t)^\rho (C_n(t)^{-1} - C_n(s)^{-1}) C_n(s)^\rho\| = O(|t-s|), \quad \rho = 1 - \alpha, \tag{2.2}$$

uniformly in n. We denote by $U_n(t,s)$ the propagator generated by $C_n(t)$. It is known that $U_n(t,s)$ is strongly convergent to $U(t,s)$ as $n \to \infty$. As is easily seen, $U_n(t,s)$ satisfies the relation

$$\begin{aligned} U_n(t,s) &= \exp(-(t-s)C_n(s)) \\ &\quad + \int_s^t U_n(t,r)(C_n(s) - C_n(r)) \exp(-(r-s)C_n(s))\, dr, \end{aligned} \tag{2.3}$$

and hence, if we set $V_n(t,s) = U_n(t,s)C_n(s)$, then it follows that

$$V_n(t,s) = \exp(-(t-s)C_n(s))C_n(s) + \int_s^t V_n(t,r)R_n(r,s)\, dr,$$

where

$$\begin{aligned} R_n(t,s) &= C_n(t)^{-1}(C_n(s) - C_n(t))C_n(s) \exp(-(t-s)C_n(s)) \\ &= (C_n(t)^{-1} - C_n(s)^{-1})C_n(s)^2 \exp(-(t-s)C_n(s)). \end{aligned}$$

By (2.2), we have

$$\|R_n(t,s)\| = O((t-s)^{-\alpha}), \quad s < t \le T,$$

uniformly in n and hence $\|R_n(t,s)\|$ is integrable as a function of t uniformly in n. If we further set

$$W_n(t,s) = V_n(t,s) - \exp(-(t-s)C_n(s))C_n(s),$$

then we have

$$W_n(t,s) = E_n(t,s) + \int_s^t W_n(t,r)R_n(r,s)\, dr, \tag{2.4}$$

where

$$E_n(t,s) = \int_s^t \exp(-(t-r)C_n(r))C_n(r)R_n(r,s)\, dr.$$

By (2.2) again, $\|C_n(t)^\rho R_n(t,s)\| = O((t-s)^{-\alpha})$ and hence $E_n(t,s)$ obeys the bound

$$\|E_n(t,s)\| = \int_s^t O((t-r)^{-\alpha})O((r-s)^{-\alpha})\,dr = O((t-s)^{1-2\alpha}),$$

which implies that $\|E_n(t,s)\|$ is also integrable as a function of t, $t \in [s,T]$. Thus the integral equation (2.4) can be solved by iteration and the solution $W_n(t,s)$ satisfies $\|W_n(t,s)\| = O((t-s)^{1-2\alpha})$, so that

$$\|V_n(t,s)\| = \|U_n(t,s)C_n(s)\| = O((t-s)^{-1})$$

uniformly in n. As previously stated, $U_n(t,s)$ strongly converges to $U(t,s)$. Since $C_n(s)$ also converges strongly to $C(s)$ on the dense set $\mathcal{D}(C(s)) = \mathcal{D}(A)$, we have that

$$U_n(t,s)C_n(s) \to U(t,s)C(s), \quad t \neq s,$$

strongly in X. This proves (1).

(2) By interpolation again, it suffices to prove the relation for the case $\gamma = 1$. We use the same argument as in the proof of (1). We set

$$\tilde{V}_n(t,s) = C_n(t)^{-1}U_n(t,s)C_n(s).$$

Then it follows from (2.3) that

$$\tilde{V}_n(t,s) = C_n(t)^{-1}C_n(s)\exp(-(t-s)C_n(s)) + \int_s^t \tilde{V}_n(t,r)R_n(r,s)\,dr$$

with the same kernel operator $R_n(t,s)$ as above. Thus we obtain that $\tilde{V}_n(t,s)$ is bounded uniformly in t, s and n, and hence

$$\|C(t)^{-1}U(t,s)C(s)\| = O(1).$$

The proof of (2) is complete.

(3) By interpolation, this follows from (1) and (2) at once. □

3. Strategy of proof of Theorem 1.1 ; three key lemmas

For brevity, we prove the main theorem for the case $t = 1$. Our aim is to evaluate the norm of difference $U(1,0) - F_N(\tau)$ with $\tau = 1/N$. Let $\varphi_0 \in C_0^\infty([0,\infty))$ be a smooth cut-off function such that $0 \le \varphi_0 \le 1$ and

$$\varphi_0(\lambda) = 1 \quad \text{for} \quad 0 \le \lambda \le 1, \quad \varphi_0(\lambda) = 0 \quad \text{for} \quad \lambda \ge 2.$$

We introduce the auxiliary operators

$$C_\epsilon(t) = A + B_\epsilon(t), \quad B_\epsilon(t) = \varphi_0(\epsilon A)B(t)\varphi_0(\epsilon A)$$

for $0 < \epsilon \ll 1$ small enough and we denote by $U_\epsilon(t,s)$, $0 \le s \le t \le 1$, the propagator generated by $C_\epsilon(t)$. As is easily seen, this propagator has the same properties as in Lemma 2.1. We further use the notations $K_{\epsilon,j}(\tau)$, $F_{\epsilon,j}(\tau)$ and $\tilde{F}_{\epsilon,N-j}(\tau)$ which are defined in the same way as in (1.2) with $B(t)$ replaced by $B_\epsilon(t)$. With these notations, the difference in question is decomposed into the sum of three operators

$$U(1,0) - F_N(\tau) = I_{\epsilon,1} + I_{\epsilon,2}(\tau) + I_{\epsilon,3}(\tau),$$

where $I_{\epsilon,1} = U(1,0) - U_\epsilon(1,0)$ and

$$I_{\epsilon,2}(\tau) = U_\epsilon(1,0) - F_{\epsilon,N}(\tau), \quad I_{\epsilon,3}(\tau) = F_{\epsilon,N}(\tau) - F_N(\tau).$$

Roughly speaking, the three difference operators are shown to obey the bounds

$$||I_{\epsilon,1}|| = O(\epsilon), \quad ||I_{\epsilon,3}(\tau)|| = O(\epsilon)$$

uniformly in τ and

$$||I_{\epsilon,2}(\tau)|| = \epsilon^{-1}O(N^{-2}(\log N)^2) + O(N^{-1}).$$

Thus we now choose ϵ as $\epsilon = N^{-1}\log N$, so that

$$||U(1,0) - F_N(\tau)|| = O(N^{-1}\log N).$$

This gives the desired error bound. The proof of the main theorem is reduced to proving the following three key lemmas. Throughout the discussion below, τ and ϵ are fixed as

$$\tau = 1/N, \quad \epsilon = N^{-1}\log N.$$

Lemma 3.1. $||I_{\epsilon,1}|| = O(N^{-1}\log N)$.

Lemma 3.2. $||I_{\epsilon,2}(\tau)|| = O(N^{-1}\log N)$.

Lemma 3.3. $||I_{\epsilon,3}(\tau)|| = O(N^{-1}\log N)$.

4. Proof of Lemma 3.1

For notational brevity, we write φ_0 and φ_∞ for $\varphi_0(\epsilon A)$ and $\varphi_\infty(\epsilon A)$, respectively, where $\varphi_\infty(\lambda) = 1 - \varphi_0(\lambda)$.

Lemma 4.1.

$$\|(U(t,s) - \exp(-(t-s)A))\varphi_\infty\| = (t-s)^{-\alpha}O(\epsilon),$$
$$\|\varphi_\infty(U(t,s) - \exp(-(t-s)A))\| = (t-s)^{-\alpha}O(\epsilon),$$
$$\|(U_\epsilon(t,s) - \exp(-(t-s)A))\varphi_\infty\| = (t-s)^{-\alpha}O(\epsilon),$$
$$\|\varphi_\infty(U_\epsilon(t,s) - \exp(-(t-s)A))\| = (t-s)^{-\alpha}O(\epsilon),$$

where all the order estimates are uniform in $0 \le s < t \le 1$.

Proof. We prove only the first relation. The same argument applies to the other relations. For brevity, we prove this for the case $s = 0$. Then the difference under consideration is written in the integral form

$$(U(t,0) - \exp(-tA))\varphi_\infty = -\int_0^t U(t,s)B(s)\exp(-sA)\varphi_\infty\, ds.$$

By definition,

$$\|\exp(-sA)\varphi_\infty\| \le e^{-s/\epsilon}$$

and by Lemma 2.1,

$$\|U(t,s)B(s)\| \le \|U(t,s)A^\alpha\| \times \|A^{-\alpha}B(s)\| = O((t-s)^{-\alpha}).$$

Hence the norm of the integrand is bounded by $O((t-s)^{-\alpha})e^{-s/\epsilon}$. This yields the desired bound $t^{-\alpha}O(\epsilon)$. □

Proof of Lemma 3.1. We again write $I_{\epsilon,1}$ in the integral form

$$I_{\epsilon,1} = U(1,0) - U_\epsilon(1,0) = \int_0^1 U_\epsilon(1,s)(B_\epsilon(s) - B(s))U(s,0)\, ds.$$

The difference $B(s) - B_\epsilon(s)$ in the integrand is represented as

$$B(s) - B_\epsilon(s) = \varphi_\infty B(s) + \varphi_0 B(s)\varphi_\infty.$$

By Lemma 4.1,

$$\|U_\epsilon(1,s)\varphi_\infty\| \le e^{-(1-s)/\epsilon} + (1-s)^{-\alpha}O(\epsilon), \quad \|\varphi_\infty U(s,0)\| \le e^{-s/\epsilon} + s^{-\alpha}O(\epsilon)$$

and by Lemma 2.1,

$$\|U_\epsilon(1,s)\varphi_0 B(s)\| = O((1-s)^{-\alpha}), \quad \|B(s)U(s,0)\| = O(s^{-\alpha}).$$

If we take account of these estimates, the desired bound $O(\epsilon) = O(N^{-1}\log N)$ can be easily obtained after a simple computation. □

5. Proof of Lemma 3.2

The proof of Lemma 3.2 is done through a series of lemmas.

Lemma 5.1. *Let $0 \le \gamma \le 1$ and let $r \in [0,1]$. Then*

$$\|A^{\gamma} \exp(-sB_{\epsilon}(r))A^{-\gamma}\| = O(1), \quad 0 \le s \le \tau,$$

is uniformly bounded.

Proof. By interpolation, it suffices to prove the lemma for the case $\gamma = 1$. For brevity, we consider only the case $r = 0$. Set $B = B(0)$ and $B_{\epsilon} = \varphi_0 B(0)\varphi_0$. Since BA^{-1} is bounded by assumption (A), we have

$$s\|AB_{\epsilon}A^{-1}\| = O(s)\|A\varphi_0\| = \epsilon^{-1}O(s) = O(1/\log N) = o(1), \quad N \to \infty.$$

This yields

$$\|A \exp(-sB_{\epsilon})A^{-1}\| \le \sum_{k=0}^{\infty}(k!)^{-1}s^k\|AB_{\epsilon}A^{-1}\|^k = O(1)$$

and the proof is complete. □

We now define

$$W_{\epsilon,j}(\tau) = U_{\epsilon}(t_j, t_{j-1}) - K_{\epsilon,j}(\tau), \quad 1 \le j \le N.$$

Lemma 5.2. *Let $W_{\epsilon,j}(\tau)$ be as above and let $\alpha \le \sigma < 1$. Then:*

(1) $\|A^{-\alpha}W_{\epsilon,j}(\tau)\| = O(\tau)$, $\|W_{\epsilon,j}(\tau)A^{-\alpha}\| = O(\tau)$.

(2) $\|A^{-\sigma}W_{\epsilon,j}(\tau)A^{\sigma}\| = O(\tau^{1-\sigma})$, $\|A^{\sigma}W_{\epsilon,j}(\tau)A^{-\sigma}\| = O(\tau^{1-\sigma})$.

(3) $\|A^{-\alpha}W_{\epsilon,j}(\tau)A\| = O(1)$, $\|AW_{\epsilon,j}(\tau)A^{-\alpha}\| = O(1)$.

Proof. We prove only the first relations in the statements (1), (2) and (3). A similar argument applies to the second ones. We also consider only the case $j = 1$ and use again the notations $B = B(0)$ and $B_{\epsilon} = B_{\epsilon}(0)$.

(1) We write $W_{\epsilon,1}(\tau)$ as the sum $W_{\epsilon,1}(\tau) = V_{\epsilon,1}(\tau) + V_{\epsilon,2}(\tau)$, where

$$V_{\epsilon,1}(\tau) = U_{\epsilon}(\tau, 0) - \exp(-\tau A), \quad V_{\epsilon,2}(\tau) = \exp(-\tau A) - K_{\epsilon,1}(\tau).$$

These two operators can be further rewritten as

$$V_{\epsilon,1}(\tau) = -\int_0^{\tau} U_{\epsilon}(\tau, s)B_{\epsilon}(s)\exp(-sA)\,ds,$$
$$V_{\epsilon,2}(\tau) = \exp(-\tau A/2)(\mathrm{Id} - \exp(-\tau B_{\epsilon}))\exp(-\tau A/2).$$

By assumption (A), it follows from Lemma 2.1 that

$$\|A^{-\alpha}V_{\epsilon,1}(\tau)\| + \|A^{-\alpha}V_{\epsilon,2}(\tau)\| = O(\tau)$$

and hence (1) is proved.

(2) We have again by assumption (A) and Lemma 2.1 that

$$\|A^{-\sigma}V_{\epsilon,1}(\tau)A^{\sigma}\| = O(1)\int_0^{\tau} s^{-\sigma}\,ds = O(\tau^{1-\sigma})$$

and also it follows that

$$\|A^{-\sigma}V_{\epsilon,2}(\tau)A^{\sigma}\| = O(\tau)\|\exp(-\tau A/2)A^{\sigma}\| = O(\tau^{1-\sigma}),$$

because $\|A^{-\sigma}(\mathrm{Id} - \exp(-\tau B_\epsilon))\| = O(\tau)$. Thus (2) is proved.

(3) The proof uses partial integration. First it is easy to see that $V_{\epsilon,2}(\tau)$ obeys the bound $\|A^{-\alpha}V_{\epsilon,2}(\tau)A\| = O(1)$. Next we consider the operator $V_{\epsilon,1}(\tau)$. We further decompose this operator as

$$A^{-\alpha}V_{\epsilon,1}(\tau)A = Q_{\epsilon,1}(\tau) + Q_{\epsilon,2}(\tau),$$

where

$$Q_{\epsilon,1}(\tau) = A^{-\alpha}\int_0^{\tau} U_\epsilon(\tau,s)B_\epsilon(d/ds)\exp(-sA)\,ds,$$
$$Q_{\epsilon,2}(\tau) = A^{-\alpha}\int_0^{\tau} U_\epsilon(\tau,s)(B_\epsilon - B_\epsilon(s))A\exp(-sA)\,ds.$$

By partial integration, it follows from Lemma 2.1 that

$$\|Q_{\epsilon,1}(\tau)\| = O(1) + \int_0^{\tau} O((\tau-s)^{\alpha-1})\|A^{\alpha}\exp(-sA)\|\,ds = O(1).$$

On the other hand, $Q_{\epsilon,2}(\tau)$ is evaluated as

$$\|Q_{\epsilon,2}(\tau)\| = O(1)\int_0^{\tau} s\|A^{1+\alpha}\exp(-sA)\|\,ds = O(1)\int_0^{\tau} s^{-\alpha}\,ds = O(\tau^{1-\alpha})$$

by use of assumption (A). This completes the proof of (3). □

We now decompose $W_{\epsilon,j}(\tau)$ into the sum $W_{\epsilon,j1}(\tau) + W_{\epsilon,j2}(\tau)$, where

$$W_{\epsilon,j1}(\tau) = U_\epsilon(t_j, t_{j-1}) - \exp(-\tau C_\epsilon(t_{j-1})),$$
$$W_{\epsilon,j2}(\tau) = \exp(-\tau C_\epsilon(t_{j-1})) - K_{\epsilon,j}(\tau)$$

with $C_\epsilon(t_{j-1}) = A + B_\epsilon(t_{j-1})$.

Lemma 5.3. *Let $W_{\epsilon,j1}(\tau)$ and $W_{\epsilon,j2}(\tau)$ be defined as above. Then:*
(1) $\|A^{-\alpha}W_{\epsilon,j1}(\tau)A^{-\alpha}\| = O(\tau^2)$.
(2) $\|A^{-1}W_{\epsilon,j2}(\tau)A^{-1}\| = \epsilon^{-1}O(\tau^3)$.

Proof. We prove the lemma only for the case $j = 1$.
(1) Let $C_\epsilon = A + B_\epsilon$ with $B_\epsilon = B_\epsilon(0)$. Then

$$W_{\epsilon,11}(\tau) = U_\epsilon(\tau,0) - \exp(-\tau C_\epsilon).$$

This is written in the integral form

$$W_{\epsilon,11}(\tau) = -\int_0^\tau U_\epsilon(\tau,s)(B_\epsilon(s) - B_\epsilon)\exp(-sC_\epsilon)\,ds.$$

Thus (1) follows from assumption (A) and Lemma 2.1 at once.
(2) The proof repeatedly uses the following commutator relation

$$[\exp(-tY), Z] = \int_0^t \exp(-sY)[Z,Y]\exp(-(t-s)Y)\,ds$$

without further references. Set

$$K_\epsilon(t) = K_{\epsilon,1}(t) = \exp(-tA/2)\exp(-tB_\epsilon)\exp(-tA/2).$$

We calculate $K'_\epsilon(t) = (d/dt)K_\epsilon(t)$ as

$$K'_\epsilon(t) = -C_\epsilon K_\epsilon(t) + R_\epsilon(t),$$

where $R_\epsilon(t) = R_{\epsilon,1}(t) + R_{\epsilon,2}(t)$ and

$$\begin{aligned} R_{\epsilon,1}(t) &= [B_\epsilon, \exp(-tA/2)]\exp(-tB_\epsilon)\exp(-tA/2), \\ R_{\epsilon,2}(t) &= \exp(-tA/2)[A/2, \exp(-tB_\epsilon)]\exp(-tA/2). \end{aligned}$$

We evaluate the norm of these two remainder operators. We further calculate the commutator appearing in the operator $R_{\epsilon,1}(t)$ as

$$\begin{aligned} [B_\epsilon, \exp(-tA/2)] &= \int_0^t \exp(-sA/2)[A/2, B_\epsilon]\exp(-(t-s)A/2)\,ds \\ &= t[A/2, B_\epsilon]\exp(-tA/2) + R_{\epsilon,3}(t), \end{aligned}$$

where

$$R_{\epsilon,3}(t) = \int_0^t [\exp(-sA/2), [A/2, B_\epsilon]]\exp(-(t-s)A/2)\,ds.$$

Since the double commutator satisfies

$$\|A^{-1}[A,[A,B_\epsilon]]A^{-1}\| = O(\epsilon^{-1}),$$

we see by Lemma 5.1 that $R_{\epsilon,3}(t)$ obeys the bound

$$\|A^{-1}R_{\epsilon,3}(t)A^{-1}\| = \epsilon^{-1}O(t^2).$$

Thus $R_{\epsilon,1}(t)$ takes the form

$$R_{\epsilon,1}(t) = t[A/2, B_\epsilon]K_\epsilon(t) + \epsilon^{-1}AO_p(t^2)A,$$

where $O_p(t^\nu)$ denotes the class of bounded operators with bound $O(t^\nu)$ as $t \to 0$. A similar argument applies to the other remainder operator $R_{\epsilon,2}(t)$. If we make use of the estimate

$$\|A^{-1}[B_\epsilon,[A,B_\epsilon]]A^{-1}\| = O(\epsilon^{-1}),$$

then we obtain

$$R_{\epsilon,2}(t) = t[B_\epsilon, A/2]K_\epsilon(t) + \epsilon^{-1}AO_p(t^2)A$$

in the same way as above. Thus

$$K'_\epsilon(t) = -C_\epsilon K_\epsilon(t) + \epsilon^{-1}AO_p(t^2)A$$

and hence the Duhamel principle, together with Lemma 5.1, yields that

$$\|A^{-1}W_{\epsilon,12}(\tau)A^{-1}\| = \|A^{-1}(\exp(-\tau C_\epsilon) - K_{\epsilon,1}(\tau))A^{-1}\| = \epsilon^{-1}O(\tau^3).$$

The proof of (2) is now complete. □

Lemma 5.4. *Let $0 \le \sigma < 1$. Then there exists $M = M_\sigma > 0$ such that*

$$\|A^\sigma F_{\epsilon,k}(\tau)\| \le M(k\tau)^{-\sigma}, \quad \|\tilde{F}_{\epsilon,N-k}(\tau)A^\sigma\| \le M((N-k)\tau)^{-\sigma}$$

for $1 \le k \le N-1$.

Proof. We prove only the first inequality. A similar argument applies to the second one. By interpolation, it suffices to prove this for the case $\sigma > \alpha$. The inequality is verified by induction on k. The case $k = 1$ is obvious. Assume that

$$\|A^\sigma F_{\epsilon,k}(\tau)\| \le M(k\tau)^{-\sigma}$$

for the case $1 \leq k \leq m-1$. Then we have by interpolation that

$$\|A^{\alpha} F_{\epsilon,k}(\tau)\| \leq M^{\delta}(k\tau)^{-\alpha} \tag{5.1}$$

for k as above, where $\delta = \alpha/\sigma < 1$. We now prove the case $k = m \leq N$. To prove this, we consider the difference

$$U_{\epsilon}(t_m, 0) - F_{\epsilon,m}(\tau) = \sum_{j=1}^{m} X_{\epsilon,jm}(\tau),$$

where

$$X_{\epsilon,jm}(\tau) = U_{\epsilon}(t_m, t_j) W_{\epsilon,j}(\tau) F_{\epsilon,j-1}(\tau)$$

with $F_{\epsilon,0}(\tau) = \mathrm{Id}$. Since $\|A^{\sigma} U_{\epsilon}(t_m, 0)\| \leq c(m\tau)^{-\sigma}$ for some $c > 0$, we have

$$\|A^{\sigma} F_{\epsilon,m}(\tau)\| \leq c(m\tau)^{-\sigma} + \sum_{j=1}^{m} \|A^{\sigma} X_{\epsilon,jm}(\tau)\|.$$

As is easily seen from Lemma 2.1,

$$\|A^{\sigma} X_{\epsilon,1m}(\tau)\| \leq \|A^{\sigma} U_{\epsilon}(t_m, t_1)\| \leq c(m\tau)^{-\sigma}$$

with another $c > 0$. By induction, it follows from Lemma 5.2 that

$$\|A^{\sigma} X_{\epsilon,mm}(\tau)\| = o(1)\|A^{\sigma} F_{\epsilon,m-1}(\tau)\| = Mo(1)(m\tau)^{-\sigma}, \quad \tau \to 0.$$

By Lemma 5.2 and (5.1), we have

$$\begin{aligned} \sum_{j=2}^{m-1} \|A^{\sigma} X_{\epsilon,jm}(\tau)\| &= O(\tau) \sum_{j=2}^{m-1} \|A^{\sigma} U_{\epsilon}(t_m, t_j)\| \times \|A^{\alpha} F_{\epsilon,j-1}(\tau)\| \\ &= O(\tau^{1-\alpha-\sigma}) M^{\delta} \sum_{j=2}^{m-1} (m-j)^{-\sigma} (j-1)^{-\alpha} \\ &= M^{\delta} O(\tau^{1-\alpha-\sigma}) m^{1-\alpha-\sigma} \leq c M^{\delta} (m\tau)^{-\sigma}. \end{aligned}$$

Since $\delta = \alpha/\sigma < 1$ strictly, we can choose $M \gg 1$ so large that the inequality in question holds true in the case $k = m$ also. Thus the proof is complete. □

Lemma 5.5. *There exists $M > 0$ such that*

$$\|A F_{\epsilon,k}(\tau)\| \leq M(k\tau)^{-1} \log N, \quad \|\tilde{F}_{\epsilon,N-k}(\tau) A\| \leq M((N-k)\tau)^{-1} \log N$$

for $1 \le k \le N-1$.

Proof. We again verify only the first inequality. It is enough to prove this for $k \gg 1$ large enough. Let $X_{\epsilon,jk}(\tau)$, $1 \le j \le k$, be as in the proof of Lemma 5.4. Then we have

$$AF_{\epsilon,k}(\tau) = AU_\epsilon(t_k,0) - \sum_{j=1}^{k} AX_{\epsilon,jk}(\tau).$$

It is easy to see that

$$\|AU_\epsilon(t_k,0)\| + \|AX_{\epsilon,1k}(\tau)\| = O((k\tau)^{-1}).$$

By Lemmas 5.2 and 5.4, we obtain

$$\|AX_{\epsilon,kk}(\tau)\| = O(1)\|A^\alpha F_{\epsilon,k-1}(\tau)\| = O((k\tau)^{-\alpha}) \le O((k\tau)^{-1}).$$

By Lemmas 5.2 and 5.4 again, the sum is evaluated as

$$\begin{aligned}\sum_{j=2}^{k-1} \|AX_{\epsilon,jk}(\tau)\| &= O(\tau)\sum_{j=2}^{k-1} \|AU_\epsilon(t_k,t_j)\| \times \|A^\alpha F_{\epsilon,j-1}(\tau)\| \\ &= O(\tau^{-\alpha})\sum_{j=2}^{k-1}(k-j)^{-1}(j-1)^{-\alpha} \\ &= O((k\tau)^{-\alpha})\log k \le M(k\tau)^{-1}\log N\end{aligned}$$

for some $M > 0$. This proves the lemma. □

We are now in a position to prove the second key lemma in question.

Proof of Lemma 3.2. Recall that $t_N = 1$. Let $X_{\epsilon,jN}(\tau)$, $1 \le j \le N$, be again as in the proof of Lemma 5.4. Then we can write the difference operator $I_{\epsilon,2}(\tau)$ in question as

$$I_{\epsilon,2}(\tau) = U_\epsilon(t_N,0) - F_{\epsilon,N}(\tau) = \sum_{j=1}^{N} X_{\epsilon,jN}(\tau).$$

By Lemmas 5.2 and 5.4, we have

$$\begin{aligned}\|X_{\epsilon,1N}(\tau)\| &\le \|U_\epsilon(t_N,t_1)A^\alpha\| \times \|A^{-\alpha}W_{\epsilon,1}(\tau)\| = O(\tau) = O(N^{-1}), \\ \|X_{\epsilon,NN}(\tau)\| &\le \|W_{\epsilon,N}(\tau)A^{-\alpha}\| \times \|A^\alpha F_{\epsilon,N-1}(\tau)\| = O(\tau) = O(N^{-1}).\end{aligned}$$

To evaluate the other operators, we now write $X_{\epsilon,jN}(\tau)$, $2 \le j \le N-1$, as

$$X_{\epsilon,jN}(\tau) = Y_{\epsilon,jN}(\tau) + Z_{\epsilon,jN}(\tau),$$

where

$$\begin{aligned} Y_{\epsilon,jN}(\tau) &= U_\epsilon(t_N, t_j) W_{\epsilon,j1}(\tau) F_{\epsilon,j-1}(\tau), \\ Z_{\epsilon,jN}(\tau) &= U_\epsilon(t_N, t_j) W_{\epsilon,j2}(\tau) F_{\epsilon,j-1}(\tau). \end{aligned}$$

By Lemmas 2.1, 5.3 and 5.4,

$$\sum_{j=2}^{N-1} \|Y_{\epsilon,jN}(\tau)\| = O(\tau^{2-2\alpha}) \sum_{j=2}^{N-1} (N-j)^{-\alpha}(j-1)^{-\alpha} = O(N^{-1}).$$

On the other hand, by Lemmas 2.1, 5.3 and 5.5,

$$\begin{aligned} \sum_{j=2}^{N-1} \|Z_{\epsilon,jN}(\tau)\| &= \epsilon^{-1} O(\tau) \log N \sum_{j=2}^{N-1} (N-j)^{-1}(j-1)^{-1} \\ &= \epsilon^{-1} O(N^{-2}(\log N)^2). \end{aligned}$$

Thus the proof of the lemma is now complete. □

6. Proof of Lemma 3.3

In this section we prove the last key lemma (Lemma 3.3). This lemma is also proved through sereval lemmas.

Lemma 6.1. *Let* $r \in [0, 1]$. *Then one has*:

$$\begin{aligned} &\|(\exp(-\tau B(r)) - \exp(-\tau B_\epsilon(r)))A^{-\alpha}\| = O(\tau), \\ &\|A^{-\alpha}(\exp(-\tau B(r)) - \exp(-\tau B_\epsilon(r)))\| = O(\tau) \end{aligned}$$

and hence

$$\|(K_j(\tau) - K_{\epsilon,j}(\tau))A^{-\alpha}\| + \|A^{-\alpha}(K_j(\tau) - K_{\epsilon,j}(\tau))\| = O(\tau)$$

Proof. The lemma is easy to prove. We shall prove the first relation. We write this difference in the integral form

$$\int_0^\tau \exp(-sB(r))(B_\epsilon(r) - B(r)) \exp(-(\tau - s)B_\epsilon(r))A^{-\alpha}\, ds.$$

Since $(B_\epsilon(r) - B(r))A^{-\alpha}$ is uniformly bounded by assumption (A), the desired bound follows from Lemma 5.1 at once. The second relation can be also proved in a similar way and the third one is obvious by definition. □

Lemma 6.2. *The difference* $K_{\epsilon,j}(\tau) - K_j(\tau)$, $1 \le j \le N$, *takes the form*

$$K_{\epsilon,j}(\tau) - K_j(\tau) = \varphi_\infty O_p(\tau)A^\alpha + A^\alpha O_p(\tau)\varphi_\infty + A^\alpha O_p(\tau^2)A^\alpha,$$

where $O_p(\tau^\nu)$ *again denotes the class of bounded operators with bound* $O(\tau^\nu)$.

Proof. For brevity, we prove the lemma only for the case $j = 1$ and write again B and B_ϵ for $B(0)$ and $B_\epsilon(0)$, respectively. Then the difference under consideration is represented as

$$K_{\epsilon,1}(\tau) - K_1(\tau) = \exp(-\tau A/2)(\exp(-\tau B_\epsilon) - \exp(-\tau B))\exp(-\tau A/2)$$

and this is further rewritten in the integral form

$$K_{\epsilon,1}(\tau) - K_1(\tau) = \sum_{j=1}^{3} \exp(-\tau A/2)\Gamma_{\epsilon,j}(\tau)\exp(-\tau A/2),$$

where

$$\begin{aligned}
\Gamma_{\epsilon,1}(\tau) &= \int_0^\tau \exp(-sB_\epsilon)\varphi_\infty B\varphi_0 \exp(-(\tau - s)B)\,ds,\\
\Gamma_{\epsilon,2}(\tau) &= \int_0^\tau \exp(-sB_\epsilon)\varphi_0 B\varphi_\infty \exp(-(\tau - s)B)\,ds,\\
\Gamma_{\epsilon,3}(\tau) &= \int_0^\tau \exp(-sB_\epsilon)\varphi_\infty B\varphi_\infty \exp(-(\tau - s)B)\,ds.
\end{aligned}$$

We analyze each operator above. If we decompose $\exp(-sB)$ as

$$\exp(-sB) = \exp(-sB_\epsilon) + (\exp(-sB) - \exp(-sB_\epsilon)),$$

then Lemmas 5.1 and 6.1 enable us to obtain that

$$B\varphi_0 \exp(-sB) = O_p(s^0)A^\alpha + \epsilon^{-\alpha}O_p(s)A^\alpha = O_p(s^0)A^\alpha,$$

because $\epsilon^{-\alpha}s \le 1$ for $0 \le s \le \tau = 1/N$. Since $A^{-\alpha}[B_\epsilon, \varphi_0]$ is bounded uniformly in ϵ, it follows from Lemma 5.1 that

$$\begin{aligned}
&[\varphi_\infty, \exp(-sB_\epsilon)] = [\exp(-sB_\epsilon), \varphi_0]\\
&= \int_0^s \exp(-\sigma B_\epsilon)[B_\epsilon, \varphi_0]\exp(-(s - \sigma)B_\epsilon)\,d\sigma = A^\alpha O_p(s).
\end{aligned}$$

A similar argument shows that $[\exp(-sB_\epsilon), \varphi_0] = O_p(s)A^\alpha$. Thus we have

$$\exp(-sB_\epsilon)\varphi_\infty = \varphi_\infty O_p(s^0) + A^\alpha O_p(s)$$

and hence

$$\Gamma_{\epsilon,1}(\tau) = \varphi_\infty O_p(\tau)A^\alpha + A^\alpha O_p(\tau^2)A^\alpha.$$

By Lemma 5.1 again, we obtain

$$\exp(-sB_\epsilon)\varphi_0 B = A^\alpha O_p(s^0), \quad \exp(-sB_\epsilon)\varphi_\infty B = A^\alpha O_p(s^0)$$

and also we have

$$[\exp(-sB) - \exp(-sB_\epsilon), \varphi_0] = O_p(s)A^\alpha$$

by Lemma 6.1. This implies that $[\exp(-sB), \varphi_0] = O_p(s)A^\alpha$ and hence we have

$$\varphi_\infty \exp(-sB) = \exp(-sB)\varphi_\infty + [\exp(-sB), \varphi_0] = O_p(s^0)\varphi_\infty + O_p(s)A^\alpha.$$

Thus it follows that

$$\Gamma_{\epsilon,2}(\tau) = A^\alpha O_p(\tau)\varphi_\infty + A^\alpha O_p(\tau^2)A^\alpha.$$

A similar argument applies to $\Gamma_{\epsilon,3}(\tau)$ and this operator is shown to take the same form as $\Gamma_{\epsilon,2}(\tau)$. The proof of the lemma is now complete. □

Lemma 6.3.

$$\|\varphi_\infty F_{\epsilon,k}(\tau)\| = O(e^{-k\tau/\epsilon}) + \epsilon O((k\tau)^{-\alpha}),$$
$$\|\tilde{F}_{\epsilon,N-k}(\tau)\varphi_\infty\| = O(e^{-(N-k)\tau/\epsilon}) + \epsilon O(((N-k)\tau)^{-\alpha})$$

for $2 \le k \le N-2$.

Proof. We prove only the first relation. Since

$$\|\varphi_\infty(U_\epsilon(t_k, 0) - F_{\epsilon,k}(\tau))\| \le \sum_{j=1}^{k} \|\varphi_\infty X_{\epsilon,jk}(\tau)\|,$$

we obtain from Lemma 5.2 that the left side obeys

$$\begin{aligned} \|\varphi_\infty&(U_\epsilon(t_k, 0) - F_{\epsilon,k}(\tau))\| \\ &= O(\tau)\sum_{j=2}^{k-1} \|\varphi_\infty U_\epsilon(t_k, t_j)\| \times \|A^\alpha F_{\epsilon,j-1}(\tau)\| \\ &\quad + O(\tau)(\|U_\epsilon(t_k, t_1)A^\alpha\| + \|A^\alpha F_{\epsilon,k-1}(\tau)\|). \end{aligned} \tag{6.1}$$

It follows from Lemma 4.1 that

$$\|\varphi_\infty U_\epsilon(t_k,t_j)\| = O(e^{-(k-j)\tau/\epsilon}) + \epsilon O(((k-j)\tau)^{-\alpha})$$

and hence we have by Lemma 5.4 that the sum on the right side of (6.1) is estimated as

$$\begin{aligned}
&O(\tau^{1-\alpha})\sum_{j=2}^{k-1}(j-1)^{-\alpha}(e^{-(k-j)\tau/\epsilon} + \epsilon((k-j)\tau)^{-\alpha}) \\
&= O(\tau^{1-\alpha})\left\{\sum_{j=2}^{l} + \sum_{j=l+1}^{k-1}\right\}(j-1)^{-\alpha}e^{-(k-j)\tau/\epsilon} + \epsilon O((k\tau)^{1-2\alpha}) \\
&= O(\tau^{1-\alpha})(k^{1-\alpha}e^{-k\tau/2\epsilon} + k^{-\alpha}(1-e^{-\tau/\epsilon})^{-1}) + \epsilon O((k\tau)^{1-2\alpha}).
\end{aligned}$$

with $l = [(k-1)/2]$, $[\ ]$ being the Gauss notation. This shows that the sum obeys the bound $\epsilon O((k\tau)^{-\alpha})$. We can easily see that the second and third terms on the right side of (6.1) also obey the same bound as above. In particular, the bound on the third term follows again from Lemma 5.4. Thus the proof is complete. □

Lemma 6.4. *Let* $0 \le \sigma < 1$. *Then there exists* $M = M_\sigma > 0$ *such that*

$$\|A^\sigma F_k(\tau)\| \le M(k\tau)^{-\sigma}, \quad \|\tilde{F}_{N-k}(\tau)A^\sigma\| \le M((N-k)\tau)^{-\sigma}$$

for $1 \le k \le N-1$.

Lemma 6.5.

$$\begin{aligned}
\|\varphi_\infty F_k(\tau)\| &= O(e^{-k\tau/\epsilon}) + \epsilon O((k\tau)^{-\alpha}), \\
\|\tilde{F}_{N-k}(\tau)\varphi_\infty\| &= O(e^{-(N-k)\tau/\epsilon}) + \epsilon O(((N-k)\tau)^{-\alpha})
\end{aligned}$$

for $2 \le k \le N-2$.

If we define $W_j(\tau)$ as

$$W_j(\tau) = U(t_j, t_{j-1}) - K_j(\tau), \quad 1 \le j \le N,$$

then it can be shown in exactly the same way as in the proof of Lemma 5.2 that this operator has the same properties as $W_{\epsilon,j}(\tau)$. This enables us to prove these two lemmas in almost the same way as in the proof of Lemmas 5.4 and 6.3. We skip the proof of the lemmas.

We are now in a position to prove the third key lemma.

Proof of Lemma 3.3. We write the difference $I_{\epsilon,3}(\tau)$ as

$$I_{\epsilon,3}(\tau) = F_{\epsilon,N}(\tau) - F_N(\tau) = \sum_{j=1}^{N} G_{\epsilon,jN}(\tau),$$

where

$$G_{\epsilon,jN}(\tau) = \tilde{F}_{\epsilon,N-j}(\tau)(K_{\epsilon,j}(\tau) - K_j(\tau))F_{j-1}(\tau).$$

By Lemmas 5.4, 6.1 and 6.4,

$$\begin{aligned} &\|G_{\epsilon,1N}(\tau)\| + \|G_{\epsilon,2N}(\tau)\| = O(\tau) = O(N^{-1}), \\ &\|G_{\epsilon,NN}(\tau)\| + \|G_{\epsilon,(N-1)N}(\tau)\| = O(N^{-1}) \end{aligned}$$

and also we obtain by Lemma 5.4 and Lemmas 6.2 $\sim$ 6.5 that

$$\begin{aligned} \|I_{\epsilon,3}(\tau)\| = O(\tau^{1-\alpha}) &\sum_{j=3}^{N-2} (j-1)^{-\alpha} e^{-(N-j)\tau/\epsilon} \\ &+ \epsilon O(\tau^{1-2\alpha}) \sum_{j=3}^{N-2} (j-1)^{-\alpha}(N-j)^{-\alpha} \\ &+ O(\tau^{2-2\alpha}) \sum_{j=3}^{N-2} j^{-\alpha}(N-j)^{-\alpha} + O(N^{-1}). \end{aligned}$$

We estimate these three sums on the right side. The first and second sums obey the bound $O(\epsilon)$ and the third one obeys the bound $O(N^{-1})$. This completes the proof. □

7. Concluding remarks

We conclude the paper by making two comments on the main theorem.

(1) The same error bound as in Theorem 1.1 remains true for other kinds of product formulas. For example, we can prove that

$$\|U(t,0) - G_N(\tau)G_{N-1}(\tau) \times \cdots \times G_2(\tau)G_1(\tau)\| = O(N^{-1}\log N)$$

uniformly in $0 \le t \le T$, where

$$G_j(\tau) = \exp(-\tau A)\exp(-\tau B(t_{j-1})), \quad t_j = j\tau, \quad \tau = t/N.$$

For brevity, we again prove this for $t = 1$. Set

$$E_N(\tau) = G_N(\tau)G_{N-1}(\tau) \times \cdots \times G_2(\tau)G_1(\tau), \quad \tau = 1/N.$$

Let $K_j(\tau)$ be as in (1.2). Then we use Theorem 1.1 to obtain that

$$\begin{aligned} E_N(\tau) &= \exp(-\tau A/2)[K_N(\tau) \times \cdots \times K_2(\tau)] \exp(-\tau A/2) \exp(-\tau B) \\ &= \exp(-\tau A/2) U(1, t_1) \exp(-\tau A/2) \exp(-\tau B) + O_p(N^{-1} \log N) \end{aligned}$$

with $B = B(t_0) = B(0)$ again. Next we calculate the commutator

$$[\exp(-\tau A/2), U(1, t_1)] = \int_0^\tau \exp(-sA/2)[U(1, t_1), A/2] \exp(-(\tau - s)A/2)\, ds.$$

By Lemma 2.1, $[U(1, t_1), A]$ is uniformly bounded, so that

$$\|[\exp(-\tau A/2), U(1, t_1)]\| = O(\tau) = O(N^{-1}).$$

This implies that

$$E_N(\tau) = U(1, t_1) \exp(-\tau A) \exp(-\tau B) + O_p(N^{-1} \log N).$$

It is easily seen from assumption (A) that

$$\exp(-\tau A) \exp(-\tau B) = \exp(-\tau A) + A^\alpha O_p(\tau) = K_1(\tau) + A^\alpha O_p(\tau).$$

Hence we have

$$E_N(\tau) = U(1, 0) + O_p(N^{-1} \log N)$$

by Lemma 2.1 and Theorem 1.1 again. Thus the desired error bound is obtained.

Similarly we can show that

$$\|U(t, 0) - \tilde{G}_N(\tau)\tilde{G}_{N-1}(\tau) \times \cdots \times \tilde{G}_2(\tau)\tilde{G}_1(\tau)\| = O(N^{-1} \log N),$$

where $\tilde{G}_j(\tau) = \exp(-\tau B(t_{j-1})) \exp(-\tau A)$.

(2) The main theorem also extends to the case in which $C(t)$ takes the form $C(t) = A(t) + B(t)$ for time dependent self-adjoint operator $A(t) \geq c > 0$ with domain $\mathcal{D}(A(t)) = \mathcal{D}(A)$, $A = A(0)$, independent of t. Suppose that $B(t)$ fulfills assumption (A) with A above. In other words, $B(t)$ is assumed to satisfy

$$\|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\| = O(|t - s|)$$

for some α, $0 \leq \alpha < 1$. If, in addition, we assume that $A(t)A(s)^{-1}$ is uniformly bounded and

$$\|A^{-\alpha}(A(t) - A(s))A^{-\alpha}\| = O(|t - s|)$$

for the same α as above, then we can show that

$$\|U(t,0) - P_N(\tau)P_{N-1}(\tau) \times \cdots \times P_2(\tau)P_1(\tau)\| = O(N^{-1}\log N)$$

with $P_j(\tau) = \exp(-\tau A(t_{j-1}))\exp(-\tau B(t_{j-1}))$. To prove this, we use

$$C_\epsilon(t) = A(t) + B_\epsilon(t), \quad B_\epsilon(t) = \varphi_0(\epsilon A(t))B(t)\varphi_0(\epsilon A(t)),$$

as an auxiliary operator. The argumet requires slight natural modifications but does not undergo any essential change. The details will be discussed elsewhere.

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