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ON THE ANALYTICITY OF STOCHASTIC FLOWS

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1. Introduction

Let $(\mathcal{C}, \mathcal{W})$ be the d-dimensional Wiener space; $\mathcal{C}$ is the space of $\mathbf{R}^d$-valued continuous functions on $[0, \infty)$ starting at 0, and $\mathcal{W}$ is the Wiener measure on $\mathcal{C}$. For $w \in \mathcal{C}$, $w(t) = \big(w^1(t), \ldots, w^d(t)\big) \in \mathbf{R}^d$ denotes its position at t. Take $V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ ($\equiv$ the space of $\mathbf{R}^N$-valued C^∞-functions on $\mathbf{R}^N$ which and whose derivatives of all orders are bounded), and consider a stochastic differential equation (SDE in abbreviation) on $\mathbf{R}^N$;

$$\begin{cases} dX(t) = \displaystyle\sum_{\alpha=1}^{d} V_\alpha(X(t))dw^\alpha(t) + V_0(X(t))dt, \\ X(0) = x, \end{cases} \tag{1.1}$$

where each $dw^\alpha(t)$, $\alpha = 1, \ldots, d$, stands for the Itô integral with respect to $w^\alpha(t)$ under $\mathcal{W}$. Let $X(t, x, w)$ be the solution to the SDE. Most studies on the stochastic flows given by $x \mapsto X(t, x, w)$ have been made in the C^∞ category, not in the analytic category. See [3, 4] and the references therein. Recently Malliavin and the second author [7] introduced a concept of analytic functions on $\mathcal{C}$ and gave several applications of it. There an example of analytic functions on $\mathcal{C}$ was given via SDE with linear coefficients. It is well known (cf. [3]) that, for a solution to an SDE, its Malliavin gradient and the Jacobian of the associated stochastic flow obey similar SDE's, and that the infinite differentiability of the solution in the sense of the Malliavin calculus relates deeply to the smoothness of the stochastic flow. Now a question arises if the solution to an SDE governed by real analytic vector fields determines a stochastic flow of analytic functions. In this paper, we shall give an affirmative answer to this question. See Theorem 2.1.

After the above observation, one may ask about the radius of convergence of the analytic function $x \mapsto X(t, x, w)$. As is easily seen (see Remark 3.7), if every V_α's are linear, then the Hessian $\partial^2 X(t, x, w)/\partial^2 x$ vanishes, and the mapping $x \mapsto X(t, x, w)$ is also linear, and hence extends to an entire function on $\mathbf{C}^N$. Thus a naïve question is if the mapping $x \mapsto X(t, x, w)$ prolongs to an entire function on $\mathbf{C}^N = \mathbf{R}^N \times \mathbf{R}^N$. We shall make a negative observation on this question in the case of 1-dimensional SDE's in Section 3; we shall see that the stochastic flow $X(t, x, w)$ determined by a

Stratonovich SDE on $\mathbf{R}$ governed by one vector field V (for the SDE, see (3.2)) does not extend to an entire function $\mathcal{W}$-a.e. if $V(x) = e^{q(x)}$ for some real polynomial q or $V(x) = \sin 2x$. See Theorem 3.1 and Example 3.15.

2. Analyticity of solutions

We prepare two classes of functions to state our result. We denote by $C^{0,\omega}([0,\infty) \times\mathbf{R}^N;\mathbf{R}^N)$ (resp. $C^{0,\infty}([0,\infty) \times \mathbf{R}^N;\mathbf{R}^N)$) the space of continuous functions $f : [0,\infty) \times \mathbf{R}^N \to \mathbf{R}^N$ such that $f(t,*)$ is analytic (resp. C^∞) for every $t \in [0,\infty)$. Our goal will be

Theorem 2.1. *Let $V_\alpha \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$, $\alpha = 0,\dots,d$. Assume that every V_α, $\alpha = 0,\dots,d$, is real analytic. Then there exists a $C^{0,\omega}([0,\infty) \times \mathbf{R}^N;\mathbf{R}^N)$-valued Wiener functional Φ on $\mathcal{C}$ such that $X(t,x,w) \equiv \Phi(w)[t,x]$ solves the SDE* (1.1).

For the proof of Theorem 2.1, we remember the following approximation of the solution to SDE (see [3]).

Lemma 2.2. *Let $W_0, W_1,\dots,W_d \in C_b^\infty(\mathbf{R}^D;\mathbf{R}^D)$. Define $Y^{(n)}(t,y,w)$, $(t,y,w) \in [0,\infty) \times \mathbf{R}^D \times \mathcal{C}$, by*

$$\tag{2.3} \begin{cases} dY^{(n)}(t) = \displaystyle\sum_{\alpha=1}^{d} W_\alpha(Y^{(n)}([2^n t]/2^n))dw^\alpha(t) + W_0(Y^{(n)}([2^n t]/2^n))dt, \\ Y^{(n)}(0) = y, \end{cases}$$

Then

$$\sup_{t\in[0,T]} \sup_{|y|\le R} \left|Y^{(m)}(t,y,w) - Y^{(n)}(t,y,w)\right| \to 0 \quad \text{in } L^p(\mathcal{W}) \quad \text{as } m,n \to \infty$$

for any $R > 0$, $T > 0$ and $p \in (1,\infty)$. Moreover, the limit determines a $C^{0,\infty}([0,\infty) \times \mathbf{R}^D;\mathbf{R}^D)$-valued Wiener functional $Y(\bullet,*,w)$ which solves the SDE

$$\begin{cases} dY(t) = \displaystyle\sum_{\alpha=1}^{d} W_\alpha(Y(t))dw^\alpha(t) + W_0(Y(t))dt, \\ Y(0) = y. \end{cases}$$

We now proceed to the proof of the theorem.

Proof of Theorem 2.1. We shall show the assertion, following the idea due to Kusuoka [5][1]. Identify $\mathbf{C}^N$ with $\mathbf{R}^N \times \mathbf{R}^N$, and, for $z \in \mathbf{C}^N$, denote $z = (x,y)$ with

[1]The argument here is much simpler than his.

$x,\ y \in \mathbf{R}^N$. Let $B(r) = \{x \in \mathbf{R}^N : |x| < r\}$. Because of the analyticity of V_α's, for each $M \in \mathbf{N}$, there exists a $\delta(M) > 0$ such that $\delta(M) > \delta(M+1)$, and V_α's prolong holomorphically to $B(M) \times B(\delta(M))$. For each $M \in \mathbf{N}$, fix a $\widetilde{V}_{\alpha,M} \in C_b^\infty(\mathbf{C}^N; \mathbf{C}^N)$ such that

$$\text{(2.4)} \qquad \begin{cases} \widetilde{V}_{\alpha,M}((x,0)) = V_\alpha(x) \quad \text{for every } x \in B(M+1) \quad \text{and} \\ \widetilde{V}_{\alpha,M} \text{ is holomorphic on } B(M) \times B(\delta(M+1)/2) \end{cases}$$

for $\alpha = 0, \dots, d$. Define $X^{(n)}(t,x,w)$ and $Z_M^{(n)}(t,z,w)$ in the same manner as $Y^{(n)}(t,,y,w)$ in Lemma 2.2 with $y = x \in \mathbf{R}^N$, $W_\alpha = V_\alpha$ and $y = z \in \mathbf{C}^N$, $W_\alpha = \widetilde{V}_{\alpha,M}$, respectively. Then the limits $X(t,x,w)$ and $Z_M(t,z,w)$ enjoy that

$$\text{(2.5)} \qquad \begin{aligned} &X(\bullet,*,w) \in C^{0,\infty}([0,\infty) \times \mathbf{R}^N; \mathbf{R}^N) \\ \text{and} \quad &Z_M(\bullet,*,w) \in C^{0,\infty}([0,\infty) \times \mathbf{C}^N; \mathbf{C}^N) \end{aligned}$$

for $\mathcal{W}$-a.e. $w \in \mathcal{C}$. Moreover, taking a subsequence if necessary, we may and will assume that

$$\text{(2.6)} \qquad \sup_{t\in[0,T]} \sup_{x\in B(R)} \left|X^{(n)}(t,x,w) - X(t,x,w)\right| \to 0,$$

$$\text{(2.7)} \qquad \sup_{t\in[0,T]} \sup_{z\in B(R)\times B(R)} \left|Z_M^{(n)}(t,z,w) - Z_M(t,z,w)\right| \to 0$$

for any $T > 0$, $R > 0$, and $M \in \mathbf{N}$ for $\mathcal{W}$-a.e. $w \in \mathcal{C}$.

Set

$$\mathcal{C}_0 = \{w \in \mathcal{C} : \text{(2.5), (2.6) and (2.7) hold for any } T > 0,\ R > 0 \text{ and } M \in \mathbf{N}\}.$$

Obviously

$$\mathcal{W}(\mathcal{C}_0) = 1.$$

Fix $T > 0$, $R > 0$, and $w \in \mathcal{C}_0$. By (2.5), we can find $M(w) \in \mathbf{N}$ such that

$$\text{(2.8)} \qquad \sup_{t\in[0,T]} \sup_{x\in B(R)} \left|X(t,x,w)\right| < M(w).$$

Then there is an $n_1(w) \in \mathbf{N}$ such that

$$\sup_{t\in[0,T]} \sup_{x\in B(R)} \left|X^{(n)}(t,x,w)\right| < M(w) + 1 \quad \text{for any } n \geq n_1(w).$$

It then follows from (2.4) that

$$Z_{M(w)+2}^{(n)}(t,(x,0),w) = X^{(n)}(t,x,w) \quad \text{for any } t \leq T, x \in B(R), \text{ and } n \geq n_1(w).$$

Hence, by (2.6) and (2.7), we have that

$$Z_{M(w)+2}(t,(x,0),w)=X(t,x,w) \quad \text{for any } t\le T \text{ and } x\in B(R). \tag{2.9}$$

Due to (2.8), (2.9), and the uniform continuity of the mapping $(t,z)\mapsto Z_{M(w)+2}(t, z,w)$ on compacts, we see the existence of $\varepsilon(w)>0$ such that

$$\mathrm{Re}Z_{M(w)+2}(t,z,w)\in B(M(w)+1) \text{ and } \mathrm{Im}Z_{M(w)+2}(t,z,w)\in B(\delta(M(w)+3)/4)$$
$$\text{for any } z\in B(R)\times B(\varepsilon(w)) \text{ and } t\le T.$$

By virtue of (2.7), there exists an $n_2(w)\in\mathbf{N}$ such that

$$\mathrm{Re}Z^{(n)}_{M(w)+2}(t,z,w)\in B(M(w)+2) \text{ and } \mathrm{Im}Z^{(n)}_{M(w)+2}(t,z,w)\in B(\delta(M(w)+3)/2)$$
$$\text{for any } z\in B(R)\times B(\varepsilon(w)), t\le T, \text{ and } n\ge n_2(w).$$

Combining this with (2.4) after observing that $Z^{(n)}_{M(w)+2}(t,z,w)$ is constructed successively by

$$\begin{aligned}
&Z^{(n)}_{M(w)+2}(t,z,w)\\
&= Z^{(n)}_{M(w)+2}\Big(\frac{k}{2^n},z,w\Big)+\sum_{\alpha=1}^{d}\widetilde{V}_{\alpha,M(w)+2}\Big(Z^{(n)}_{M(w)+2}\Big(\frac{k}{2^n},z,w\Big)\Big)\Big\{w^\alpha(t)-w^\alpha\Big(\frac{k}{2^n}\Big)\Big\}\\
&\quad+\widetilde{V}_{0,M(w)+2}\Big(Z^{(n)}_{M(w)+2}\Big(\frac{k}{2^n},z,w\Big)\Big)\Big\{t-\frac{k}{2^n}\Big\}, \quad \text{for } t\in\Big[\frac{k}{2^n},\frac{k+1}{2^n}\Big],
\end{aligned}$$

we see that the mapping

$$B(R)\times B(\varepsilon(w))\ni z\mapsto Z^{(n)}_{M(w)+2}(t,z,w)$$

is holomorphic for every $t\le T$ and $n\ge n_2(w)$. By (2.7), so is the mapping

$$B(R)\times B(\varepsilon(w))\ni z\mapsto Z_{M(w)+2}(t,z,w).$$

Thus (2.9) implies that the mapping

$$B(R)\ni x\mapsto X(t,x,w)$$

is analytic. □

3. One dimensional SDE

We shall study two cases where the stochastic flow $\mathbf{R}^N\ni x\mapsto X(t,x,w)\in\mathbf{R}^N$ does not extend to an entire function on $\mathbf{C}^N$. In both cases, we shall deal with the situation where $d=1$. In such a case, we shall write just $dw(t)$ for $dw^1(t)$.

Our first case is

Theorem 3.1. *Let $V \in C_b^\infty(\mathbf{R} : \mathbf{R})$, $a \in \mathbf{R}$, and $X(\bullet, *, w)$ be the stochastic flow associated with the SDE*

$$dX(t) = V(X(t)) \circ dw(t) + aV(x(t))dt, \quad X(0) = x \in \mathbf{R}, \tag{3.2}$$

where $\circ$ stands for the Stratonovich integral. Assume that there exists a polynomial $q : \mathbf{R} \to \mathbf{R}$ such that q is not a constant function and enjoys that $V = e^q$. Then

$$\mathcal{W}\Big(\{w : x \mapsto X(t, x, w) \text{ prolongs to an entire function on } \mathbf{C}\}\Big) = 0 \text{ for any } t > 0.$$

REMARK 3.3. $V = e^q$ as above is C_b^∞ if and only if $q(x) = \sum_{n=0}^k a_n x^n$ satisfies that k is even and $a_k < 0$.

For the proof, we shall prepare the following sufficient condition so that $x \mapsto X(t, x, w)$ is not a polynomial.

Lemma 3.4. *Let $V_0, \dots, V_d \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ and $Y(t, y, w)$ be the stochastic flow associated with the Stratonovich SDE*

$$\begin{cases} dY(t) = \displaystyle\sum_{\alpha=1}^{d} V_\alpha(Y(t)) \circ dw^\alpha(t) + V_0(Y(t))dt, \\ Y(0) = y \in \mathbf{R}^N. \end{cases} \tag{3.5}$$

If $\partial^2 V_\alpha^k / \partial y^i \partial y^j (y) \neq 0$ for some $1 \le \alpha \le d$, $1 \le i, j, k \le N$, and $y \in \mathbf{R}^N$, where $V_\alpha = (V_\alpha^1, \dots, V_\alpha^N)$, then

$$\mathcal{W}\Big(\{w : y \mapsto Y^i(t, y, w) \text{ is a polynomial for any } 1 \le i \le N\}\Big) = 0, \tag{3.6}$$

where $Y^i(t, y, w)$ is the i-th component of $Y(t, y, w)$.

REMARK 3.7. If $\partial^2 V_\alpha^k / \partial y^i \partial y^j (y) = 0$ for every $1 \le \alpha \le d$, $1 \le i, j, k \le N$, and $y \in \mathbf{R}^N$, then, by (3.9) below, we see that

$$\mathcal{W}\Big(\{w : y \mapsto Y^i(t, y, w) \text{ is a polynomial of order at most 1 for any } 1 \le i \le N\}\Big) = 1.$$

Proof of Lemma 3.4. It is known (cf. [4, p.163]) that

$$\lim_{|y|\to\infty} \frac{|Y(t,y,w)|}{(1+|y|)^{3/2}} = 0 \quad \text{for } \mathcal{W}\text{-a.e. } w \in \mathcal{C}.$$

We therefore have that

$$\begin{aligned}
&\mathcal{W}\Big(\{w : y \mapsto Y^k(t,y,w) \text{ is a polynomial for any } 1 \le k \le N\}\Big) \\
&= \mathcal{W}\Big(\{w : y \mapsto Y^k(t,y,w) \text{ is a polynomial} \\
&\qquad\qquad\qquad \text{of order at most 1 for any } 1 \le k \le N\}\Big) \\
&= \mathcal{W}\left(\left\{w : \frac{\partial^2 Y^k(t,*,w)}{\partial y^i \partial y^j} \equiv 0 \text{ for any } 1 \le i,j,k \le N\right\}\right).
\end{aligned}$$

Thus it suffices to show that

$$\mathcal{W}\left(\left\{w : \frac{\partial^2 Y^k(t,*,w)}{\partial y^i \partial y^j} \equiv 0 \text{ for any } 1 \le i,j,k \le N\right\}\right) = 0. \tag{3.8}$$

To see (3.8), set

$$\begin{aligned}
J(t,y,w) &= \left(J^i_j(t,y,w)\right)_{1\le i,j\le N} = \left(\frac{Y^i(t,y,w)}{\partial y^j}\right)_{1\le i,j\le N}, \\
\widehat{J}(t,y,w) &= \left(\widehat{J}^i_j(t,y,w)\right)_{1\le i,j\le N} = J(t,y,w)^{-1}, \\
K(t,y,w) &= \left(K^k_{ij}(t,y,w)\right)_{1\le i,j,k\le N} = \left(\frac{Y^k(t,y,w)}{\partial y^i \partial y^j}\right)_{1\le i,j,k\le N}.
\end{aligned}$$

It is easily seen (cf. [3, 4]) that

$$\begin{aligned}
dJ^i_j(t,y) &= \sum_{\alpha=0}^{d}\sum_{r=1}^{N} \frac{\partial V^i_\alpha}{\partial y^r}(Y(t,y))J^r_j(t,y) \circ dw^\alpha(t), \\
d\widehat{J}^i_j(t,y) &= -\sum_{\alpha=0}^{d}\sum_{r=1}^{N} \widehat{J}^i_r(t,y)\frac{\partial V^r_\alpha}{\partial y^j}(Y(t,y)) \circ dw^\alpha(t), \\
dK^k_{ij}(t,y) &= \sum_{\alpha=0}^{d}\sum_{r=1}^{N} \frac{\partial V^k_\alpha}{\partial y^r}(Y(t,y))K^r_{ij}(t,y) \circ dw^\alpha(t) \\
&\quad + \sum_{\alpha=0}^{d}\sum_{r,s=1}^{N} \frac{\partial^2 V^k_\alpha}{\partial y^r \partial y^s}(Y(t,y))J^r_i(t,y)J^s_j(t,y) \circ dw^\alpha(t),
\end{aligned}$$

and

$$J(0,y) = \widehat{J}(0,y) = I \text{ and } K(0,y) = \mathbf{0},$$

where we have used the convention that $dw^0(t) = dt$, and I and $\mathbf{0}$ denote the unit element of $\mathbf{R}^N \otimes \mathbf{R}^N$ and the zero element of $\mathbf{R}^N \otimes (\mathbf{R}^N \otimes \mathbf{R}^N)$, respectively. Then it follows that

$$K^k_{ij}(t,y,w) = \sum_{r=1}^N J^k_r(t,y,w) L^r_{ij}(t,y,w), \tag{3.9}$$

where

$$L^k_{ij}(t,y,w) = \sum_{\alpha=0}^d \sum_{p,q,r=1}^N \int_0^t \widehat{J^k_p}(s,y,w) \times \frac{\partial^2 V^p_\alpha}{\partial y^q \partial y^r}(Y(s,y,w)) J^q_i(s,y,w) J^r_j(s,y,w) \circ dw^\alpha(s).$$

Since

$$\mathcal{W}\Big(\{w : J(t,y,w)\widehat{J}(t,y,w) = I \text{ for any } y \in \mathbf{R}\}\Big) = 1,$$

we have that

$$\mathcal{W}\Big(\Big\{w : \frac{\partial^2 Y^k(t,*,w)}{\partial y^i \partial y^j} \equiv 0 \text{ for any } 1 \le i,j,k \le N\Big\}\Big) = \mathcal{W}\Big(\{w : L^k_{ij}(t,*,w) \equiv 0 \text{ for any } 1 \le i,j,k \le N\}\Big).$$

Hence, to see (3.8), it suffices to show that

$$\mathcal{W}\Big(\{w : L^k_{ij}(t,y,w) = 0\}\Big) = 0 \quad \text{for some } 1 \le i,j,k \le N \text{ and } y \in \mathbf{R}^N. \tag{3.10}$$

To this end, set $\mathcal{R} = \mathbf{R}^N \times (\mathbf{R}^N \otimes \mathbf{R}^N) \times (\mathbf{R}^N \otimes \mathbf{R}^N) \times \big(\mathbf{R}^N \otimes (\mathbf{R}^N \otimes \mathbf{R}^N)\big)$, and denote its coordinate by $(y, J, \widehat{J}, L) = ((y^i)_{1\le i\le N}, (J^j_i)_{1\le i,j\le N}, (\widehat{J}^j_i)_{1\le i,j\le N}, (L^k_{ij})_{1\le i,j,k\le N})$. Define $\tilde{V}_\alpha \in C^\infty(\mathcal{R};\mathcal{R})$, $\alpha = 0,\dots,d$, by

$$\tilde{V}_\alpha((y,J,\widehat{J},L)) = \Bigg(\Big(V^i_\alpha(y)\Big)_{1\le i\le N}, \Big(\sum_{r=1}^N \frac{\partial V^i_\alpha}{\partial y^r}(y) J^r_j\Big)_{1\le i,j\le N}, \Big(-\sum_{r=1}^N \widehat{J}^i_r \frac{\partial V^r_\alpha}{\partial y^j}(y)\Big)_{1\le i,j\le N}, \Big(\sum_{p,q,r=1}^N \widehat{J^k_p} \frac{\partial^2 V^p_\alpha}{\partial y^q \partial y^r}(y) J^q_i J^r_j\Big)_{1\le i,j,k\le N}\Bigg).$$

Then $\Phi(t,y,w) \equiv \big(Y(t,y,w), J(t,y,w), \widehat{J}(t,y,w), L(t,y,w)\big)$ obeys an SDE

$$d\Phi(t,y) = \sum_{\alpha=0}^d \tilde{V}_\alpha(\Phi(t,y)) \circ dw^\alpha(t), \quad \Phi(0,y) = (y, I, I, \mathbf{0}).$$

Choose $1 \leq \alpha \leq d$, $1 \leq i, j, k \leq N$, and $y \in \mathbf{R}^N$ so that $(\partial^2 V_\alpha^k / \partial y^i \partial y^j)(y) \neq 0$. Define the projection $\pi : \mathcal{R} \to \mathbf{R}$ by $\pi((y, J, \widehat{J}, L)) = L_{ij}^k$. Then $\pi\big(\tilde{V}_\alpha(y, I, I, \mathbf{0})\big) = (\partial^2 V_\alpha^k / \partial y^i \partial y^j)(y) \neq 0$. Taking an advantage of the partial hypoellipticity argument (cf. [6, 9]), we see that the law of $L_{ij}^k(t, y)$ on $\mathbf{R}$ is absolutely continuous with respect to the Lebesgue measure, and hence that (3.10) holds. □

Proof of Theorem 3.1. Set

$$F(x) = \int_0^x e^{-q(y)} dy, \quad x \in \mathbf{R}.$$

Since $V = e^q$ is bounded, F' is bounded from below by positive constant, and hence F is a strictly increasing function with $\lim_{x \to \pm\infty} F(x) = \pm\infty$. Thus F admits an inverse function $F^{-1} : \mathbf{R} \to \mathbf{R}$. It is then an easy matter to see that

$$X(t, x, w) = F^{-1}\big(F(x) + w(t) + at\big), \quad t \in [0, \infty), x \in \mathbf{R}, w \in \mathcal{C}. \tag{3.11}$$

We shall show the assertion of the theorem by reductio ad absurdum. Hence suppose that

$$\mathcal{W}\Big(\big\{w : x \mapsto X(t, x, w) \text{ extends to an entire function}\big\}\Big) > 0.$$

Choose $w \in \mathcal{C}$ such that $x \mapsto X(t, x, w)$ extends to an entire function, and we shall write the extension by $X(t, z, w)$, $z \in \mathbf{C}$. Observe that F also extends to an entire function, say F again. Then (3.11) implies that

$$F(X(t, z, w)) = F(z) + w(t) + at \quad \text{for every } z \in \mathbf{C}. \tag{3.12}$$

The order $\rho(f)$ of entire function $f : \mathbf{C} \to \mathbf{C}$ is given (cf. [1]) by

$$\rho(f) = \limsup_{r \to \infty} \frac{\log \log \max_{|z|=r} |f(z)|}{\log r} = \limsup_{n \to \infty} \frac{n \log n}{\log(1/|a_n|)},$$

where $f(z) = \sum_{n=0}^\infty a_n z^n$. As is easily seen, $\rho(f) = \rho(f')$. Moreover, if we represent as $q(x) = \sum_{n=0}^{k_q} b_n x^n$, where $b_{k_q} \neq 0$, then it is also straightforward to see that $\rho(e^{-q}) = k_q$. Hence we have that

$$\rho(F) = k_q < \infty. \tag{3.13}$$

Remember Pólya's theorem ([2, 8]) ; if f and g are entire functions and $\rho(f \circ g) < \infty$, then either g is a polynomial or $\rho(f) = 0$. Due to (3.6) in Lemma 3.4 with $d = N = 1$, $V_1 = V$, and $V_0 = aV$, we see that $X(t, z, w)$ is not a polynomial. Hence, by (3.12) and (3.13), we obtain that $k_q = 0$, which contradicts to that q is not a constant function. □

REMARK 3.14. In [7], an analytic function u on $\mathcal{C}$ was defined so that $\mathbf{R} \ni \xi \mapsto u(w + \xi h)$ extends to an entire function for any $w \in \mathcal{C}$ and $h \in H$, H being the Cameron-Martin subspace of $\mathcal{C}$. Consider the same situation as described in Theorem 3.1. Then the expression (3.11) implies that $\xi \mapsto X(t, x, w + \xi h)$ does not extend to an entire function for any $t \in [0, \infty)$, $x \in \mathbf{R}$, $w \in \mathcal{C}$, and $h \in H$. Namely, if it did, F^{-1} should prolong to an entire function. Then F should extend to an injective entire function, which would yield a contradiction since F extends to a transcendental entire function by definition.

We shall give another example where $X(t, *, w)$ does not extend to an entire function $\mathcal{W}$-a.e., to which Theorem 3.1 is not applicable.

EXAMPLE 3.15. Consider an SDE on $\mathbf{R}$

$$dX(t) = \sin(2X(t)) \circ dw(t), \quad X(0) = x. \tag{3.16}$$

It is straightforward to see that

$$X(t, x, w) = \begin{cases} x, & \text{if } x \in \frac{\pi}{2}\mathbf{Z}, \\ \operatorname{Arctan}\Big((\tan x)e^{2w(t)}\Big) + k\pi, & \text{if } -\frac{\pi}{2} + k\pi < x < \frac{\pi}{2} + k\pi, k \in \mathbf{Z}, \end{cases}$$

solves the SDE (3.16). Then, by a direct computation, we obtain that

$$\frac{\partial X(t, x, w)}{\partial x} = \frac{e^{2w(t)}}{1 + \left(e^{4w(t)} - 1\right)\sin^2 x},$$

which does not extends to an entire function, and hence $x \mapsto X(t, x, w)$ does not prolong to an entire function.

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