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ASYMPTOTIC COMPLETENESS FOR HAMILTONIANS WITH TIME-DEPENDENT ELECTRIC FIELDS

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1. Introduction

We consider the Schrödinger equation with time-dependent electric field

$$i\partial_t u(t,x) = H(t)u(t,x) \quad \text{on} \quad \mathbb{H} = L^2(\mathbb{R}^\nu) \quad (\nu \geq 1). \tag{1.1}$$

Here $H(t)$ is called Stark Hamiltonian of the form

$$H(t) = -\frac{1}{2}\Delta - E(t)\cdot x + V(x)$$

with electric field $E(t) = E + e(t)$. E is a nonzero constant vector in $\mathbb{R}^\nu$ and the perturbation $e(t) \to 0$ as $|t| \to \infty$. $V(x)$ is a multiplicative operator of a real valued function decaying as $|x| \to \infty$. We also denote $H(t) - V$ as $H_0(t)$. As is well-known, $H_0(t)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ for each $t \in \mathbb{R}$. (See [15].) In this paper we assume V is smooth and short range. (*i.e.* There exists $\epsilon > 0$ such that $V(x) = O(|x|^{-1/2-\epsilon})$ as $|x| \to \infty$.) $H(t)$ is also self-adjoint on $\mathbb{H}$ since V is relatively bounded with respect to $H_0(t)$. With some suitable conditions on $V(x)$ and $e(t)$, $H(t)$ generates a unique unitary propagator $\{U(t,s)\}_{-\infty<t,s<\infty}$ such that $U(t,s)$ is a solution of (1.1). We give the description of it later in detail. We also denote the unitary propagator generated by $H_0(t)$ as $\{U_0(t,s)\}$. If $e(t) = 0$, $U(t,s) = e^{-i(t-s)H}$ where $H = -(1/2)\Delta - E\cdot x + V(x)$.

Since $V(x)$ is decaying as $|x| \to \infty$, one expects the following. For any $\phi \in \mathbb{H}$, there exists $\phi^\pm$, $\psi^\pm \in \mathbb{H}$ such that

$$||U_0(t,s)\phi - U(t,s)\phi^\pm|| \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{1.2}$$

$$||U(t,s)\phi - U_0(t,s)\psi^\pm|| \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{1.3}$$

In other words, any solution of the free equation approaches to that of the perturbed equation (1.1) as $t \to \pm\infty$, and the similar fact holds by exchanging the free Hamiltonian $H_0(t)$ and the perturbed Hamiltonian $H(t)$. Since $U_0(t,s)$ and $U(t,s)$ are unitary, the above formulas (1.2) and (1.3) mean the existence of the following strong limits.

$$(1.4)\qquad \tilde{W}^{\pm}(s) = s - \lim_{t\to\pm\infty} U(t,s)^* U_0(t,s),$$

$$(1.5)\qquad W^{\pm}(s) = s - \lim_{t\to\pm\infty} U_0(t,s)^* U(t,s).$$

We say the wave operator $\tilde{W}(s)$ is complete if the inverse wave operator $W^{\pm}(s)$ in (1.5) exists. This agrees with the usual definition of the asymptotic completeness for the case of time-independent perturbation.

Let us recall some known results about the asymptotic completeness of Stark Hamiltonians. Researches for Schrödinger operators with electric fields have been made mainly for D.C. and A.C. Stark effects. Asymptotic completeness for A.C. Stark Hamiltonian, in which case $E(t)\cdot x = (\cos t)x_1$, was first proved by Howland and Yajima in [9] and [17]. In these papers they prove the asymptotic completeness in the following way. Since $H(t)$ is periodic in time, the unitary propagator satisfies

$$U(t+2\pi, s+2\pi) = U(t,s) \quad \text{for all} \quad t,s \in \mathbb{R}.$$

So they define a semi-group on $L^2(\mathbb{T}; L^2(\mathbb{R}^\nu))$ $(\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z})$ by

$$\begin{aligned} (U(\sigma)\phi)(t) &= U(t,t-\sigma)\phi(t-\sigma), \\ (U_0(\sigma)\phi)(t) &= U_0(t,t-\sigma)\phi(t-\sigma) \quad \text{for} \quad \phi(t) \in L^2(\mathbb{T}; L^2(\mathbb{R}^\nu)). \end{aligned}$$

We can easily see that $U(\sigma)$ and $U_0(\sigma)$ are generated by the self-adjoint operators $K = -i(d/dt) + H(t)$ and $K_0 = K - V$ on $L^2(\mathbb{T}; \mathbb{R}^\nu)$. They prove the asymptotic completeness for $H(t)$ by reducing it to that for K and K_0. These results were extended to the 3-body case by Nakamura [13]. The asymptotic completeness of modified wave operator for long-range potentials, which decay slowly $|x| \to \infty$, was proved by Kitada-Yajima [10]. Recently the asymptotic completeness for $E(t) = E + (\cos t)\mu$ is proved by Møller [11] (μ is small enough compared with the main field E).

As for D.C. Stark Hamiltonian, in which case $E(t) = E$, the asymptotic completeness for long-range many-particle systems was proved by Adachi and Tamura in [2] and [3]. In these papers they show the propagation estimates, which describes the decay of the solution $e^{-itH}\phi$ for some direction as $t \to \infty$. They prove it by using the commutator technique of E. Mourre [12]. It is based on the the following form inequality appearing in Appendix B in [8]. For all $\lambda \in \mathbb{R}$ and $\epsilon > 0$, there exist $\delta > 0$ such that

$$(1.6)\qquad f(H)i[H,A]f(H) \geq (|E| - \epsilon)f(H)^2 \quad \text{for all} \quad f \in C_0^\infty([\lambda-\delta, \lambda+\delta]).$$

Here A is a self-adjoint operator, which is equal to $(E/|E|)\cdot(-i\nabla)$. A is called a conjugate operator. With this positivity of the form, the propagation estimates is shown in the following way. For example suppose $V(x)$ is smooth and short range. Then we have

$$\left\| \chi\left(\left| \frac{x}{t^2} - E \right| \geq \epsilon \right) e^{-itH} f(H)\phi \right\| = O(t^{-1-\epsilon_0})$$

with characteristic function χ and $\phi \in L^{2,(1+\epsilon_0)/2}(\mathbb{R}^\nu)$. Here $L^{2,p}(\mathbb{R}^\nu)$ is a Hilbert space with weight $\langle x \rangle^p$. With this estimate we can see that $V(x)e^{-itH}\phi \in L^1(\mathbb{R}_t)$. So the existence and the asymptotic completeness of the wave operator is easily obtained by use of the following expression, which is called Cook's method.

$$e^{itH_0}e^{-itH}\phi = \phi - i\int_0^t e^{i\theta H_0}V(x)e^{-i\theta H}\phi d\theta. \tag{1.7}$$

The aim of this paper is to show the asymptotic completeness for the Schrödinger operators with time-dependent electric field, which tends to non zero constant vector as $t \to \infty$. To do so we modify the commutator method and show some propagation estimates for the constant electric fields to the Schrödinger operator of the form (1.1) allowing $e(t)$ to be nonperiodic but small as $t \to \infty$. Combining Cook's method and these results, we prove the existence and the asymptotic completeness of wave operators. We consider two cases. The first one is that the directional derivative of $V(x)$ along E is relatively small for the main field $|E|$. The second case is that this condition is not satisfied, but a stronger decay in t is assumed for the perturbation $e(t)$.

Thorough this paper, we assume $V(x)$ is smooth and short-range.
i.e. $V(x) \in C^\infty(\mathbb{R}^\nu)$ and there exists $\delta_0 > 1/2$ such that

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\delta_0 - |\alpha|} \quad \text{for all} \quad \alpha \tag{1.8}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

Let us state more precisely the assumption and results. Either of the following two assumptions are supposed on $V(x)$ and $e(t)$.

ASSUMPTION 1.1. We assume that

$$|E| > \sup_{x \in \mathbb{R}^\nu} \frac{E}{|E|} \cdot \nabla_x V(x), \tag{1.9}$$

and that there exist $c(t) \in C^2(\mathbb{R})$ and $\eta_0 > 0$ satisfying

$$|\dot{c}(t)| = O(t^{-\eta_0}) \quad \text{as} \quad t \to \infty, \tag{1.10}$$

$$\ddot{c}(t) = e(t) \quad \left(\dot{c}(t) = \frac{d}{dt}c(t) \right). \tag{1.11}$$

With this Assumption we write

$$b(t) = -\dot{c}(t), \tag{1.12}$$

$$a(t) = -\frac{1}{2}\int_0^t \{|\dot{c}(\theta)|^2 - 2E\cdot c(\theta)\}d\theta. \tag{1.13}$$

ASSUMPTION 1.2. $e(t)$ is a continuous integrable function on $\mathbb{R}_+$. $b(t)$ defined by

$$b(t) = \int_t^\infty e(\theta)d\theta. \tag{1.14}$$

satisfies $|b(t)| = O(t^{-u_0})$ as $t \to \infty$ for some $u_0 > 5/2$.

Under this Assumption we put

$$c(t) = \int_t^\infty b(\theta)d\theta, \quad a(t) = \frac{1}{2}\int_t^\infty \{|b(\theta)|^2 - 2E\cdot c(\theta)\}d\theta. \tag{1.15}$$

Let us add some words related to the above assumptions. If the directional derivative of $V(x)$ is relatively small for the main field E, Mourre's inequality holds without localization. i.e. There exists $C > 0$ and $i[H(t), A(t)] \geq C$ holds for sufficiently large t. We assume the decay order of $e(t)$ in Assumption 1.1 in this case. If the directional derivative of $V(x)$ is not small enough, we can not neglect the derivative of $f(H(t))$ with respect to t. It is closely related with the decay of the perturbation $e(t)$ as $t \to \infty$. So we need stronger condition for the decay of the perturbation $e(t)$, in which case we assume Assumption 1.2. In each of these Assumptions, $H(t)$ is essentially self-adjoint on $D(|x|)\cap H^2(\mathbb{R}^\nu)$. And we can construct unique unitary propagator satisfying the following properties. (See [18].)

For all $t, t', s \in \mathbb{R}$,

$$U(t,t) = I, \quad U(t,s)U(s,t') = U(t,t'), \tag{1.16}$$

$$\frac{d}{dt}U(t,s) = -iH(t)U(t,s). \tag{1.17}$$

We also denote the unitary propagator associated with $H_0(t)$ as $U_0(t,s)$. With these unitary operators we define the wave operators in the same way as in (1.4) and (1.5). Our main result is the following.

Theorem 1.3. *Suppose* Assumption 1.1 *or* 1.2 *holds. Then the operators* $W^+(s)$ *and* $\tilde{W}^+(s)$ *exist for all* $s \in \mathbb{R}$.

REMARK 1.4. Theorem 1.3 holds as $t \to -\infty$, if we replace ∞ in Assumption 1.1 and 1.2 by $-\infty$.

2. Translated Hamiltonians

To show the main theorem, we have to treat the time-dependent part '$e(t) \cdot x$', which is not bounded. For this purpose we introduce another Hamiltonian $\hat{H}(t)$, which is obtained by translating $H(t)$ in both x and p spaces. In Sections 3 and 4, we apply the commutator method for $\hat{H}(t)$ and prove the propagation estimates for the propagator associated with $\hat{H}(t)$ instead of $H(t)$. The existence of the wave operators for $H(t)$ and $H_0(t)$ is obtained by showing it for translated Hamiltonians.

DEFINITION 2.1.

$$\hat{H}(t) = -\frac{1}{2}\Delta - E \cdot x + V(x - c(t)). \tag{2.1}$$

We denote $\hat{H}_0 = \hat{H}(t) - V(x - c(t))$.

We can also construct unique unitary propagators $\hat{U}(t,s)$ and $\hat{U}_0(t,s)$, generated by $\hat{H}(t)$ and $\hat{H}_0$. We remark that $U(t,s)$ and $\hat{U}(t,s)$ ($U_0(t,s)$ and $\hat{U}_0(t,s)$) are related through the following relation. It is based on 'Avron-Herbst formula' with a slight modification. (For example, see [4].)

(Avron-Herbst formula)

$$U(t,s) = \tau(t)\hat{U}(t,s)\tau^*(s), \tag{2.2}$$

where

$$\tau(t) = \exp(-ia(t))\exp(-ib(t)\cdot x)\exp(ic(t)\cdot p), \quad p = -i\nabla_x. \tag{2.3}$$

We show the propagation estimates for $\hat{U}(t,s)$ and $\hat{U}_0(t,s)$ instead of $U(t,s)$ and $U_0(t,s)$. Once we prove them, the estimates for $U(t,s)$ and $U_0(t,s)$ can be easily obtained by using the fact that $g(x)e^{-ip\cdot c(t)} = e^{-ip\cdot c(t)}g(x - c(t))$ holds for an operator of multiplication by a function $g(x)$. Before going to the propagation estimates by use of the commutator method, let us recall the well-known formula of functional calculus in [7].

Let $f \in C^\infty(\mathbb{R})$ be a function such that for some $m_0 \in \mathbb{R}$

$$|f^{(k)}(t)| \le C_k(1 + |t|)^{m_0 - k}, \quad {}^\forall k \in \mathbb{N} \cup \{0\}. \tag{2.4}$$

Then we can construct an almost analytic extension $\tilde{f}(z)$ of $f(t)$ satisfying

$$\tilde{f}(t) = f(t), \quad t \in \mathbb{R}, \tag{2.5}$$

$$|\partial_{\bar{z}}\tilde{f}| \le C_N|\mathrm{Im}z|^N\langle z\rangle^{m_0-1-N}, \quad {}^\forall N \in \mathbb{N}, \tag{2.6}$$

$$\mathrm{supp}\tilde{f}(z) \subset \{z; |\mathrm{Im}z| \le 1 + |\mathrm{Re}z|\}. \tag{2.7}$$

We remark that supp$\tilde{f}$ is compact in $\mathbb{C}$ if $f \in C_0^\infty(\mathbb{R})$ (See Appendix in [5]). Further, if (2.4) holds with $m_0 < 0$ we can rewrite a function of a self-adjoint operator A in the following integral

$$f(A) = \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{f}(z)(z-A)^{-1}dz \wedge d\bar{z}. \tag{2.8}$$

With this form, we can compute the commutator of an operator P and $g(A)$ in the following way.

For operators P and Q, we define $ad_Q^0(P) = P$. For $m \in \mathbb{N}$, we define $ad_Q^m(P) = [ad_Q^{m-1}(P), Q]$ inductively. Now we take Q as the resolvent of A. Then we have

$$ad_A^n(P)(z-A)^{-1} = (z-A)^{-1}ad_A^n(P) + (z-A)^{-1}ad_A^{n+1}(P)(z-A)^{-1} \tag{2.9}$$

by using the resolvent equation. With these results, we have the following Lemma.

Lemma 2.2. *Let A and P be linear operators on $\mathbb{H}$. Suppose A is self-adjoint and relatively bounded with respect to P. Suppose that the form $ad_Q^m(P)$ extends to a bounded operator for $1 \le m \le n$. Then for any $g \in C^\infty(\mathbb{R})$ satisfying* (2.4) *with $m_0 < n$, we have*

$$Pg(A) = \sum_{m=0}^{n-1} \frac{g^{(m)}(A)}{m!}ad_A^m(P) + \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{g}(z)\mathbb{R}^r_{n,A,P}(z)dz \wedge d\bar{z}, \tag{2.10}$$

where $\mathbb{R}^r_{n,A,P}(z) = (z-A)^{-n}ad_A^n(P)(z-A)^{-1}$, and

$$g(A)P = \sum_{m=0}^{n-1} ad_A^m(P)\frac{(-1)^m}{m!}g^{(m)}(A) + \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{g}(z)\mathbb{R}^l_{n,A,P}(z)dz \wedge d\bar{z}, \tag{2.11}$$

where $\mathbb{R}^l_{n,A,P}(z) = (z-A)^{-1}ad_A^n(P)(A-z)^{-n}$ and $\tilde{g}(z)$ denotes an almost analytic extension of $g(x)$.

We introduce smeared cut off functions, which are needed to write down the propagation estimate. For a, $b \in \mathbb{R}$, we call $F_{\ge a}(t)$, $F_{\le b}(t) \in C^\infty(\mathbb{R})$ smeared cut off functions if they satisfy for some $\delta > 0$

$$F_{\ge a}(t) = 1 \quad \text{for} \quad t \ge a+\delta \quad \text{and} \quad F_{\ge a}(t) = 0 \quad \text{for} \quad t \le a-\delta,$$
$$F_{\le b}(t) = 1 \quad \text{for} \quad t \le b-\delta \quad \text{and} \quad F_{\le b}(t) = 0 \quad \text{for} \quad t \ge b+\delta,$$

For simplicity's sake, in the following we write $F(t \ge a)$, $F(t \le b)$ instead of $F_{\ge a}(t)$, $F_{\le b}(t)$.

3. Propagation estimates I

As we have mentioned in Section 1, Mourre's inequality plays important role in showing the propagation estimate. The aim of this section is to prove Theorem 3.6 and finally to show the main theorem under Assumption 1.1. We rewrite the condition (1.9) as follows.

$$|E| - \sup_{x\in\mathbb{R}^\nu} \omega \cdot \nabla V(x) > 0 \quad \text{with} \quad \omega = \frac{E}{|E|}. \tag{3.1}$$

We denote the left hand side of the above inequality by E_0. Next we define the conjugate operator $A_0(t)$ as follows. Especially we denote $A_0(0)$ as A_0.

$$A_0(t) = \omega \cdot p - E_0 t \quad \text{where} \quad p = -i\nabla_x \tag{3.2}$$

For a time-dependent self-adjoint operator $A(t)$, we denote Heisenberg derivative $d_t A(t) + i[H(t), A(t)]$ as $DA(t)$ with $d_t A(t) = (d/dt)A(t)$. From (3.1) we have $DA_0(t) \geq 0$ in the form sense. We show the propagation estimate by use of the commutator method, which is based on the formula of Lemma 2.2. Before that we have to show that the propagator $\hat{U}(t,s)$ leaves $D(A_0)$ invariant since the conjugate operator $A_0(t)$ is not bounded.

Lemma 3.1. *We denote the set of bounded operators in* $\mathbb{H}$ *as* $\mathfrak{B}(\mathbb{H})$. *Let* $s \in \mathbb{R}$ *and* $h \in C_0^\infty(\mathbb{R})$ *and* $0 \leq \delta \leq 2$ *be given. Then*

(i) *For* $1 \leq n \leq 4$, *the form* $ad^n_{A_0(t)}(h(\hat{H}(t)))$ *is extended to a bounded operator on* $\mathbb{H}$. *Moreover* $(\hat{H}(t)+i)ad^n_{A_0(t)}(h(\hat{H}(t)))$ *and* $ad^n_{A_0(t)}(h(\hat{H}(t)))(\hat{H}(t)+i)$ *are continuous* $\mathfrak{B}(\mathbb{H})$*-valued functions of* t *which are uniformly bounded in* $t > 0$.

(ii) $\langle A_0\rangle^\delta \hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-\delta}$ *is a continuous* $\mathfrak{B}(\mathbb{H})$*-valued function of* t.

(iii) $(-A_0(t))^\delta F((A_0(t)/t) \leq -\epsilon)\langle A_0\rangle^{-\delta}$ *is a* $\mathfrak{B}(\mathbb{H})$*-valued continuous function of* t.

Proof. It is easily seen that $ad^n_{A_0(t)}(\hat{H}(t))$ is bounded for $n \geq 1$. So we use the boundness of these operators in the formula of functional calculus which appeared in Lemma 2.2. (i) is easily obtained by use of (2.8). To prove (ii), by an interpolation we have only to prove the case $\delta = 2$. By a straight forward computation we rewrite the commutator.

$$\begin{aligned}
&A_0^2\hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-2} \\
&= \hat{U}(t,s)A_0^2 h(\hat{H}(s))\langle A_0\rangle^{-2} - 2ad^1_{A_0}(\hat{U}(t,s))A_0 h(\hat{H}(s))\langle A_0\rangle^{-2} \\
&\quad + ad^2_{A_0}(\hat{U}(t,s))h(\hat{H}(s))\langle A_0\rangle^{-2}.
\end{aligned}$$

So the boundness is obtained if we show $ad^k_{A_0}(\hat{U}(t,s)) \in \mathfrak{B}(\mathbb{H})$ for $k = 1,\ 2$. We rewrite $ad^1_{A_0}(\hat{U}(t,s))$ in the following way

$$\begin{aligned} ad^1_{A_0}(\hat{U}(t,s)) &= -\hat{U}(t,s)\{\hat{U}(s,t)A_0\hat{U}(t,s) - A_0\} \\ &= -\int_s^t \hat{U}(t,\theta)i[\hat{H}(\theta),A_0]\hat{U}(\theta,s)d\theta. \end{aligned}$$

Since $[\hat{H}(\theta),A_0]$ is bounded, we can easily see that $ad^1_{A_0}(\hat{U}(t,s)) \in \mathfrak{B}(\mathbb{H})$. As for the double commutator, we use the expression

$$\begin{aligned} & ad^2_{A_0}(\hat{U}(t,s)) \\ &= -i \sum_{\alpha_1+\alpha_2+\alpha_3=1} \int_s^t ad^{\alpha_1}_{A_0}(\hat{U}(t,\theta))ad^{\alpha_2+1}_{A_0}(\hat{H}(\theta))ad^{\alpha_3}_{A_0}(\hat{U}(\theta,s))d\theta. \end{aligned}$$

With this form, we can see that $ad^2_{A_0}(\hat{U}(t,s))$ is bounded. Therefore (ii) for $\delta = 2$ follows from (i). To prove (iii) we have only to show that $\|(-A_0)^2\langle A_0\rangle^{-2}\|$ is locally bounded in t. However this is obvious. □

Secondly we introduce a set of functions $\mathfrak{F}_{\beta,\alpha,\epsilon}$ of those functions $g_{\beta,\alpha,\epsilon}(x,t) \in \mathfrak{F}_{\beta,\alpha,\epsilon}$, which are obtained by smoothly cutting off the part $x > -\epsilon t$ of a function $-t^{-\beta}(-x)^{\alpha}$.

Definition 3.2. Given β, $\alpha \geq 0$ and $\epsilon > 0$, we denote by $\mathfrak{F}_{\beta,\alpha,\epsilon}$ the set of function g of the form $g(x,t) = g_{\beta,\alpha,\epsilon}(x,t) = -t^{-\beta}(-x)^{\alpha}\chi(x/t)$ defined for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, where $\chi \in C^{\infty}(\mathbb{R})$ and satisfies the following properties:

$$\begin{aligned} &\chi(x) = 1 \quad \text{for} \quad x < -2\epsilon, \quad \chi(x) = 0 \quad \text{for} \quad x > -\epsilon. \\ &\frac{d}{dx}\chi(x) \leq 0 \quad \text{and} \quad \alpha\chi(x) + x\frac{d}{dx}\chi(x) = \tilde{\chi}(x)^2 \quad \text{for some} \\ &\quad \tilde{\chi} \in C^{\infty}(\mathbb{R}), \quad \text{with} \quad \tilde{\chi} \geq 0. \end{aligned}$$

From Lemma 3.1 we can see that $(-g_{\beta,\alpha,\epsilon}(A_0(t),t))^{1/2}\hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-\alpha/2}$ is a bounded operator for $0 \leq \alpha \leq 4$. Further the following Lemma shows that it is bounded uniformly in t. We remark that Corollary 3.4, which is easily obtained by using Lemma 3.3, is closely related with the propagation estimate.

Lemma 3.3. *Let β_0, $\epsilon > 0$, $0 < \alpha_0 \leq 4$, and $h \in C_0^{\infty}(\mathbb{R})$ be given. Then*

$$\begin{aligned} &\|(-g_{\beta,\alpha,\epsilon}(A_0(t),t))^{1/2}\hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-\alpha/2}\|_{\mathfrak{B}(\mathbb{H})} \\ &\quad = O(1) \quad \textit{as} \quad t \to \infty \end{aligned} \tag{3.3}$$

for $(\beta,\alpha) = (0,1),\dots,(0,\alpha_0'),(\beta_0,\alpha_0)$ $(\alpha_0' = \max\{m \in \mathbb{N} | m < \alpha_0\})$. *If* $\alpha_0 \le 1$, (3.3) *holds for* $(\beta,\alpha) = (\beta_0,\alpha_0)$.

Corollary 3.4. *Under the same conditions in* Lemma 3.3, *we have the following result*: *Let* $\epsilon > 0$ *and* $0 \le \theta \le 1$ *be given. Then*

$$\begin{aligned}(3.4)\qquad &\|(-g_{0,\alpha(1-\theta),\epsilon}(A_0(t),t))^{1/2}\hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-\alpha/2}\|_{\mathfrak{B}(\mathbb{H})}\\ &= O(t^{(\beta-\alpha\theta)/2}) \quad as \quad t \to \infty\end{aligned}$$

for $(\beta,\alpha) = (0,1),\dots,(0,\alpha_0'),(\beta_0,\alpha_0)$ $(= (\beta_0,\alpha_0)$ *if* $\alpha_0 \le 1)$.

Corollary 3.4 easily follows from Lemma 3.3 and the following inequality.

$$(3.5)\qquad -t^{-\beta}(\epsilon t)^{\alpha\theta} g_{0,\alpha(1-\theta),2\epsilon}(x,t) \le -g_{\beta,\alpha,\epsilon}(x,t).$$

Once we prove Lemma 3.3, we have the propagation estimate with respect to p, which corresponds to the momentum of the solution. With this estimate we have the propagation estimate with respect to x. And finally we have Theorem 1.3 by using Cook's method.

For simplicity, we denote the square of the left hand side of (3.3) as $G_{\beta,\alpha,\epsilon}(t)$. In the following Lemma, we prove the integrability of $-(d/dt)G_{\beta,\alpha,\epsilon}(t)$ by using induction on α.

Proof of Lemma 3.3. We set $\zeta(t) = \hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-\alpha/2}\phi$ with $\phi \in \mathbb{H}$. We denote the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}} = \langle\cdot,\cdot\rangle$ and $\langle\zeta(t),P\zeta(t)\rangle$ as $\langle P\rangle_t$ for an operator P. First we use the formula of functional calculus and decompose $-(d/dt)G_{\beta,\alpha,\epsilon}(t)$ as follows.

$$(3.6)\qquad -\frac{d}{dt}G_{\beta,\alpha,\epsilon}(t) \equiv \langle I_1\rangle_t + \langle I_2\rangle_t$$

$$(3.7)\qquad = \langle\zeta(t), i[\hat{H}(t), g_{\beta,\alpha,\epsilon}(A_0(t),t)]\zeta(t)\rangle$$

$$(3.8)\qquad + \left\langle\zeta(t), \frac{d}{dt}g_{\beta,\alpha,\epsilon}(A_0(t),t)\zeta(t)\right\rangle.$$

We compute the commutator appeared in the second line by use of Lemma 2.2. Then we can rewrite it as follows.

$$(3.9)\qquad I_1 = i\sum_{m=1}^{3}(m!)^{-1}g^{(m)}(A_0(t),t)ad^m_{A_0(t)}(\hat{H}(t))$$

$$(3.10)\qquad + \frac{1}{2\pi}\int_{\mathbb{C}}\partial_{\bar{z}}\tilde{g}(z,t)R^r_{4,A_0(t),\hat{H}(t)}dz\wedge d\bar{z},$$

with $g^{(m)}(x,t)=\partial_x^m g(x,t)$.

As for I_2 we rewrite $g(A_0(t),t)$ by using integral representation (2.8):

$$I_2=\left(\frac{\partial}{\partial t}g\right)(A_0(t),t)+g^{(1)}(A_0(t),t)d_tA_0(t). \tag{3.11}$$

Here $d_tA_0(t)$ is the derivative of $A_0(t)$ with respect to t, which is equal to $-E_0$.

We combine I_1 and I_2 by using the Heisenberg derivative $DA_0(t)$.

$$\begin{aligned} I_1+I_2 &= \left(\frac{\partial}{\partial t}g\right)(A_0(t),t)+\sum_{m=1}^{3}(m!)^{-1}g^{(m)}(A_0(t),t)ad^m_{A_0(t)}(DA_0(t)), \\ &\quad +\frac{1}{2\pi}\int_{\mathbb{C}}\partial_{\bar z}\tilde g(z,t)R^r_{4,A_0(t),\hat H(t)}dz\wedge d\bar z. \end{aligned} \tag{3.12}$$

From Definition 3.2, we have $(g^{(1)}(x,t))^{1/2}\in C^\infty$ and $((\partial/\partial t)g)(x,t)\geq 0$. We denote $(g^{(1)}(x,t))^{1/2}$ as $g_h(x,t)$. We proceed to compute the sum in (3.12). We use (2.11) again to decompose $g^{(1)}(A_0(t),t)ad^1_{A_0(t)}(DA_0(t))$ into E_2,E_3,E_4 that are given below. As for the rest of the sum

$$\sum_{m=2}^{3}(m!)^{-1}g^{(m)}(A_0(t),t)ad^m_{A_0(t)}(DA_0(t)),$$

we commute $g^{(m)}(A_0(t),t)$ and $ad^m_{A_0(t)}(DA_0(t))$ and get the terms E_5, E_6 given below.

At last we have

$$-\frac{d}{dt}G_{\beta,\alpha,\epsilon}(t)=\langle E_1\rangle_t+\ldots+\langle E_7\rangle_t \tag{3.13}$$

where

$$E_1=\left(\frac{\partial}{\partial t}g\right)(A_0(t),t), \tag{3.14}$$

$$E_2=g_h(A_0(t),t)DA_0(t)g_h(A_0(t),t), \tag{3.15}$$

$$E_3=g_h(A_0(t),t)\sum_{m=1}^{3}ad^m_{A_0(t)}(DA_0(t))\frac{(-1)^m}{m!}g_h^{(m)}(A_0(t),t), \tag{3.16}$$

$$E_4=g_h(A_0(t),t)\left\{\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar z}\tilde g_h(z,t)R^l_{4,A_0(t),DA_0(t)}dz\wedge d\bar z\right\}, \tag{3.17}$$

$$E_5=\sum_{m=2}^{3}(m!)^{-1}j_m(A_0(t),t)k_m(A_0(t),t) \tag{3.18}$$

$$\times\left\{\sum_{m_1=0}^{4-m}ad^{m_1}_{A_0(t)}(DA_0(t))\frac{(-1)^{m_1}}{m_1!}j_m^{(m_1)}(A_0(t),t)\right\}, \tag{3.19}$$

$$E_6 = \sum_{m=2}^{3}(m!)^{-1}j_m(A_0(t),t)k_m(A_0(t),t) \tag{3.20}$$

$$\times\left\{\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar{z}}\tilde{j}_m(z)R^l_{5-m,A_0(t),DA_0(t)}(z)dz\wedge d\bar{z}\right\}, \tag{3.21}$$

$$E_7 = \frac{1}{2\pi}\int_{\mathbb{C}}\partial_{\bar{z}}\tilde{g}(z,t)R^r_{4,A_0(t),\hat{H}(t)}(z)dz\wedge d\bar{z}, \tag{3.22}$$

with

$$\begin{aligned} j_m(x,t) &= g_{\beta/2,((\alpha-m)/2)_+,\epsilon/2}(x,t),\\ k_m(x,t) &= \frac{t^\beta}{m!}(-x)^{-(\alpha-m)_+}g^{(m)}(x,t),\\ H_m &= ad^{m-1}_{A_0(t)}(DA_0(t)). \end{aligned}$$

$\langle E_1\rangle_t$ and $\langle E_2\rangle_t$ do not decay as $t\to\infty$, but it is negligible since they are positive in the form sense. We note that $DA_0(t)$ is positive by virtue of (3.1). To show (3.3), we have only to show the integrability of $\langle E_3\rangle_t,\dots,\langle E_7\rangle_t$ and that whose integrations are dominated from above by $\|\phi\|^2$.

We show the integrability by induction on α_0. Suppose $0<\alpha_0\le 1$ and $(\beta,\alpha)=(\beta_0,\alpha_0)$. Then we can easily see that

$$\|g^{(1)}_{\beta_0,\alpha_0,\epsilon}(A_0(t),t)\|_{\mathfrak{B}(\mathbb{H})}\le Ct^{-\beta_0+\alpha_0-1}.$$

With this estimate and the fact that $|\partial_{\bar{z}}\tilde{g}(z,t)|\le C\langle z/t\rangle^{\alpha_0-6}|\mathrm{Im}z/t|^5t^{-\beta_0-1}$, we have

$$\begin{aligned} &\langle E_3\rangle_t,\langle E_5\rangle_t = O(t^{-\beta_0+\alpha_0-2})\|\phi\|^2,\\ &\langle E_4\rangle_t,\langle E_6\rangle_t,\langle E_7\rangle_t = O(t^{-\beta_0+\alpha_0-5})\|\phi\|^2. \end{aligned}$$

Each term appearing above is integrable. We have

$$\frac{d}{dt}G_{\beta_0,\alpha_0,\epsilon}(t)\le Ct^{-\beta_0-1}\|\phi\|^2, \tag{3.23}$$

which implies $G_{\beta_0,\alpha_0,\epsilon}(t)=O(1)\|\phi\|^2$ since $G_{\beta_0,\alpha_0,\epsilon}(t)$ is non-negative. So we obtain

$$\|(-g_{\beta_0,\alpha_0,\epsilon}(A_0(t),t))^{1/2}\zeta(t)\| = O(1)\|\phi\| \tag{3.24}$$

under the restriction $0<\alpha_0\le 1$.

Next we show the integrability under the condition $1<\alpha_0\le 2$. In this case $g^{(1)}_{\beta_0,\alpha_0,\epsilon}(A_0(t),t)$ is not bounded in $t>0$. To show the integrability we derive the following estimate at first.

$$\|(-g_{0,1,\epsilon}(A_0(t),t))^{1/2}\hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-1/2}\|_{\mathfrak{B}(\mathbb{H})} = O(1). \tag{3.25}$$

To show it we put $\beta_0 = 4/5$, $\alpha_0 = 1$ into (3.24). Then we have

$$\|(-g_{4/5,1,\epsilon}(A_0(t),t))^{1/2}\hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-1/2}\|_{\mathfrak{B}(\mathbb{H})} = O(1).$$

Applying the inequality (3.5) to the above estimate, we can sharpen this estimate compared with (3.24). We have

$$\|(-g_{0,0,\epsilon}(A_0(t),t))^{1/2}\hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-1/2}\|_{\mathfrak{B}(\mathbb{H})} = O(t^{-1/10}).$$

We put this estimate into $\langle E_3\rangle_t, \ldots, \langle E_7\rangle_t$ with $(\beta_0, \alpha_0) = (0,1)$. Then we obtain (3.25). Again we put the estimate (3.25) into $\langle E_3\rangle_t, \ldots, \langle E_7\rangle_t$.

At last we have

$$\langle E_3\rangle_t, \langle E_5\rangle_t = O(t^{-\beta_0+\alpha_0-3})\|\phi\|^2$$
$$\langle E_4\rangle_t, \langle E_6\rangle_t, \langle E_7\rangle_t = O(t^{-\beta_0+\alpha_0-5})\|\phi\|^2$$

for $1 < \alpha_0 \leq 2$ and $\beta_0 > 0$.

So we also have the integrability under the restriction $1 < \alpha_0 \leq 2$. For larger α_0, we obtain the integrability by repeating these arguments. (See Theorem 2.4 in [16].) We have proved Lemma 3.3. □

By virtue of Corollary 3.4 we have the following Lemma.

Lemma 3.5. *For all $\epsilon > 0$ and $0 < u \leq 2$,*

$$\left\| F\left(\frac{\omega \cdot p}{t} \leq E_0 - \epsilon\right)\hat{U}(t,s)h(\hat{H}(s))\langle x\rangle^{-u/2}\right\|_{\mathfrak{B}(\mathbb{H})} = O(t^{-u}). \tag{3.26}$$

Proof. By the same argument as in the proof of Lemma 3.3 we obtain

$$\|(-g_{0,\alpha_0,\epsilon}(A_0(t),t))^{1/2}\hat{U}(t,s)h(\hat{H}(s))\langle A_0\rangle^{-\alpha_0/2}\|_{\mathfrak{B}(\mathbb{H})} = O(1). \tag{3.27}$$

We remark that $\langle A_0\rangle^{-u}$ can be replaced by $\langle x\rangle^{-u/2}$, due to the fact that $h(\hat{H}(s))p^2$ $\langle x\rangle^{-1} \in \mathfrak{B}(\mathbb{H})$. In addition we rewrite $\alpha_0/2$ as u. Combining the above estimate and Corollary 3.4, we have (3.26). □

With this Lemma, we have the following theorem, so-called minimal acceleration estimate.

Theorem 3.6. *There exists $\sigma > 0$ such that for all $0 < u \leq 2$*

$$\left\| F\left(\frac{|x|}{t^2} \leq \sigma\right)\hat{U}(t,s)h(\hat{H}(s))\langle x\rangle^{-u/2}\right\|_{\mathfrak{B}(\mathbb{H})} = O(t^{-L}) \quad as \quad t \to \infty, \tag{3.28}$$

with $L = \min\{u, 3/2, 1 + \eta_0\}$.

Proof. We set $\xi(t) = F((|x|/t^2) \leq \sigma)\hat{U}(t,s)h(\hat{H}(s))\langle x\rangle^{-u/2}\phi$. Then

$$\begin{aligned} ||(\omega\cdot p)^2\xi(t)||^2 &\geq \langle \xi(t), (\omega\cdot p)^4 F\left(\frac{A_0(t)}{t} \geq -\epsilon\right)\xi(t)\rangle \\ &\geq (E_0-\epsilon)^4 t^4 \langle \xi(t), F\left(\frac{A_0(t)}{t} \geq -\epsilon\right)\xi(t)\rangle. \end{aligned}$$

We use the decomposition $F(\cdot \geq -\epsilon) = I - F(\cdot \leq -\epsilon)$ to see that

$$||(\omega\cdot p)^2\xi(t)||^2 \geq (E_0-\epsilon)^4 t^4 ||\xi(t)||^2 - (E_0-\epsilon)^4 t^4 \left\langle \xi(t), F\left(\frac{A_0(t)}{t} \leq -\epsilon\right)\xi(t)\right\rangle.$$

We put the estimate (3.26) of Lemma 3.5 into this inequality. Using the commutator estimate $||[F((A_0(t)/t) \leq -\epsilon), F((|x|/t^2) \leq \sigma)]||_{\mathfrak{B}(\mathbb{H})} = O(t^{-3})$, we have

$$||(\omega\cdot p)^2\xi(t)||^2 \geq (E_0-\epsilon)^4 t^4 ||\xi(t)||^2 - C(E_0-\epsilon)^4\{t^{4-2u} + t\}||\phi||^2. \tag{3.29}$$

On the other hand, we use the inequality $(1/2)(\omega\cdot p)^2 \leq \hat{H}(t) + E\cdot x - V(x-c(t))$ to obtain

$$||(\omega\cdot p)^2\xi(t)|| \leq 2||\hat{H}(t)\xi(t)|| + 2||(E\cdot x)\xi(t)|| + C||\phi||. \tag{3.30}$$

We have

$$||\hat{H}(t)\xi(t)||_{\mathfrak{B}(\mathbb{H})} \leq C\left(1 + \int_s^t |\dot{c}(\theta)|d\theta\right)||\phi||,$$

since $||[\hat{H}(t), F((|x|/t^2) \leq \sigma)]||_{\mathfrak{B}(\mathbb{H})}$ is bounded uniformly in t, and

$$\hat{H}(t)\hat{U}(t,s) - \hat{U}(t,s)\hat{H}(s) = -\int_s^t \hat{U}(t,\theta)\dot{c}(\theta)\cdot\nabla V(x-c(\theta))\hat{U}(\theta,s)d\theta \tag{3.31}$$

Then it follows from (3.30) and (3.31) that

$$||(\omega\cdot p)^2\xi(t)|| \leq Ct^{1-\eta_0}||\phi|| + C\sigma t^2||\xi(t)|| + C||\phi||. \tag{3.32}$$

Combining (3.29), (3.32) and Assumption 1.1, we have

$$\{(E_0-\epsilon)^4 - C\sigma^2\}||\xi(t)||^2 \leq C(t^{-2-2\eta_0} + t^{-2u} + t^{-3})||\phi||^2. \tag{3.33}$$

We thus obtain Theorem 3.6, by taking $\sigma > 0$ sufficiently small. □

4. Propagation estimates II

In Section 3 we have given the propagation estimate under the condition that the directional derivative of $V(x)$ is small compared with the main field $|E|$. In this section we give the estimate without this assumption for V. Instead we assume Assumption 1.2. The form inequality $DA_0(t) \geq 0$ does not hold since E_0 is negative. But we can see that the following inequality

$$f(\hat{H}(t))i[\hat{H}(t), A]f(\hat{H}(t)) \geq (|E| - \epsilon)f(\hat{H}(t))^2 \tag{4.1}$$

holds for $t \gg 1$ if we take support of f suitably. Before showing it, we note the relation between $\hat{H}(t)$ and the time-independent Hamiltonian H. By an elementary calculation we can see that

$$\hat{H}(t) = e^{-ic(t)\cdot p}(H - E \cdot c(t))e^{ic(t)\cdot p}.$$

This implies that

$$f(\hat{H}(t)) = e^{-ic(t)\cdot p} f_T(H) e^{ic(t)\cdot p} \quad \text{for all} \quad f \in C_0^\infty(\mathbb{R}), \tag{4.2}$$

where $f_T(\cdot) = f(\cdot - E \cdot c(t))$.

Combining this relation and (1.6), we have the following lemma.

Lemma 4.1. *Let $\lambda \in \mathbb{R}$ and $\epsilon > 0$ be given. Then there exists $\delta_1 > 0$ having the following property. For every real valued function $f \in C_0^\infty([\lambda - \delta_1, \lambda + \delta_1])$, there exists $T_0 > 0$ such that*

$$f(\hat{H}(t))i[\hat{H}(t), A]f(\hat{H}(t)) \geq (|E| - \epsilon)f(\hat{H}(t))^2 \quad \textit{for} \quad t > T_0, \tag{4.3}$$

with $A = \omega \cdot p$.

Proof. We take $\delta > 0$ as in (1.6) and $f \in C_0^\infty([\lambda - \delta/2, \lambda + \delta/2])$. Then $f_T \in C_0^\infty([\lambda - \delta, \lambda + \delta])$ and $f_T(H)$ satisfies (1.6) for sufficiently large t since $E \cdot c(t) \to 0$ as $t \to \infty$. So we use the relation (4.2) to obtain (4.3). □

With this inequality we give maximal and minimal acceleration estimates under Assumption 1.2. To do so we define some conjugate operators.

DEFINITION 4.2.

$$A_1(t) = \omega \cdot p - E_1 t, \tag{4.4}$$

$$A_2(t) = vt - \langle x \rangle^{1/2}. \tag{4.5}$$

Using Lemma 4.1 we can see that $f(\hat{H}(t))DA_j(t)f(\hat{H}(t)) \geq 0$ if we choose $E_1 < |E|$ and v sufficiently large. We denote $\langle p \rangle$ by A_1, and $-A_2(0)$ as A_2. For these operators we have the following lemma, which implies that $\hat{U}(t,s)$ leaves $D(A_j)$ invariant.

Lemma 4.3. *Let $0 < \delta \leq 2$, and $h \in C_0^\infty(\mathbb{R})$ be given. Then the following properties hold for $j = 1, 2$.*

(i) *For $1 \leq n \leq 4$ the form $ad^n_{A_j(t)}(h(\hat{H}(t)))$ extends to a bounded operator on $\mathbb{H}$. Moreover $(\hat{H}(t)+i)ad^n_{A_j(t)}(h(\hat{H}(t)))$ and $ad^n_{A_j(t)}(h(\hat{H}(t)))(\hat{H}(t)+i)$ are continuous $\mathfrak{B}(\mathbb{H})$-valued uniformly bounded functions of t.*

(ii) *${A_j}^{\delta} f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)){A_j}^{-\delta}$ is a $\mathfrak{B}(\mathbb{H})$-valued continuous function of t.*

(iii) *$(-A_j(t))^{\delta}F((A(t)/t) \leq -\epsilon){A_j}^{-\delta}$ is a continuous $\mathfrak{B}(\mathbb{H})$-valued function of t.*

(iv) *$(d/dt)\big(h(\hat{H}(t))\big)$ exists in $\mathfrak{B}(\mathbb{H})$ for $1 \leq n \leq 3$ and the form $ad^n_{A_j(t)}((d/dt)h(\hat{H}(t)))$ extends to a bounded operator on $\mathbb{H}$.*

Proof. For $A(t) = A_1(t)$ the proof is almost the same as that of Lemma 3.1. We have only to give the proof for $A(t) = A_2(t)$. (i) and (iii) are easily obtained by the same argument of the proof in Lemma 3.1. (iv) is obtained by using the formula (2.8).

To prove (ii) we show that $A_2^2 f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s))A_2^{-2} \in \mathfrak{B}(\mathbb{H})$. By using commutators, we rewrite it as

$$\begin{aligned}&f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)) - 2ad^1_{A_2}(f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)))A_2^{-1}\\&+ad^2_{A_2}(f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)))A_2^{-2}.\end{aligned}$$

So it is sufficient to show

$$ad^k_{A_2}(f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)))A_2^{-k} \in \mathfrak{B}(\mathbb{H}) \quad (k=1,2).$$

For the case $k = 1$ we rewrite the commutator as follows.

$$\begin{aligned}&ad^1_{A_2}(f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)))A_2^{-1}\\&= \sum_{m_1+m_2+m_3=1} ad^{m_1}_{A_2}(f(\hat{H}(t)))ad^{m_2}_{A_2}(\hat{U}(t,s))ad^{m_3}_{A_2}(h(\hat{H}(s)))A_2^{-1}.\end{aligned} \tag{4.6}$$

If $m_2 = 0$, the boundness is obtained from (i). For the term $m_2 = 1$ we rewrite it as

$$\begin{aligned}&f(\hat{H}(t))ad^1_{A_2}(\hat{U}(t,s))h(\hat{H}(s))A_2^{-1}\\&= -if(\hat{H}(t))\left\{\int_s^t \hat{U}(t,\theta)ad^1_{A_2}(\hat{H}(\theta))\hat{U}(\theta,s)d\theta\right\}h(\hat{H}(s))A_2^{-1}.\end{aligned} \tag{4.7}$$

We can see that $ad^1_{A_2}(f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)))A_2^{-1}$ is bounded since $\langle p\rangle h(\hat{H}(s))A_2^{-1}$ and $\langle p\rangle\hat{U}(t,s)\langle p\rangle^{-1} \in \mathfrak{B}(\mathbb{H})$ (See Theorem 1 in [14].). So we have shown that $ad^1_{A_2}(\hat{U}(t,s))A_2^{-1} \in \mathfrak{B}(\mathbb{H})$. This implies the boundness of the single commutator $ad^1_{A_2}(f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)))A_2^{-1}$.

As for the case $k=2$, we have

$$\begin{aligned} &ad^2_{A_2}(f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)))A_2^{-2} \\ &= \sum_{m_1+m_2+m_3=2} C_{m_1,m_2,m_3} ad^{m_1}_{A_2}(f(\hat{H}(t)))ad^{m_2}_{A_2}(\hat{U}(t,s))ad^{m_3}_{A_2}(h(\hat{H}(s)))A_2^{-2}. \end{aligned}$$

Since we have proved the boundness of the single commutator, we have only to show the boundness of the term $m_2=2$. To see that we rewrite the double commutator as follows.

$$\begin{aligned} &ad^2_{A_2}(\hat{U}(t,s)) \\ &= -i \sum_{l_1+l_2+l_3=1} \int_s^t ad^{l_1}_{A_2}(\hat{U}(t,s))ad^{l_2+1}_{A_2}(\hat{H}(\theta))ad^{l_3}_{A_2}(\hat{U}(t,s))d\theta. \end{aligned} \tag{4.8}$$

If $l_2 = 1$, the boundness can be easily seen, since $ad^2_{A_2}(\hat{H}(\theta))$ is a bounded operator. For the term $l_2 = 0$ we obtain the boundness from the fact that $\langle p\rangle ad^1_{A_2}(\hat{U}(\theta))\langle p\rangle^{-1}$ and $\langle p\rangle h(\hat{H}(s))A_2^{-1} \in \mathfrak{B}(\mathbb{H})$. So we have shown the boundness of $f(\hat{H}(t))ad^2_{A(t)}(U(t,s))h(\hat{H}(s))A_2^{-2}$, which implies the assertion (ii). □

From Lemma 4.3 we obtain $(-g_{\beta,\alpha,\epsilon}(A_j(t),t))^{1/2}f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s))A_2^{-\alpha/2}$ is bounded for $\beta>0$ and $0\le\alpha\le4$. What remains to be shown is that it is bounded uniformly in t as we have done in Section 3. To see this we prove the following lemma which is closely related with the propagation estimate under Assumption 1.2.

Lemma 4.4. *Suppose $h\in C_0^\infty(\mathbb{R})$, β_0, $\epsilon>0$, and $f\in C_0^\infty(\mathbb{R})$ satisfies (4.3).*
(i) *Let $0<\alpha_0\le\min\{u_0,3\}$ be given. Then*

$$\begin{aligned} &\|(-g_{\beta,\alpha,\epsilon}(A_1(t),t))^{1/2}f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s))A_1^{-\alpha/2}\|_{\mathfrak{B}(\mathbb{H})} \\ &= O(1) \quad as \quad t\to\infty, \end{aligned} \tag{4.9}$$

for $(\beta,\alpha)=(0,1),\dots,(0,\alpha_0'),(\beta_0,\alpha_0)$ (for $(\beta,\alpha)=(\beta_0,\alpha_0)$ if $\alpha_0\le1$).
(ii) *Let $0<\alpha_0\le3$ be given. Then*

$$\begin{aligned} &\|(-g_{\beta,\alpha,\epsilon}(A_2(t),t))^{1/2}f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s))A_2^{-\alpha/2}\|_{\mathfrak{B}(\mathbb{H})} \\ &= O(1) \quad as \quad t\to\infty, \end{aligned} \tag{4.10}$$

for $(\beta,\alpha) = (0,1),\ldots,(0,\alpha_0'),(\beta_0,\alpha_0)\quad (= (\beta_0,\alpha_0)$ *if* $\alpha_0 \le 1)$.

By using (3.5) one can prove the following

Corollary 4.5. *Under the same conditions as in* Lemma 4.4, *we have the following result: Let* $\epsilon > 0$ *and* $0 \le \theta \le 1$ *be given.*

$$\begin{aligned} &\|(-g_{0,\alpha(1-\theta),\epsilon}(A_j(t),t))^{1/2} f(\hat H(t))\hat U(t,s)h(\hat H(s))A_j^{-\alpha/2}\|_{\mathfrak{B}(\mathbb{H})} \\ &\quad = O(t^{(\beta-\alpha\theta)/2}) as \quad t\to\infty \quad (j=1,2) \end{aligned} \tag{4.11}$$

for $(\beta,\alpha) = (0,1),\ldots,(0,\alpha_0'),(\beta_0,\alpha_0)\quad (= (\beta_0,\alpha_0)$ *if* $\alpha_0 \le 1)$

Proof of Lemma 4.4. We also denote the square of the left hand side of (4.9) as $\tilde G_{\beta,\alpha,\epsilon}(t)$. Similarly to Lemma 3.3 we prove this Lemma by investigating the integrability of $-(d/dt)\tilde G_{\beta,\alpha,\epsilon}(t)$. We write $\zeta(t) = f(\hat H(t))\hat U(t,s)h(\hat H(s))A^{-\alpha/2}\phi$ and $\hat\zeta(t) = \hat U(t,s)h(\hat H(s))A^{-\alpha/2}\phi$ with $\phi\in\mathbb{H}$. We write $\langle\zeta(t),P\zeta(t)\rangle$ as $\langle p\rangle_t$. Further we denote $\langle\hat\zeta(t),P\hat\zeta(t)\rangle$ by $\langle\langle p\rangle\rangle_t$ for an operator P. Then

$$\begin{aligned} -\frac{d}{dt}\tilde G_{\beta,\alpha,\epsilon}(t) &\equiv \langle I_1\rangle_t + 2\Re\langle\langle I_2\rangle\rangle_t + \langle I_3\rangle_t \\ &= \langle\zeta(t), i[\hat H(t), g(A(t),t)]\zeta(t)\rangle \\ &\quad +2\Re\left\langle\hat\zeta(t),\left(\frac{d}{dt}f(\hat H(t))\right)g(A(t),t)f(\hat H(t))\hat\zeta(t)\right\rangle \\ &\quad +\left\langle\zeta(t),\frac{d}{dt}g(A(t),t)\zeta(t)\right\rangle, \end{aligned}$$

where $\Re z = (1/2)(z+\bar z)$ for $z\in\mathbb{C}$.

As we have stated in Section 1 we cannot neglect the derivative $(d/dt)f(\hat H(t))$. We take $f_2\in C_0^\infty(\mathbb{R})$ which is identically equal to 1 on the support of the function f. (We denote it as $f\subset f_2$.) Further we choose $f_1\in C_0^\infty(\mathbb{R})$ such that $f_2\subset f_1$. By virtue of (2.10) and the fact that $\zeta(t) = f_1(\hat H(t))\zeta(t)$, we can rewrite I_1 as follows.

$$\begin{aligned} I_1 &= i\sum_{m=1}^{3}(m!)^{-1}g^{(m)}(A(t),t)ad^m_{A(t)}(f_1(\hat H(t))\hat H(t)) \\ &\quad +\frac{1}{2\pi}\int_{\mathbb{C}}\partial_{\bar z}\tilde g(z,t)(z-A(t))^{-4}ad^4_{A(t)}(f_1(\hat H(t))\hat H(t))(z-A(t))^{-1}dz\wedge d\bar z. \end{aligned}$$

We use (2.8) and rewrite $(d/dt)f(\hat H(t))$ by computing the derivative of the resolvent. Then

$$I_2 = -\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar z}\tilde f(z)(z-\hat H(t))^{-1}\dot c(t)\cdot\nabla V(x-c(t))(z-\hat H(t))^{-1}dz\wedge d\bar z$$

$$\times g_{\beta,\alpha,\epsilon}(A(t),t)f(\hat{H}(t)).$$

Applying (2.8) again, we see that

$$\begin{aligned} I_3 = &\left(\frac{\partial}{\partial t}g\right)(A(t),t) + \sum_{m=1}^{3}(m!)^{-1}g^{(m)}(A(t),t)ad^m_{A(t)}(d_tA(t)) \\ &+\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar{z}}\tilde{g}(z,t)(z-A(t))^{-4}ad^3_{A(t)}(d_tA(t))(z-A(t))^{-1}dz\wedge d\bar{z}. \end{aligned}$$

Denoting $d_tA(t)+i[f_1(\hat{H}(t))\hat{H}(t),A(t)]$ by $D_1A(t)$, we sum I_1 and I_3 as follows.

$$\begin{aligned} I_1+I_3 = &\left(\frac{\partial}{\partial t}g\right)(A(t),t) \\ &+i\sum_{m=1}^{3}(m!)^{-1}g^{(m)}(A(t),t)ad^{m-1}_{A(t)}(D_1A(t)) \\ &+\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar{z}}\tilde{g}(z,t)(z-A(t))^{-4}ad^3_{A(t)}(D_1A(t))(z-A(t))^{-1}dz\wedge d\bar{z}, \end{aligned}$$

with $g^{(m)} = g^{(m)}(A(t),t)$. We compute the term $g^{(1)}(A(t),t)D_1A(t)$ appearing in the second line. We sandwich $g^{(1)}(A(t),t)D_1A(t)$ by $f(\hat{H}(t))$ and rewrite $I_0 = f(\hat{H}(t))g^{(1)}(A(t),t)D_1A(t)f(\hat{H}(t))$ by using the commutator. Then we have

$$\begin{aligned} I_0 = &f(\hat{H}(t))g_hf_2(\hat{H}(t))D_1A(t)f_2(\hat{H}(t))g_hf(\hat{H}(t)) \\ &+f(\hat{H}(t))g_h(I-f_2(\hat{H}(t)))D_1A(t)f_2(\hat{H}(t))g_hf(\hat{H}(t)) \\ &+f(\hat{H}(t))g_h[g_h,D_1A(t)f_2(\hat{H}(t))]f(\hat{H}(t)). \end{aligned}$$

We remark that $f(\hat{H}(t))g_h(I-f_2(\hat{H}(t)))$ is equal to $f(\hat{H}(t))[f_2(\hat{H}(t)),g_h]$ and $f_2(\hat{H}(t))D_1A(t)f_2(\hat{H}(t)) = f_2(\hat{H}(t))DA(t)f_2(\hat{H}(t))$ since $f_2 \subset f_1$. We compute these commutators by use of Lemma 2.2, and

$$\begin{aligned} I_0 = &f(\hat{H}(t))g_hf_2(\hat{H}(t))DA(t)f_2(\hat{H}(t))g_hf(\hat{H}(t)) \\ &+f(\hat{H}(t))\sum_{m=1}^{3}(m!)^{-1}g_h^{(m)}ad^m_{A(t)}(f_2(\hat{H}(t)))D_1A(t)f_2(\hat{H}(t))g_hf(\hat{H}(t)) \\ &+f(\hat{H}(t))\left\{\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar{z}}\tilde{g}_h(z,t)(z-A(t))^{-4}\right. \\ &\left.\times\, ad^4_{A(t)}(f_2(\hat{H}(t)))(z-A(t))^{-1}dz\wedge d\bar{z}\right\}D_1A(t)f_2(\hat{H}(t))g_hf(\hat{H}(t)) \\ &+f(\hat{H}(t))g_h\sum_{m=1}^{3}\frac{(-1)^m}{m!}ad^m_{A(t)}(D_1A(t)f_2(\hat{H}(t)))g_h^{(m)}f(\hat{H}(t)) \end{aligned}$$

$$
\begin{aligned}
&+f(\hat{H}(t))g_h\left\{\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar z}\tilde{g_h}(z,t)(z-A(t))^{-1}ad^4_{A(t)}(D_1A(t)f_2(\hat{H}(t)))\right.\\
&\qquad\left.\times(z-A(t))^{-4}dz\wedge d\bar z\right\}f(\hat{H}(t)).\\
&\equiv f(\hat{H}(t))\{E_2+\ldots+E_6\}f(\hat{H}(t))
\end{aligned}
$$

We show the uniform boundness of $\tilde{G}_{\beta,\alpha,\epsilon}(t)$ for $0<\alpha\le 1$ and for larger α by induction. To do this method, we cannot neglect the term $g^{(m)}(A(t),t)ad^{m-1}_{A(t)}(D_1A(t))$ in I_1+I_3. So we also compute the commutator of $g^{(m)}(A(t),t)$ and $D_1A(t)$. Finally we have the following expressions.

$$
\begin{aligned}
&-\frac{d}{dt}\{\|(-g_{\beta,\alpha,\epsilon}(A(t),t))^{1/2}f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s))A^{-\alpha/2}\phi\|^2\}\\
&=\langle E_1\rangle_t+\ldots+\langle E_9\rangle_t+2\Re\langle\langle E_{10}\rangle\rangle_t.
\end{aligned}\tag{4.12}
$$

Here

$$
\begin{aligned}
E_1&=\left(\frac{\partial}{\partial t}g\right)(A(t),t),\\
E_2&=g_hf_2(\hat{H}(t))DA(t)f_2(\hat{H}(t))g_h,\\
E_3&=\sum_{m=1}^{3}(m!)^{-1}g_h^{(m)}ad^m_{A(t)}(f_2(\hat{H}(t)))D_1A(t)f_2(\hat{H}(t))g_h,\\
E_4&=\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar z}\tilde{g_h}(z,t)(z-A(t))^{-4}ad^4_{A(t)}(f_2(\hat{H}(t)))(z-A(t))^{-1}dz\wedge d\bar z\\
&\quad\times D_1A(t)f_2(\hat{H}(t))g_h,\\
E_5&=g_h\sum_{m=1}^{3}\frac{(-1)^m}{m!}g_h^{(m)}ad^m_{A(t)}(D_1A(t)f_2(\hat{H}(t)))g_h^{(m)},\\
E_6&=\frac{1}{2\pi i}g_h\int_{\mathbb{C}}\partial_{\bar z}\tilde{g_h}(z,t)(z-A(t))^{-1}ad^4_{A(t)}(D_1A(t)f_2(\hat{H}(t)))(z-A(t))^{-4}dz\wedge d\bar z,\\
E_7&=\sum_{m=2}^{3}(m!)^{-1}j_m(A(t),t)k_m(A(t),t)\sum_{m_1=0}^{3-m}ad^{m_1}_{A(t)}(H_m)\frac{(-1)^{m_1}}{m_1!}j_m^{(m_1)}(A(t),t),\\
E_8&=\sum_{m=2}^{3}j_m(A(t),t)k_m(A(t),t)\\
&\quad\times\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar z}\tilde{j_m}(z,t)(z-A(t))^{-1}ad^{4-m}_{A(t)}(H_m)(A(t)-z)^{m-4}dz\wedge d\bar z,\\
E_9&=\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar z}\tilde{g}(z,t)(z-A(t))^{-4}ad^3_{A(t)}(D_1A(t))(z-A(t))^{-1}dz\wedge d\bar z,\\
E_{10}&=-\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar z}\tilde{f}(z)(z-\hat{H}(t))^{-1}\dot{c}(t)\cdot\nabla V(x-c(t))(z-\hat{H}(t))^{-1}dz\wedge d\bar z
\end{aligned}
$$

$$\times g_{\beta,\alpha,\epsilon}(A(t),t)f(\hat{H}(t)),$$

where

$$\begin{aligned} j_m(x,t) &= g_{\beta/2,((\alpha-m)/2)_+,\epsilon/2}(x,t), \\ k_m(x,t) &= \frac{t^\beta}{m!}(-x)^{-(\alpha-m)_+}g^{(m)}(x,t), \\ H_m &= ad_{A(t)}^{m-1}(D_1A(t))f_2(\hat{H}(t)). \end{aligned}$$

Here the terms E_7 and E_8 are obtained in the following way. We rewrite $g^{(m)}(x,t) = j_m(x,t)k_m(x,t)j_m(x,t)$ and compute the commutators of H_m and them. Since $\tilde{G}_{\beta,\alpha,\epsilon}(t)$ is non-negative, it remains to show that each term in (4.12) is either integrable with respect to t or non-negative.

Suppose $0 < \alpha_0 \leq 1$ and $(\beta,\alpha) = (\beta_0,\alpha_0)$. By the same argument of Section 3, we can see that each of $\langle E_3\rangle_t, \ldots, \langle E_9\rangle_t$ is integrable with respect to t. Now we show the integrability of the remaining term $\langle E_{10}\rangle_t$.

For $A(t) = A_1(t)$, $\langle E_{10}\rangle_t = O(t^{-\beta_0-u_0+1})$ from Assumption 1.2. As for the case $A(t) = A_2(t)$, we obtain $\langle E_{10}\rangle_t = O(t^{-\beta_0-u_0-2})$. It is due to the facts

$$\begin{aligned} &\|\langle x\rangle^{3/2}(z-\hat{H}(t))^{-1}\langle x\rangle^{-3/2}\|_{\mathfrak{B}(\mathbb{H})} \leq C|\mathrm{Im} z|^3, \\ &\|\langle x\rangle^{-3/2}g_{\beta_0,\alpha_0,\epsilon}(A_2(t),t)\|_{\mathfrak{B}(\mathbb{H})} = O(t^{-\beta_0+\alpha_0-3}). \end{aligned}$$

For larger α_0 we prove this Lemma by using induction on α_0. So we obtain Lemma 4.4. □

Finally we give propagation estimates under Assumption 1.2. It is obtained by the same argument as in Lemma 3.5. We choose the conjugate operator $A(t) = A_1(t)$. From the inequality (4.9) we have

$$\left\|F\left(\frac{\omega\cdot p}{t} \leq E_1 - \epsilon\right)\hat{U}(t,s)h(\hat{H}(s))\langle x\rangle^{-u/2}\right\|_{\mathfrak{B}(\mathbb{H})} = O(t^{-u})$$

for $0 < u \leq \min\{u_0/2, 3/2\}$.

By the same argument as in the proof of Theorem 3.6, we have the minimal acceleration estimate under Assumption 1.2

Theorem 4.6. *Suppose the conditions in* Lemma 4.4 *are satisfied. Let* $0 < u \leq \min\{u_0/2, 3/2\}$ *be given. Then there exists* $\sigma > 0$ *such that*

$$\left\|F\left(\frac{|x|}{t^2} \leq \sigma\right)f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s))\langle x\rangle^{-u/2}\right\| = O(t^{-L}) \quad \text{as} \quad t\to\infty \tag{4.13}$$

where $L = \min\{u, 3/2\}$.

We also have the following estimate under Assumption 1.2, which is called maximal acceleration estimate.

Theorem 4.7. *There exists sufficiently large number $M > 0$ such that for all $0 < u \leq 3/2$,*

$$\left\| F\left(\frac{|x|}{t^2} \geq M\right) f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s))\langle x\rangle^{-u/2}\right\| = O(t^{-u}) \quad as \quad t \to \infty. \tag{4.14}$$

Proof. We recall the fact that $p\langle x\rangle^{-1/2} f(\hat{H}(t)) \in \mathfrak{B}(\mathbb{H})$. We apply Lemma 4.4 with the conjugate operator $A_2(t)$. Then we have $f(\hat{H}(t))DA_2(t)f(\hat{H}(t)) \geq 0$ holds for $v \gg 1$. So we can see that Theorem 4.7 holds by use of Lemma 4.4 and Corollary 4.5. □

Proof of Theorem 1.3

For both cases of Assumption 1.1 and Assumption 1.2, we apply Cook's method to show the existence of the strong limit of $\hat{U}_0(t,s)^*\hat{U}(t,s)$ and that of $W^+(s)$ and $\tilde{W}^+(s)$. At first we assume that Assumption 1.1 holds. In the same way as in (1.7) we rewrite $\hat{U}_0(t,s)^*\hat{U}(t,s)\phi$ as follows.

$$\hat{U}_0(t,s)^*\hat{U}(t,s)\psi = \psi - i\int_s^t \hat{U}_0^*(s,\theta)V(x - c(\theta))\hat{U}(\theta,s)\psi d\theta.$$

Here we choose $\psi = h(\hat{H}(t))\langle x\rangle^{-u/2}\phi$ with $\phi \in \mathbb{H}$, $h \in C_0^\infty(\mathbb{R})$, and $u > 1$. To see the existence of $s - \lim_{t\to\infty} \hat{U}_0(t,s)^*\hat{U}(t,s)$, we have only to show the integrability of $\|V(x - c(\theta))\hat{U}(\theta,s)\phi\|$ with respect to $\theta \in \mathbb{R}_+$. We remark that it is sufficient to show the integrability when $\phi = h(\hat{H}(s))\langle x\rangle^{-u}\psi$ with $\psi \in \mathbb{H}$ and $h \in C_0^\infty(\mathbb{R})$.

From Assumption 1.1 we have $|c(t)| = o(t)$ as $t \to \infty$. We split $V(x - c(t))$ into $V(x-c(t))F((|x|/t^2) \leq \sigma)+V(x-c(t))F((|x|/t^2) \geq \sigma)$. Combining Theorem 3.6 and this decomposition, we have $\|V(x - c(\theta))\hat{U}(\theta,s)\psi\| \in L^1(d\theta,\mathbb{R}_+)$. This implies the existence of $s - \lim_{t\to\infty} \hat{U}_0(t,s)^*\hat{U}(t,s)$ and that of $W^+(s)$. We also obtain the existence of $\tilde{W}^+(s)$ in the same way.

Suppose Assumption 1.2 holds. In a similar way to the above arguement we can easily see the existence of $s - \lim_{t\to\infty} \hat{U}_0(t,s)^* f(\hat{H}(t))\hat{U}(t,s)$ for each $f \in C_0^\infty(\mathbb{R})$. To show the existence of $s - \lim_{t\to\infty} \hat{U}_0(t,s)^*\hat{U}(t,s)$, we define a bounded operator as follows. Let $M > 1$ be given. We take $\rho \in C_0^\infty([-1,1])$ such that $\rho(x) \equiv 1$ for $x \in [-1/2,1/2]$. We denote $\rho(x/M)$ as $\rho_M(x)$ and

$$P_M(s) = \int_s^\infty \hat{U}_0(\theta,s)^*\frac{d}{d\theta}\rho_{2M}(\hat{H}(\theta))\hat{U}(\theta,s)d\theta.$$

We remark that

$$\left|\frac{d}{d\theta}\rho_M(\hat{H}(\theta))\right| \le CM^{-1}|\dot{c}(\theta)|,$$

which follows from the fact that an almost analytic extension of $\rho_M(x)$ satisfies

$$|\partial_{\bar{z}}\tilde{\rho}_M(z)| \le CM^{-1}\left\langle\frac{z}{M}\right\rangle^{-3}\left|\frac{\mathrm{Im}z}{M}\right|^2.$$

In the same way as in (4.7) we have

$$\begin{aligned}
&\hat{U}_0(t,s)^*(I-\rho_{2M}(\hat{H}(t))\hat{U}(t,s)\rho_M(\hat{H}(s))) \\
&= \hat{U}_0(t,s)^*\hat{U}(t,s)\{\rho_{2M}(\hat{H}(s)) - \hat{U}(s,t)\rho_{2M}(\hat{H}(t))\hat{U}(t,s)\}\rho_M(\hat{H}(s)) \\
&= \hat{U}_0(t,s)^*\hat{U}(t,s)\left\{-P_M(s) + \int_t^\infty \hat{U}(s,\theta)\frac{d}{d\theta}\rho_{2M}(\hat{H}(\theta))\hat{U}(\theta,s)d\theta\right\}\rho_M(\hat{H}(s)).
\end{aligned}$$

By an elementary calculus we have

$$\begin{aligned}
&\hat{U}_0(t,s)^*\hat{U}(t,s)\{I+P_M(s)\}\rho_M(\hat{H}(s)) \\
&= \hat{U}_0(t,s)^*\rho_M(\hat{H}(t))\hat{U}(t,s)\rho_M(\hat{H}(s)) + o(1).
\end{aligned}$$

This proves the existence of the strong limit $s-\lim_{t\to\infty}\hat{U}_0(t,s)^*\hat{U}(t,s)\{I+P_M(s)\}\rho_M(\hat{H}(s))$ for each $M>1$. Since $\{I+P_M(s)\}\rho_M(\hat{H}(s))\phi \to \phi$ for each $\phi \in \mathbb{H}$ as $M \to \infty$, we can also see the existence of $s-\lim_{t\to\infty}\hat{U}_0(t,s)^*\hat{U}(t,s)$.

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