BILINEAR IDENTITY FOR q-HYPERGEOMETRIC INTEGRALS

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1. Introduction

In this paper we describe a bilinear identity satisfied by certain multidimensional q-hypergeometric integrals. We give two equivalent forms of the identity, cf. Theorems 2.9 and 3.7. We call this identity the *hypergeometric Riemann identity*. In the one-dimensional case the q-hypergeometric integrals can be expressed in terms of the basic hypergeometric series $_n\varphi_{n-1}$ and the hypergeometric Riemann identity reduces to bilinear identities for these series, see Section 7.

There are two interpretations of the hypergeometric Riemann identity which make it a subject of our interest. First, the hypergeometric Riemann identity is an analogue of the Riemann bilinear relation for the twisted (co)homology groups defined by the reciprocally dual local systems, see [3], [7]. An appropriate form of the hypergeometric Riemann identity is given by Theorem 2.9. From this point of view, the trigonometric and elliptic hypergeometric spaces correspond respectively to the top cohomology and homology groups, the hypergeometric pairings provide natural duality between them and the Shapovalov pairings play the role of the intersection forms. This analogy can be seen directly from the explicit formulae. But in fact, there is much deeper similarity, including the dual discrete local systems, the difference twisted de Rham complex etc., see [1], [11]. The deformation of the Riemann bilinear relation for the hyperelliptic Riemann surfaces was obtained in [9].

The second interpretation of the hypergeometric Riemann identity comes from the representation theory of quantum affine algebras, namely, via the quantum Knizhnik-Zamolodchikov (qKZ) equation. The qKZ equation is a remarkable system of difference equations introduced in [4]. It is a natural deformation of the famous differential Knizhnik-Zamolodchikov equation inheriting many of its nice properties.

In [11] Varchenko and the author constructed all solutions of the qKZ equa-

tion with values in a tensor product of $U_q(\mathfrak{sl}_2)$ Verma modules with generic highest weights using the q-hypergeometric integrals. The space of solutions of the qKZ equation was indentified in [11] with the tensor product of the corresponding evaluation Verma modules over the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$. The dual version of that construction is also available: the system of difference equations is dual to the qKZ equation, its solutions take values in the dual space to the tensor product of $U_q(\mathfrak{sl}_2)$-modules and the space of solutions can be identified with the dual space to the tensor product of $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules. In this context the hypergeometric Riemann identity means that the q-hypergeometric solutions of the qKZ and dual qKZ equations transform the natural pairing of the spaces of solutions to the natural pairing of the target spaces; namely, given respective solutions Ψ and Ψ^* of the qKZ and dual qKZ equations we have that

$$\langle \, Value \; \Psi^*, \; Value \; \Psi \, \rangle_{\text{ target spaces}} \; = \; \langle \Psi^*, \Psi \rangle_{\text{ spaces of solutions}} \cdot$$

In particular, we can say that the hypergeometric Riemann identity is a deformation of Gaudin-Korepin's formula for norms of the Bethe vectors [6], cf. [13].

In the main part of the paper we make no references to the deformed (co)homology theory and limit ourselves to only a few remarks about the representation theory; for more conceptual point of view see [11], [12].

The results of [11] are crucial for our proof of the hypergeometric Riemann identity. It is shown in [11] that a matrix formed by certain hypergeometric integrals satisfies a system of linear difference equations, cf. Theorem 4.1 in this paper, which is equivalent to the qKZ equation. Moreover, this matrix has a finite limit in a suitable asymptotic zone and the limit is a triangular matrix, cf. Propositions 4.3, 4.4. To prove the hypergeometric Riemann identity we first use the system of difference equations and its dual system, and show that it is enough to check the identity as the parameters tends to limit in the asymptotic zone. The remaining verification is quite straightforward since all the matrices involved have constant triangular asymptotics in the asymptotic zone.

In this paper we consider the so-called *trigonometric* or *multiplicative* case, which involves the q-hypergeometric integrals. There is the rational version of the story which involves the multidimensional hypergeometric integrals of Mellin-Barnes type, formulae being written in terms of the gamma function, rational and trigonometric functions. The rational case can be considered as the logarithmic deformation taking the intermediate place between the classical and trigonometric deformed cases. The rational case of the qKZ equation and related deformed (co)homologies were studied in [12]. The rational version of the hypergeometric Riemann identity can be obtained similarly to the trigonometric one using results of [12]. It will appear separately.

The paper is organized as follows. In Section 2 we give definitions and formulate the main result of the paper – the hypergeometric Riemann identity, cf. Theorem 2.9. The equivalent form of the identity is given in Section 3, cf. Theorem 3.7. In Section 4 we describe a system of difference equations satisfied by the q-hypergeometric integrals and study their behaviour in a suitable asymptotic zone. The final step of the proof of the hypergeometric Riemann identity is made in Section 5. We give the restricted version of the hypergeometric Riemann identity in Section 6 and consider the one-dimensional example in Section 7.

There are three Appendices in the paper. Appendix A contains the proof of Proposition 2.8. Some of the determinant formulae from [11] relevant to this paper are reproduced in Appendix B. In Appendix C we explain that the q-hypergeometric integrals which we are using in the paper essentially coincide with the symmetric A-type Jackson integrals.

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2. The hypergeometric Riemann identity

Basic notations

Let $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$. Fix a nonzero complex number p such that $|p| < 1$. Set

$$p^{\mathbb{Z}} = \{p^s \mid s \in \mathbb{Z}\}.$$

Let $(u)_{\infty} = (u; p)_{\infty} = \prod_{s=0}^{\infty}(1 - p^s u)$ and let $\theta(u) = (u)_{\infty}(pu^{-1})_{\infty}(p)_{\infty}$ be the Jacobi theta-function.

Fix a nonnegative integer ℓ. Take nonzero complex numbers η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ called *parameters*. Say that the parameters are generic if for any $r = 0, \ldots, \ell - 1$ and any $k, m = 1, \ldots, n$, we have

(2.1)
$$\eta^{r+1} \notin p^{\mathbb{Z}}, \qquad \eta^{\pm r} x_k/x_m \notin p^{\mathbb{Z}}, \quad \eta^{\pm r} y_k/y_m \notin p^{\mathbb{Z}} \quad \text{for} \quad k \neq m,$$
$$\eta^{\pm r} x_k/y_m \notin p^{\mathbb{Z}}.$$

All over the paper we assume that the parameters are generic, unless otherwise stated.

For any function $f(t_1, \ldots, t_\ell)$ and any permutation $\sigma \in \mathbf{S}_\ell$ set

$$(2.2) \qquad [f]_\sigma(t_1, \ldots, t_\ell) = f(t_{\sigma_1}, \ldots, t_{\sigma_\ell}) \prod_{\substack{1 \leqslant a < b \leqslant \ell \\ \sigma_a > \sigma_b}} \frac{t_{\sigma_b} - \eta t_{\sigma_a}}{\eta t_{\sigma_b} - t_{\sigma_a}} \,,$$

$$(2.3) \qquad [\![f]\!]_\sigma(t_1, \ldots, t_\ell) = f(t_{\sigma_1}, \ldots, t_{\sigma_\ell}) \prod_{\substack{1 \leqslant a < b \leqslant \ell \\ \sigma_a > \sigma_b}} \frac{\eta \theta(\eta^{-1} t_{\sigma_b}/t_{\sigma_a})}{\theta(\eta t_{\sigma_b}/t_{\sigma_a})} \,.$$

Each of the formulae defines an action of the symmetric group $\mathbf{S}_\ell$.

For any $\mathfrak{l} = (\mathfrak{l}_1, \ldots, \mathfrak{l}_n) \in \mathbb{Z}_{\geqslant 0}^n$ set $\mathfrak{l}^m = \mathfrak{l}_1 + \ldots + \mathfrak{l}_m$, $m = 1, \ldots, n$. Set

$$\mathscr{Z}_\ell^n = \{ \mathfrak{l} \in \mathbb{Z}_{\geqslant 0}^n \mid \mathfrak{l}^n = \ell \} \,.$$

For any $\mathfrak{l}, \mathfrak{m} \in \mathscr{Z}_\ell^n$, $\mathfrak{l} \neq \mathfrak{m}$, say that

$$(2.4) \qquad \mathfrak{l} \ll \mathfrak{m} \qquad \text{if} \qquad \mathfrak{l}^k \leqslant \mathfrak{m}^k \qquad \text{for any} \quad k = 1, \ldots, n-1 \,.$$

REMARK. We can identify $\mathfrak{l} \in \mathbb{Z}_{\geqslant 0}^n$ with a partition $\mathfrak{l}^n \geqslant \ldots \geqslant \mathfrak{l}^1$. The introduced above ordering on $\mathscr{Z}_\ell^n$ coincides with the inverse dominance ordering for the corresponding partitions.

For any $x \in \mathbb{C}^n$ and $\mathfrak{l} \in \mathscr{Z}_\ell^n$ we define the point $x \triangleright \mathfrak{l}[\eta] \in \mathbb{C}^{\times \ell}$ as follows:

$$(2.5) \quad x \triangleright \mathfrak{l}[\eta] = \left(\eta^{1-\mathfrak{l}_1} x_1, \eta^{2-\mathfrak{l}_1} x_1, \ldots, x_1, \eta^{1-\mathfrak{l}_2} x_2, \ldots, x_2, \ldots, \eta^{1-\mathfrak{l}_n} x_n, \ldots, x_n \right),$$

For any function $f(t_1, \ldots, t_\ell)$ and a point $t^\star = (t_1^\star, \ldots, t_\ell^\star)$ we define the multiple residue $\operatorname{Res} f(t)\big|_{t=t^\star}$ by the formula

$$(2.6) \qquad \operatorname{Res} f(t)\big|_{t=t^\star} = \operatorname{Res} \left(\ldots \operatorname{Res} f(t_1, \ldots, t_\ell)\big|_{t_\ell = t_\ell^\star} \cdots \right)\Big|_{t_1 = t_1^\star} \,.$$

We often use in the paper the following compact notations:

$$t = (t_1, \ldots, t_\ell) \,, \qquad x = (x_1, \ldots, x_n) \,, \qquad y = (y_1, \ldots, y_n) \,.$$

For any vector space V we denote by V^* the dual vector space, and for a linear operator A we denote by A^* the dual operator.

In this paper we extensively use results from [11]. We have the following correspondence of parameters $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ in this paper and parameters $\xi_1, \ldots, \xi_n$, $z_1, \ldots, z_n$ in [11]:

$$(2.7) \qquad x_m = \xi_m z_m \,, \qquad y_m = \xi_m^{-1} z_m \,, \qquad m = 1, \ldots, n \,.$$

The hypergeometric integral

Let $\widetilde{\Phi}(t; x; y; \eta)$ be the following function:

$$\widetilde{\Phi}(t; x; y; \eta) = \prod_{m=1}^{n} \prod_{a=1}^{\ell} \frac{1}{(x_m/t_a)_\infty \, (t_a/y_m)_\infty} \prod_{\substack{a,b=1 \\ a \neq b}}^{\ell} \frac{1}{(\eta^{-1}t_a/t_b)_\infty} \, .$$

For any function $f(t_1, \ldots, t_\ell)$ holomorphic in $\mathbb{C}^{\times \ell}$ we define below the *hypergeometric integral* $Int(f\widetilde{\Phi})$.

Assume that $|\eta| > 1$ and $|x_m| < 1$, $|y_m| > 1$, $m = 1, \ldots, n$. Then we set

$$(2.8) \qquad Int[x; y; \eta](f\widetilde{\Phi}) = \frac{1}{(2\pi i)^\ell} \int_{\mathbb{T}^\ell} f(t) \, \widetilde{\Phi}(t; x; y; \eta) \, (dt/t)^\ell$$

where $(dt/t)^\ell = \prod_{a=1}^{\ell} dt_a/t_a$ and $\mathbb{T}^\ell = \{t \in \mathbb{C}^\ell \mid |t_1| = 1, \ldots, |t_\ell| = 1\}$. We define $Int[x; y; \eta](f\widetilde{\Phi})$ for arbitrary values of the parameters η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ by the analytic continuation with respect to the parameters.

Proposition 2.1. *For generic values of the parameters η, $x_1, \ldots, x_n$, y_1, $\ldots, y_n$, see (2.1), the hypergeometric integral $Int[x; y; \eta](f\widetilde{\Phi})$ is well defined and is a holomorphic function of the parameters.*

Proof. For generic values of the parameters η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ singularities of the integrand $f(t) \widetilde{\Phi}(t; x; y; \eta)$ are at most at the following hyperplanes:

$$(2.9) \qquad t_a = 0, \qquad t_a = p^s x_m, \qquad t_a = p^s y_m, \qquad t_a = p^{-s}\eta t_b,$$

$a, b = 1, \ldots, \ell$, $a \neq b$, $m = 1, \ldots, n$, $s \in \mathbb{Z}_{\geqslant 0}$. The number of edges (nonempty intersections of the hyperplanes) of configuration (2.9) and dimensions of the edges are always the same for nonzero generic values of the parameters. Therefore, the topology of the complement in $\mathbb{C}^\ell$ of the union of the hyperplanes (2.9) does not change if the parameters are nonzero generic.

The rest of the proof is similar to the proof of Theorem 5.7 in [12]. $\qquad \square$

It is clear from the proof of Proposition 2.1 that for generic value of the parameters the hypergeometric integral $Int[x; y; \eta](f\widetilde{\Phi})$ can be represented as an integral

$$(2.10) \qquad Int[x; y; \eta](f\widetilde{\Phi}) = \frac{1}{(2\pi i)^\ell} \int_{\widetilde{\mathbb{T}}^\ell[x;y;\eta]} f(t) \, \widetilde{\Phi}(t; x; y; \eta) \, (dt/t)^\ell$$

where $\widetilde{\mathbb{T}}^\ell[x; y; \eta]$ is a suitable deformation of the torus $\mathbb{T}^\ell$ which does not depend on f.

REMARK. In what follows we are using the hypergeometric integrals $Int(f\widetilde{\Phi})$ only for symmetric functions f which have a certain particular form. In this case the hypergeometric integrals coincide with the symmetric A-type Jackson integrals, cf. Appendix C.

The hypergeometric spaces and the hypergeometric pairing

Let $\mathcal{F}[x; \eta; \ell]$ be the space of rational functions $f(t_1, \ldots, t_\ell)$ such that the product

$$f(t_1, \ldots, t_\ell) \prod_{a=1}^{\ell} t_a^{-1} \prod_{m=1}^{n} \prod_{a=1}^{\ell} (t_a - x_m) \prod_{1 \leqslant a < b \leqslant \ell} \frac{\eta t_a - t_b}{t_a - t_b}$$

is a symmetric polynomial of degree less than n in each of the variables $t_1, \ldots, t_\ell$. Elements of the space $\mathcal{F}[x; \eta; \ell]$ are invariant with respect to action (2.2) of the symmetric group $\mathbf{S}_\ell$. Set

$$\mathcal{F}'[x; \eta; \ell] = \left\{ f(t_1, \ldots, t_\ell) \mid t_1 \ldots t_\ell \, f(t_1, \ldots, t_\ell) \in \mathcal{F}[x; \eta^{-1}; \ell] \right\}.$$

The spaces $\mathcal{F}$ and $\mathcal{F}'$ are called the *trigonometric hypergeometric spaces.*

REMARK. We have quite a few motivations to call the spaces $\mathcal{F}$ and $\mathcal{F}'$ trigonometric, though no trigonometric functions will appear actually in the paper.

Fix $\alpha \in \mathbb{C}^\times$. Let $\mathcal{F}_{ell}[\alpha; x; \eta; \ell]$ be the space of functions $g(t_1, \ldots, t_\ell)$ such that

$$g(t_1, \ldots, t_\ell) \prod_{m=1}^{n} \prod_{a=1}^{\ell} \theta(t_a/x_m) \prod_{1 \leqslant a < b \leqslant \ell} \frac{\theta(\eta t_a/t_b)}{\theta(t_a/t_b)}$$

is a symmetric holomorphic function of $t_1, \ldots, t_\ell$ in $\mathbb{C}^{\times \ell}$ and

$$g(t_1, \ldots, pt_a, \ldots, t_\ell) = \alpha \eta^{2-2a} g(t_1, \ldots, t_\ell).$$

Elements of the space $\mathcal{F}_{ell}[\alpha; x; \eta; \ell]$ are invariant with respect to action (2.3) of the symmetric group $\mathbf{S}_\ell$. Set

$$\mathcal{F}'_{ell}[\alpha; x; \eta; \ell] = \mathcal{F}_{ell}[\alpha^{-1}; x; \eta^{-1}; \ell].$$

The spaces $\mathcal{F}_{ell}$ and $\mathcal{F}'_{ell}$ are called the *elliptic hypergeometric spaces.*

REMARK. The parameter α here is related to the parameter κ in [11]: $\alpha = \kappa \eta^{\ell-1} \prod_{m=1}^{n} \xi_m^{-1}$, cf. (2.7).

In what follows we do not indicate explicitly all arguments for the hypergeometric spaces and related maps if it causes no confusion. The suppressed arguments are supposed to be the same for all the spaces and maps involved.

REMARK. The trigonometric hypergeometric spaces can be considered as degenerations of the elliptic hypergeometric spaces as $p \to 0$ and then $\alpha \to 0$. In this limit the spaces $\mathcal{F}_{ell}[\alpha]$ and $\mathcal{F}'_{ell}[\alpha]$ degenerate into the spaces $\mathcal{F}$ and $\mathcal{F}'$, respectively. Because of this correspondence we use two slightly different versions of the trigonometric hypergeometric spaces.

Proposition 2.2([11]). *For any* α, η, $x_1, \ldots, x_n$ *we have that*

$$\dim \mathcal{F}[x; \eta; \ell] = \dim \mathcal{F}'[x; \eta; \ell] = \dim \mathcal{F}_{ell}[\alpha; x; \eta; \ell] = \binom{n + \ell - 1}{n - 1}.$$

Let $\Phi(t; x; y; \eta)$ be the following function:

$$(2.11) \qquad \Phi(t; x; y; \eta) = \prod_{m=1}^{n} \prod_{a=1}^{\ell} \frac{(t_a/x_m)_\infty}{(t_a/y_m)_\infty} \prod_{1 \leqslant a < b \leqslant \ell} \frac{(\eta t_a/t_b)_\infty}{(\eta^{-1} t_a/t_b)_\infty}.$$

We call the function $\Phi(t; x; y; \eta)$ the *phase function*. Notice that

$$\Phi(t; y; x; \eta^{-1}) = \big(\Phi(t; x; y; \eta)\big)^{-1}.$$

The hypergeometric integral (2.10) induces the *hypergeometric pairings* of the trigonometric and elliptic hypergeometric spaces:

$$(2.12) \qquad I[\alpha; x; y; \eta] : \mathcal{F}_{ell}[\alpha; x; \eta] \otimes \mathcal{F}[x; \eta] \to \mathbb{C},$$

$$f \otimes g \mapsto \frac{1}{\ell!} \, Int \, [x; y; \eta]\big(fg \, \Phi(\cdot; x; y; \eta)\big),$$

$$I'[\alpha; x; y; \eta] : \mathcal{F}'_{ell}[\alpha; y; \eta] \otimes \mathcal{F}'[y; \eta] \to \mathbb{C},$$

$$f \otimes g \mapsto \frac{1}{\ell!} \, Int \, [y; x; \eta^{-1}]\big(fg \, \Phi(\cdot; y; x; \eta^{-1})\big).$$

We also consider these pairings as linear maps from the elliptic hypergeometric spaces to the dual spaces of the trigonometric hypergeometric spaces, denoting them by the same letters:

$$(2.13) \qquad I[\alpha; x; y; \eta] : \mathcal{F}_{ell}[\alpha; x; \eta] \to \big(\mathcal{F}[x; \eta]\big)^*,$$

$$I'[\alpha; x; y; \eta] : \mathcal{F}'_{ell}[\alpha; y; \eta] \to \big(\mathcal{F}'[y; \eta]\big)^*.$$

REMARK. In this paper we multiply the hypergeometric pairings by an additional factor $\dfrac{1}{(2\pi i)^\ell \ell!}$ compared with the hypergeometric pairing in [11].

Proposition 2.3([11]). *Let the parameters* η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ *be generic.*
Assume that

$$\alpha \neq p^s \eta^r, \qquad \alpha \eta^{2-2\ell} \prod_{m=1}^{n} x_m / y_m \neq p^{-s-1} \eta^{-r}, \qquad r = 0, \ldots, \ell-1, \quad s \in \mathbb{Z}_{\geqslant 0}.$$

Then the hypergeometric pairing $I[\alpha; x; y; \eta] : \mathcal{F}_{ell}[\alpha; x; \eta] \to \left(\mathcal{F}[x; \eta]\right)^*$ *is nondegenerate.*

The statement follows from Corollary B.3.

Corollary 2.4. *Let the parameters* η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ *be generic.*
Assume that

$$\alpha \neq p^{-s-1} \eta^r, \qquad \alpha \eta^{2-2\ell} \prod_{m=1}^{n} x_m / y_m \neq p^{s} \eta^{-r}, \qquad r = 0, \ldots, \ell-1, \quad s \in \mathbb{Z}_{\geqslant 0}.$$

Then the hypergeometric pairing $I'[\alpha; x; y; \eta] : \mathcal{F}'_{ell}[\alpha; y; \eta] \to \left(\mathcal{F}'[y; \eta]\right)^*$ *is nondegenerate.*

Proof. Let π be the following map: $\pi : f(t_1, \ldots, t_\ell) \mapsto t_1 \ldots t_\ell f(t_1, \ldots, t_\ell)$. Then the next diagram is commutative:

$$(2.14) \qquad \begin{array}{ccc} \mathcal{F}_{ell}[p^{-1}\alpha^{-1}; y; \eta^{-1}] & \xrightarrow{\;I[p^{-1}\alpha^{-1}; y; x; \eta^{-1}]\;} & \left(\mathcal{F}[y; \eta^{-1}]\right)^* \\ \pi \downarrow & & \downarrow \pi^* \\ \mathcal{F}'_{ell}[\alpha; y; \eta] & \xrightarrow[\;I'[\alpha; x; y; \eta]\;]{} & \left(\mathcal{F}'[y; \eta]\right)^* \end{array}$$

and the vertical arrows are invertible, which proves the statement. $\qquad\square$

The Shapovalov pairing

Let the points $x \triangleright \mathfrak{l}[\eta] \in \mathbb{C}^{\times \ell}$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, be defined by (2.5). For any function $f(t_1, \ldots, t_\ell)$ set

$$Res[x; \eta](f) = \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} \mathrm{Res}\left(t_1^{-1} \ldots t_\ell^{-1} f(t_1, \ldots, t_\ell)\right)\Big|_{t = x \triangleright \mathfrak{m}[\eta]}.$$

Lemma 2.5([11]). *For any* $f \in \mathcal{F}[x; \eta; \ell]$ *and* $g \in \mathcal{F}'[y; \eta; \ell]$ *we have*

$$Res[x; \eta](fg) = (-1)^\ell Res[y; \eta^{-1}](fg) = \frac{1}{(2\pi i)^\ell \ell!} \int\limits_{\widetilde{\mathbb{T}}^\ell[x; y; \eta]} f(t)\, g(t)\, (dt/t)^\ell$$

where $\widetilde{\mathbb{T}}^\ell[x; y; \eta]$ *is the deformation of the torus* $\mathbb{T}^\ell$ *defined by (2.10).*

Lemma 2.6([11]). *For any $f \in \mathcal{F}_{ell}[\alpha; x; \eta; \ell]$ and $g \in \mathcal{F}'_{ell}[\alpha; y; \eta; \ell]$ we have*

$$Res[x; \eta](fg) = (-1)^{\ell} Res[y; \eta^{-1}](fg).$$

We define the *Shapovalov pairings* of the trigonometric and elliptic hypergeometric spaces as follows:

(2.15) $$S[x; y; \eta] : \mathcal{F}'[y; \eta; \ell] \otimes \mathcal{F}[x; \eta; \ell] \to \mathbb{C},$$

$$f \otimes g \mapsto Res[x; \eta](fg),$$

$$S_{ell}[\alpha; x; y; \eta] : \mathcal{F}'_{ell}[\alpha; y; \eta; \ell] \otimes \mathcal{F}_{ell}[\alpha; x; \eta; \ell] \to \mathbb{C},$$

$$f \otimes g \mapsto Res[x; \eta](fg).$$

We also consider these pairings as linear maps, denoting them by the same letters:

(2.16) $$S[x; y; \eta] : \mathcal{F}'[y; \eta] \to \left(\mathcal{F}[x; \eta]\right)^{*},$$

$$S_{ell}[\alpha; x; y; \eta] : \mathcal{F}'_{ell}[\alpha; y; \eta] \to \left(\mathcal{F}_{ell}[\alpha; x; \eta]\right)^{*}.$$

Proposition 2.7. *Let the parameters η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ be generic. Then the Shapovalov pairing $S[x; y; \eta] : \mathcal{F}'[y; \eta] \to \left(\mathcal{F}[x; \eta]\right)^{*}$ is nondegenerate.*

The statement follows from Lemma 3.5.

Proposition 2.8. *Let the parameters η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ be generic. Assume that*

$$\alpha\,\eta^{-r} \notin p^{\mathbb{Z}}, \qquad \alpha\,\eta^{r+2-2\ell} \prod_{m=1}^{n} x_m/y_m \notin p^{\mathbb{Z}}, \qquad r = 0, \ldots, \ell - 1.$$

Then the Shapovalov pairing $S_{ell}[\alpha; x; y; \eta] : \mathcal{F}'_{ell}[\alpha; y; \eta] \to \left(\mathcal{F}_{ell}[\alpha; x; \eta]\right)^{*}$ is nondegenerate.

The statement follows from Proposition A.4.

The hypergeometric Riemann identity

In this section we formulate the main result of the paper, the hypergeometric Riemann identity which involves both the hypergeometric and Shapovalov pairings, see Theorem 2.9. We prove this result in Section 5.

Theorem 2.9. *Let the parameters* η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ *be generic. Then the following diagram is commutative:*

$$
\begin{array}{ccc}
\mathcal{F}_{ell}[\alpha; x; \eta] & \xrightarrow{\;\; I[\alpha; x; y; \eta] \;\;} & \left(\mathcal{F}[x; \eta]\right)^* \\[2mm]
{\scriptstyle \left(S_{ell}[\alpha; x; y; \eta]\right)^*} \Big\downarrow & & \Big\downarrow {\scriptstyle (-1)^\ell \left(S[x; y; \eta]\right)^{-1}} \\[2mm]
\left(\mathcal{F}'_{ell}[\alpha; y; \eta]\right)^* & \xleftarrow[\;\; \left(I'[\alpha; x; y; \eta]\right)^* \;\;]{} & \mathcal{F}'[y; \eta]
\end{array}
$$

REMARK. Given bases of the hypergeometric spaces, Theorem 2.9 translates into bilinear relations for the corresponding hypergeometric integrals. In the next section we describe an important example of the bases — the bases given by the weight functions (3.1), (3.2).

3. Tensor coordinates on the hypergeometric spaces

In this section we give an equivalent form of the hypergeometric Riemann identity, see Theorem 3.7.

Bases of the hypergeometric spaces

For any $\mathfrak{l} \in \mathcal{Z}_\ell^n$ define the functions $w_{\mathfrak{l}}$ and $W_{\mathfrak{l}}$ by the formulae:

(3.1)

$$
w_{\mathfrak{l}}(t; x; y; \eta) = \prod_{m=1}^{n} \prod_{s=1}^{\mathfrak{l}_m} \frac{1-\eta}{1-\eta^s} \sum_{\sigma \in \mathsf{S}_\ell} \Big[\prod_{m=1}^{n} \prod_{a \in \Gamma_m} \Big(\frac{t_a}{t_a - x_m} \prod_{1 \leqslant l < m} \frac{t_a - y_l}{t_a - x_l} \Big) \Big]_\sigma ,
$$

$$
W_{\mathfrak{l}}(t; \alpha; x; y; \eta) =
$$

$$
= \prod_{m=1}^{n} \prod_{s=1}^{\mathfrak{l}_m} \frac{\theta(\eta)}{\theta(\eta^s)} \sum_{\sigma \in \mathsf{S}_\ell} \Big[\!\Big[\prod_{m=1}^{n} \prod_{a \in \Gamma_m} \Big(\frac{\theta(\eta^{2a-2} \alpha_m^{-1} t_a / x_m)}{\theta(t_a / x_m)} \prod_{1 \leqslant l < m} \frac{\theta(t_a / y_l)}{\theta(t_a / x_l)} \Big) \Big]\!\Big]_\sigma
$$

where $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \ldots, \mathfrak{l}^m\}$ and $\alpha_m = \alpha \prod_{1 \leqslant l < m} x_l / y_l$, $m = 1, \ldots, n$. Set

(3.2)

$$
w'_{\mathfrak{l}}(t; x; y; \eta) = \prod_{m=1}^{n} y_m^{\mathfrak{l}_m} \prod_{a=1}^{\ell} t_a^{-1} w_{\mathfrak{l}}(t; y; x; \eta^{-1}) ,
$$

$$
W'_{\mathfrak{l}}(t; \alpha; x; y; \eta) = W_{\mathfrak{l}}(t; \alpha^{-1}; y; x; \eta^{-1}) .
$$

The functions $w_{\mathfrak{l}}$, $w'_{\mathfrak{l}}$ and $W_{\mathfrak{l}}$, $W'_{\mathfrak{l}}$ are called the *trigonometric* and *elliptic weight functions*, respectively.

REMARK. In this paper we use a slightly different normalization of the trigonometric weight functions compared with [11].

Proposition 3.1([11]). *Let the parameters η, $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ be generic. Then the functions $\{w_{\mathfrak{l}}(t;x;y;\eta)\}_{\mathfrak{l}\in\mathcal{Z}_\ell^n}$ form a basis in the trigonometric hypergeometric space $\mathcal{F}[x;\eta]$.*

The statement follows from Proposition B.1.

Corollary 3.2. *Under the above assumptions the functions $\{w'_{\mathfrak{l}}(t;x;y;\eta)\}_{\mathfrak{l}\in\mathcal{Z}_\ell^n}$ form a basis in the trigonometric hypergeometric space $\mathcal{F}'[y;\eta]$.*

Proposition 3.3([11]). *Let the parameters η, $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ be generic. Assume that*

$$(3.3)\qquad \alpha\,\eta^{-r}\prod_{1\leqslant l\leqslant m}x_l/y_l \notin p^{\mathbb{Z}}, \qquad m=1,\ldots,n-1, \quad r=0,\ldots,2\ell-2\,.$$

Then the functions $\{W_{\mathfrak{l}}(t;\alpha;x;y;\eta)\}_{\mathfrak{l}\in\mathcal{Z}_\ell^n}$ form a basis in the elliptic hypergeometric space $\mathcal{F}_{ell}[\alpha;x;\eta]$.

The statement follows from Proposition A.3.

Corollary 3.4. *Under the above assumptions the functions $\{W'_{\mathfrak{l}}(t;\alpha;x;y;\eta)\}_{\mathfrak{l}\in\mathcal{Z}_\ell^n}$ form a basis in the elliptic hypergeometric space $\mathcal{F}'_{ell}[\alpha;x;\eta]$.*

The next lemma shows that the bases $\{w_{\mathfrak{l}}\}$ and $\{w'_{\mathfrak{l}}\}$ of the trigonometric hypergeometric space are biorthogonal with respect to the Shapovalov pairing (2.15), and the same holds for the bases $\{W_{\mathfrak{l}}\}$, $\{W'_{\mathfrak{l}}\}$ of the elliptic hypergeometric spaces.

Lemma 3.5.

$$S\left(w'_{\mathfrak{l}}, w_{\mathfrak{m}}\right) = \delta_{\mathfrak{l}\mathfrak{m}}\prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1}\frac{(1-\eta)\,\eta^s\,y_m}{(1-\eta^{s+1})\,(x_m-\eta^s y_m)}\,,$$

$$S_{ell}\left(W'_{\mathfrak{l}}, W_{\mathfrak{m}}\right) = \delta_{\mathfrak{l}\mathfrak{m}}\prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1}\frac{\eta^s\,\theta(\eta)\,\theta(\eta^s\alpha_{\mathfrak{l},m}^{-1})\,\theta(\eta^{1-s-\mathfrak{l}_m}\alpha_{\mathfrak{l},m}\,x_m/y_m)}{\theta'(1)\,\theta(\eta^{s+1})\,\theta(\eta^{-s}x_m/y_m)}$$

where $\alpha_{\mathfrak{l},m} = \alpha\displaystyle\prod_{1\leqslant j<m}\eta^{-2\mathfrak{l}_j}x_j/y_j$ *and* $\theta'(1) = \dfrac{d}{du}\theta(u)\big|_{u=1} = -(p)_\infty^3$.

Proof. The formulae are respectively equivalent to formulae (C.9) and (C.4) in [11]. $\qquad\square$

The tensor coordinates and the hypergeometric maps

Let $V = \displaystyle\bigoplus_{\mathfrak{m}\in\mathcal{Z}_\ell^n}\mathbb{C}v_{\mathfrak{m}}$ and let $V^* = \displaystyle\bigoplus_{\mathfrak{m}\in\mathcal{Z}_\ell^n}\mathbb{C}v_{\mathfrak{m}}^*$ be the dual space. Denote by $\langle\,,\rangle$ the canonical pairing: $\langle v_{\mathfrak{l}}^*, v_{\mathfrak{m}}\rangle = \delta_{\mathfrak{l}\mathfrak{m}}$.

Introduce the *tensor coordinates* on the hypergeometric spaces, cf. [11], [14]. They are the following linear maps:

$$B[x; y; \eta] : V^* \to \mathcal{F}[x; \eta] , \qquad\qquad B_{ell}[\alpha; x; y; \eta] : V^* \to \mathcal{F}_{ell}[\alpha; x; \eta] ,$$

$$v_{\mathfrak{m}}^* \mapsto w_{\mathfrak{m}}(t; x; y; \eta) , \qquad\qquad v_{\mathfrak{m}}^* \mapsto W_{\mathfrak{m}}(t; \alpha; x; y; \eta) ,$$

$$B'[x; y; \eta] : V^* \to \mathcal{F}'[y; \eta] , \qquad\qquad B'_{ell}[\alpha; x; y; \eta] : V^* \to \mathcal{F}'_{ell}[\alpha; y; \eta] ,$$

$$v_{\mathfrak{m}}^* \mapsto w'_{\mathfrak{m}}(t; x; y; \eta) , \qquad\qquad v_{\mathfrak{m}}^* \mapsto W'_{\mathfrak{m}}(t; \alpha; x; y; \eta) .$$

Under the assumptions of Propositions 3.1 and 3.3 the tensor coordinates are isomorphisms of the respective vector spaces.

REMARK. The tensor coordinates used in this paper differ from the tensor coordinates in [11] by normalization factors.

The tensor coordinates and the Shapovalov pairings (2.16) induce bilinear forms

$$(,)[x; y; \eta] : V \otimes V \to \mathbb{C} , \qquad\qquad ((,))[\alpha; x; y; \eta] : V^* \otimes V^* \to \mathbb{C} ,$$

$$(u, v) = \left\langle B'^{-1} S^{-1} B^{*-1} u, v \right\rangle , \qquad\qquad ((u, v)) = (-1)^\ell \left\langle u, (B_{ell})^* S_{ell} B'_{ell} v \right\rangle .$$

We omit the common arguments in the second line. The explicit formulae for these pairings are:

$$(3.4) \qquad (v_{\mathfrak{l}}, v_{\mathfrak{m}}) = \delta_{\mathfrak{l}\mathfrak{m}} \prod_{m=1}^n \prod_{s=0}^{\mathfrak{l}_m - 1} \frac{(1 - \eta^{s+1})(x_m - \eta^s y_m)}{(1 - \eta)\, \eta^s y_m} ,$$

$$(3.5) \qquad ((v_{\mathfrak{l}}^*, v_{\mathfrak{m}}^*)) = \delta_{\mathfrak{l}\mathfrak{m}} \prod_{m=1}^n \prod_{s=0}^{\mathfrak{l}_m - 1} \frac{\eta^s \theta(\eta)\, \theta(\eta^s \alpha_{\mathfrak{l},m}^{-1})\, \theta(\eta^{1-s-\mathfrak{l}_m} \alpha_{\mathfrak{l},m} x_m / y_m)}{\theta(\eta^{s+1})\, \theta(\eta^{-s} x_m / y_m)\, (p)_\infty^3}$$

where $\alpha_{\mathfrak{l},m} = \alpha \prod_{1 \leqslant j < m} \eta^{-2\mathfrak{l}_j} x_j / y_j$, cf. Lemma 3.5. These formulae imply the next proposition.

Proposition 3.6. *Let the parameters* η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ *be generic. Then the form* $(\,,)$ *is nondegenerate. The form* $((\,,))$ *is nondegenerate provided that*

$$\alpha \eta^{-r} \notin p^{\mathbb{Z}} , \qquad \alpha \eta^{r+2-2\ell} \prod_{m=1}^n x_m / y_m \notin p^{\mathbb{Z}} , \qquad r = 0, \ldots, \ell - 1 ,$$

and

$$\alpha \eta^{-r} \prod_{1 \leqslant l \leqslant m} x_l / y_l \notin p^{\mathbb{Z}} , \qquad m = 1, \ldots, n-1 , \quad r = 0, \ldots, 2\ell - 2 .$$

REMARK. The space V can be identified with a weight subspace in a tensor product of $U_q(\mathfrak{sl}_2)$ Verma modules, the form $(\,,)$ coinciding with the tensor product of the corresponding $U_q(\mathfrak{sl}_2)$ Shapovalov forms. The space V also can be identified with a weight subspace in a tensor product of evaluation Verma modules over the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$. The last space has a certain natural bilinear form which is an elliptic analogue of the tensor product of the Shapovalov forms. The form $(\!(\,,)\!)$ on V^* and the "elliptic Shapovalov form" on V correspond to each other.

Consider the following linear maps:

$$\bar{I}[\alpha;x;y;\eta] : V^* \to V\,, \qquad\qquad \bar{I}'[\alpha;x;y;\eta] : V^* \to V\,,$$

$$\bar{I} = B^* I B_{ell}\,, \qquad\qquad\qquad \bar{I}' = (B')^* I' B'_{ell}\,,$$

where I and I' are given by (2.13) and we omit the common arguments in the second line. We call $\bar{I}$ and $\bar{I}'$ the *hypergeometric maps*.

Theorem 2.9 is equivalent to the following statement.

Theorem 3.7. *Let the parameters η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ be generic. Then the hypergeometric maps $\bar{I}[\alpha;x;y;\eta]$, $\bar{I}'[\alpha;x;y;\eta]$ respect the forms $(\,,)[x;y;\eta]$, $(\!(\,,)\!)[\alpha;x;y;\eta]$. That is, for any $u,v \in V^*$ we have*

$$(\!(u,v)\!)[\alpha] = \bigl(\bar{I}'[\alpha]\,u,\bar{I}[\alpha]\,v\bigr)\,.$$

4. Difference equations and asymptotics

In this section we describe a system of difference equations satisfied by the hypergeometric maps and asymptotics of the hypergeometric maps in a suitable asymptotic zone of the parameters $x_1, \ldots, x_n$, $y_1, \ldots, y_n$.

Difference equations for the hypergeometric maps

Let $L_k[x;y;\eta]$ and $L'_k[x;y;\eta]$, $k = 1, \ldots, n$, be linear operators acting in the trigonometric hypergeometric spaces $\mathcal{F}[x;y;\eta]$ and $\mathcal{F}'[x;y;\eta]$, respectively. The operators are defined by their actions on the bases of the trigonometric weight functions:

$$L_k[\alpha;x;y;\eta]\,w_{\mathfrak{l}}(\cdot\,;x;y;\eta) = \Bigl(\alpha\eta^{1-\ell}\prod_{m=1}^{n} x_m/y_m\Bigr)^{l^k} w_{k_{\mathfrak{l}}}(\cdot\,;{}^k x;{}^k y;\eta)\,,$$

$$L'_k[\alpha;x;y;\eta]\,w'_{\mathfrak{l}}(\cdot\,;x;y;\eta) = \Bigl(\alpha\eta^{1-\ell}\prod_{m=1}^{n} x_m/y_m\Bigr)^{-l^k} w'_{k_{\mathfrak{l}}}(\cdot\,;{}^k x;{}^k y;\eta)\,, \quad\cdot$$

where $\quad {}^k\mathfrak{l} = (\mathfrak{l}_{k+1}, \ldots, \mathfrak{l}_n, \mathfrak{l}_1, \ldots, \mathfrak{l}_k)$, $\quad {}^k x = (x_{k+1}, \ldots, x_n, x_1, \ldots, x_k)$, ${}^k y = (y_{k+1}, \ldots, y_n, y_1, \ldots, y_k)$.

Using the tensor coordinates we introduce operators $K_m, K'_m \in \operatorname{End}(V)$, $m = 1, \ldots, n$, by the formulae:

$$(4.1) \qquad K_m[\alpha; x; y; \eta] = \left((B[x; y; \eta])^{-1} L_m[\alpha; x; y; \eta] \, B[x; y; \eta] \right)^*,$$

$$K'_m[\alpha; x; y; \eta] = \left((B'[x; y; \eta])^{-1} L'_m[\alpha; x; y; \eta] \, B'[x; y; \eta] \right)^*.$$

We also define operators $M_m[\alpha; x; y; \eta] \in \operatorname{End}(V^*)$, $m = 1, \ldots, n$:

$$(4.2) \quad M_m[\alpha; x; y; \eta] v_{\mathfrak{l}}^* = \mu_{\mathfrak{l}, m}[\alpha; x; y; \eta] v_{\mathfrak{l}}^*, \qquad \mu_{\mathfrak{l}, m} = \left(\alpha \eta^{1 - \mathfrak{l}^m} \prod_{1 \leqslant j \leqslant m} x_j / y_j \right)^{-\mathfrak{l}^m}.$$

Let T_m^h, $m = 1, \ldots, n$, be the multiplicative shift operators acting on functions of $x_1, \ldots, x_n$, $y_1, \ldots, y_n$:

$$(T_m^h f)(x_1, \ldots, x_n; y_1, \ldots, y_n)$$
$$= f(hx_1, \ldots, hx_m, x_{m+1}, \ldots, x_n; hy_1, \ldots, hy_m, y_{m+1}, \ldots, y_n).$$

Set $\quad T_m = T_m^p$, $m = 1, \ldots, n$.

Theorem 4.1([11]). *The hypergeometric map* $\bar{I}[\alpha; x; y; \eta]$ *satisfies the following system of difference equations:*

$$T_m \bar{I}[\alpha; x; y; \eta] = K_m[\alpha; x; y; \eta] \, \bar{I}[\alpha; x; y; \eta] \, M_m[\alpha; x; y; \eta], \qquad m = 1, \ldots, n.$$

Corollary 4.2. *The hypergeometric map* $\bar{I}'[\alpha; x; y; \eta]$ *satisfies the following system of difference equations:*

$$T_m \bar{I}'[\alpha; x; y; \eta] = K'_m[\alpha; x; y; \eta] \, \bar{I}'[\alpha; x; y; \eta] \left(M_m[\alpha; x; y; \eta] \right)^{-1}, \, m = 1, \ldots, n.$$

The last claim results from the commutativity of diagram (2.14) and formulae (3.2).

REMARK. The numbers $\mu_{\mathfrak{l}, m}$ are related to the transformation properties of the elliptic weight functions:

$$T_m W_{\mathfrak{l}} = \mu_{\mathfrak{l}, m} \prod_{1 \leqslant j \leqslant m} (x_j / y_j)^{\ell} W_{\mathfrak{l}}, \qquad T_m W'_{\mathfrak{l}} = \mu_{\mathfrak{l}, m}^{-1} \prod_{1 \leqslant j \leqslant m} (x_j / y_j)^{-\ell} W'_{\mathfrak{l}}.$$

REMARK. The system of difference equations $T_m \Psi = K_m \Psi$, $m = 1, \ldots, n$, can be identified with the qKZ equation with values in a weight subspace $(= V)$ of a tensor product of $U_q(\mathfrak{sl}_2)$ Verma modules. Its solutions have the form $\Psi = \bar{I} Y$, where $Y \in \operatorname{End}(V^*)$ solves the system of difference equations $T_m Y = M_m^{-1} Y$. Notice that the operators $M_1, \ldots, M_n$ are invariant with respect to the shift operators $T_1^h, \ldots, T_n^h$ for any nonzero h. The factor Y plays the role of an adjusting map in [11].

The system $T_m \Psi' = K'_m \Psi'$, $m = 1, \ldots, n$, corresponds to the dual qKZ equation, if we identify the spaces V and V^* using the Shapovalov form $(,)$.

Asymptotics of the hypergeometric maps

Let $\mathbb{A}$ be the following asymptotic zone of the parameters $x_1, \ldots, x_n$, $y_1, \ldots, y_n$:

$$\mathbb{A} = \left\{ \begin{array}{ll} |x_m/x_{m+1}| \ll 1, & m = 1, \ldots, n-1 \\ |x_m/y_m| \simeq 1, & m = 1, \ldots, n \end{array} \right\}.$$

We say that $(x; y)$ tends to limit in $\mathbb{A}$ and write $(x; y) \rightrightarrows \mathbb{A}$ if

$$x_m/x_{m+1} \to 0, \qquad m = 1, \ldots, n-1,$$

and the ratios x_m/y_m and y_m/x_m remain bounded for any $m = 1, \ldots, n$. If a function $f(x; y)$ has a finite limit as $(x; y) \rightrightarrows \mathbb{A}$, we denote this limit by $\lim_{\mathbb{A}} f$. Notice that the limit $\lim_{\mathbb{A}} f$ can depend on $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, but it is invariant with respect to the shift operators $T_1^h, \ldots, T_n^h$ for any nonzero h.

REMARK. The operators $K_1, \ldots, K_n$, $K_1', \ldots, K_n'$, cf. (4.1), have finite limits as $(x; y) \rightrightarrows \mathbb{A}$, and the limits are respectively lower and upper triangular with respect to the basis $\{v_{\mathfrak{l}}\}_{\mathfrak{l} \in \mathcal{Z}_\ell^n}$ and ordering (2.4). The diagonal parts of the limits $\lim_{\mathbb{A}} K_m^{-1}$ and $\lim_{\mathbb{A}} K_m'$ are equal to M_m^*.

Define the functions $I_{\mathfrak{l}\mathfrak{m}}(x, y)$ and $I_{\mathfrak{l}\mathfrak{m}}'(x, y)$ by the formulae:

$$I_{\mathfrak{l}\mathfrak{m}}(x, y) = I[\alpha; x; y]\big(W_{\mathfrak{l}}(\cdot; \alpha; x; y), w_{\mathfrak{m}}(\cdot; x; y)\big),$$

$$I_{\mathfrak{l}\mathfrak{m}}'(x, y) = I'[\alpha; x; y]\big(W_{\mathfrak{l}}'(\cdot; \alpha; x; y), w_{\mathfrak{m}}'(\cdot; x; y)\big).$$

Proposition 4.3([11]). *For any* $\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_\ell^n$ *the hypergeometric integral* $I_{\mathfrak{l}\mathfrak{m}}(x; y)$ *has a finite limit as* $(x; y) \rightrightarrows \mathbb{A}$. *Moreover* $\lim_{\mathbb{A}} I_{\mathfrak{l}\mathfrak{m}} = 0$ *unless* $\mathfrak{l} \ll \mathfrak{m}$ *or* $\mathfrak{l} = \mathfrak{m}$, *cf.* (2.4), *and*

$$\lim_{\mathbb{A}} I_{\mathfrak{l}\mathfrak{l}} = \prod_{m=1}^{n} \prod_{s=0}^{\mathfrak{l}_m - 1} \frac{\eta^{-s}\, (\eta^{-1})_\infty\, (\eta^s \alpha_{\mathfrak{l},m}^{-1})_\infty\, (p\eta^{1-s-\mathfrak{l}_m} \alpha_{\mathfrak{l},m} x_m/y_m)_\infty}{(\eta^{-s-1})_\infty\, (\eta^{-s} x_m/y_m)_\infty\, (p)_\infty} .$$

Recall that $\alpha_{\mathfrak{l},m} = \alpha \prod_{1 \leqslant j < m} \eta^{-2\mathfrak{l}_j} x_j/y_j$.

Proposition 4.4. *For any* $\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_\ell^n$ *the hypergeometric integral* $I_{\mathfrak{l}\mathfrak{m}}'(x; y)$ *has a finite limit as* $(x; y) \rightrightarrows \mathbb{A}$. *Moreover* $\lim_{\mathbb{A}} I_{\mathfrak{l}\mathfrak{m}}' = 0$ *unless* $\mathfrak{l} \gg \mathfrak{m}$ *or* $\mathfrak{l} = \mathfrak{m}$, *cf.* (2.4), *and*

$$\lim_{\mathbb{A}} I_{\mathfrak{l}\mathfrak{l}}' = \prod_{m=1}^{n} \prod_{s=0}^{\mathfrak{l}_m - 1} \frac{-\eta^{-s} \alpha_{\mathfrak{l},m}\, (\eta)_\infty\, (p\eta^{-s} \alpha_{\mathfrak{l},m})_\infty\, (\eta^{s-1+\mathfrak{l}_m} \alpha_{\mathfrak{l},m}^{-1} y_m/x_m)_\infty}{(\eta^{s+1})_\infty\, (\eta^s y_m/x_m)_\infty\, (p)_\infty} .$$

The proof is similar to the proof of Proposition 4.3.

Corollary 4.5. *For any* α, η *the hypergeometric maps* $\bar{I}[\alpha; x; y; \eta]$ *and* $\bar{I}'[\alpha; x; y; \eta]$ *have finite limits as* $(x; y) \rightrightarrows \mathbb{A}$*. Moreover,*

$$\lim_{\mathbb{A}} \bar{I} v_{\mathfrak{l}}^* = v_{\mathfrak{l}} \lim_{\mathbb{A}} I_{\mathfrak{l}\mathfrak{l}} + \sum_{\mathfrak{m} \gg \mathfrak{l}} v_{\mathfrak{m}} \lim_{\mathbb{A}} I_{\mathfrak{l}\mathfrak{m}}$$

and

$$\lim_{\mathbb{A}} \bar{I}' v_{\mathfrak{l}}^* = v_{\mathfrak{l}} \lim_{\mathbb{A}} I'_{\mathfrak{l}\mathfrak{l}} + \sum_{\mathfrak{m} \ll \mathfrak{l}} v_{\mathfrak{m}} \lim_{\mathbb{A}} I'_{\mathfrak{l}\mathfrak{m}} \,.$$

5. Proof of the hypergeometric Riemann identity

In this section we prove the bilinear identity for the hypergeometric integrals. Its equivalent forms are given by Theorems 2.9 and 3.7. We will prove the latter theorem.

Proof of Theorem 3.7. Let $G_{\mathfrak{l}\mathfrak{m}}(\alpha; x; y; \eta) = \big(\bar{I}'[\alpha; x; y; \eta] v_{\mathfrak{l}}^*, \bar{I}[\alpha; x; y; \eta] v_{\mathfrak{m}}^*\big)[x; y; \eta]\,.$

We have to prove that

(5.1) $$G_{\mathfrak{l}\mathfrak{m}}(\alpha; x; y; \eta) = \big(\!\!\big(v_{\mathfrak{l}}^*, v_{\mathfrak{m}}^*\big)\!\!\big)[\alpha; x; y; \eta]\,, \qquad \mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_{\ell}^n\,.$$

Since both sides of the above equality are analytic functions of α, we can assume that α is generic. In particular, we will use the next statement.

Lemma 5.1. *Let* α *be generic. Let* $\mu_{\mathfrak{l}, k}$ *be defined by* (4.2)*. If* $\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_{\ell}^n$ *are such that* $\mu_{\mathfrak{l}, k} = \mu_{\mathfrak{m}, k}$ *for any* $k = 1, \ldots, n$*, then* $\mathfrak{l} = \mathfrak{m}$*.*

From the definitions of operators L_m, L'_m and the Shapovalov pairing S it is easy to see that

$$S[x; y; \eta] = \big(L_m[\alpha; x; y; \eta]\big)^* S[x; y; \eta] L'_m[\alpha; x; y; \eta]\,, \qquad m = 1, \ldots, n\,.$$

Therefore, for any $u, v \in V$ we have

(5.2) $$\big(K'_m[\alpha; x; y; \eta] u, K_m[\alpha; x; y; \eta] v\big)[x; y; \eta] = (u, v)[x; y; \eta]\,, \qquad m = 1, \ldots, n\,.$$

Hence, the function $G_{\mathfrak{l}\mathfrak{m}}(\alpha; x; y; \eta)$ satisfies a system of difference equations

(5.3) $$T_k G_{\mathfrak{l}\mathfrak{m}} = \mu_{\mathfrak{l}, k}^{-1} \mu_{\mathfrak{m}, k} G_{\mathfrak{l}\mathfrak{m}}\,, \qquad k = 1, \ldots, n\,,$$

see Theorem 4.1, Corollary 4.2 and formulae (4.2), (5.2). On the other hand, Corollary 4.5 shows that the function $G_{\mathfrak{l}\mathfrak{m}}(x; y)$ has a finite limit as $(x; y) \rightrightarrows \mathbb{A}$, and equations (5.3) imply that

$$\lim_{\mathbb{A}} G_{\mathfrak{l}\mathfrak{m}} = \mu_{\mathfrak{l}, k}^{-1} \mu_{\mathfrak{m}, k} \lim_{\mathbb{A}} G_{\mathfrak{l}\mathfrak{m}}\,, \qquad k = 1, \ldots, n\,.$$

Therefore, $\lim_{\mathbb{A}} G_{\mathfrak{l}\mathfrak{m}} = 0$ for $\mathfrak{l} \neq \mathfrak{m}$ by Lemma 5.1.

Using once again equations (5.3) we obtain that $G_{\mathfrak{l}\mathfrak{m}}(x; y) = 0$ for $\mathfrak{l} \neq \mathfrak{m}$, and $G_{\mathfrak{l}\mathfrak{l}}(x; y) = \lim_{\mathbb{A}} G_{\mathfrak{l}\mathfrak{l}}$. In particular, the functions $G_{\mathfrak{l}\mathfrak{m}}(x; y)$ are invariant with respect to the shift operators $T_1^h, \ldots, T_n^h$ for any nonzero h.

Obviously, the right hand side of (5.1) enjoys the same properties: $((v_{\mathfrak{l}}^*, v_{\mathfrak{m}}^*)) = 0$ for $\mathfrak{l} \neq \mathfrak{m}$, and $((v_{\mathfrak{l}}^*, v_{\mathfrak{l}}^*))$ is invariant with respect to $T_1^h, \ldots, T_n^h$. Hence, it remains to show that

$$\lim_{\mathbb{A}} G_{\mathfrak{l}\mathfrak{l}} = ((v_{\mathfrak{l}}^*, v_{\mathfrak{l}}^*)), \qquad \mathfrak{l} \in \mathcal{Z}_{\ell}^n,$$

which is a straightforward calculation using formulae (3.4), (3.5), Propositions 4.3, 4.4 and Corollary 4.5. Theorem 3.7 is proved. $\qquad\square$

6. The restricted hypergeometric Riemann identity

Let us write down the hypergeometric Riemann identity using the bases of the weight functions in the hypergeometric spaces:

$$(6.1) \qquad \sum_{\mathfrak{m} \in \mathcal{Z}_{\ell}^n} M_{\mathfrak{m}} \, I'(W_{\mathfrak{l}}', w_{\mathfrak{m}}') \, I(W_{\mathfrak{n}}, w_{\mathfrak{m}}) = \delta_{\mathfrak{l}\mathfrak{n}} N_{\mathfrak{l}}, \qquad \mathfrak{l}, \mathfrak{n} \in \mathcal{Z}_{\ell}^n,$$

$$M_{\mathfrak{m}} = \prod_{i=1}^{n} \prod_{s=0}^{\mathfrak{m}_i - 1} \frac{(1 - \eta^{s+1})(x_i - \eta^s y_i)}{(1 - \eta) \eta^s y_i},$$

$$N_{\mathfrak{l}} = \prod_{m=1}^{n} \prod_{s=0}^{\mathfrak{l}_m - 1} \frac{\eta^s \theta(\eta) \theta(\eta^s \alpha_{\mathfrak{l},m}^{-1}) \theta(\eta^{1-s-\mathfrak{l}_m} \alpha_{\mathfrak{l},m} x_m / y_m)}{\theta'(1) \theta(\eta^{s+1}) \theta(\eta^{-s} x_m / y_m)}$$

where $\alpha_{\mathfrak{l},m} = \alpha \prod_{1 \leqslant j < m} \eta^{-2\mathfrak{l}_j} x_j / y_j$, cf. Theorem 2.9 and Lemma 3.5. Formula (6.1) holds for generic values of the parameters η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, cf. (2.1), and all the coefficients $N_{\mathfrak{l}}$, $\mathfrak{l} \in \mathcal{Z}_{\ell}^n$, are clearly regular in this case. Suppose now that $x_1 = \eta^r y_1$ for some nonnegative integer r less than ℓ, thus violating (2.1), and all other assumptions (2.1) hold. Then the coefficients $N_{\mathfrak{l}}$ with $\mathfrak{l}_1 \leqslant r$ remain regular while the coefficients $N_{\mathfrak{l}}$ with $\mathfrak{l}_1 > r$ have a pole at $x_1 = \eta^r y_1$. This suggests that if the parameters η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ are slightly nongeneric, then some of the hypergeometric integrals can survive and still satisfy a certain version of the hypergeometric Riemann identity. We study such a possibility in this section.

Fix integers $\ell_1, \ldots, \ell_n$ such that $1 \leqslant \ell_m \leqslant \ell$, $m = 1, \ldots, n$. Set

$$\bar{\mathcal{Z}}_{\ell}^n = \{\mathfrak{l} \in \mathcal{Z}_{\ell}^n \mid \mathfrak{l}_m \leqslant \ell_m, \quad m = 1, \ldots, n\}.$$

Assume that for any $r = 0, \ldots, \ell - 1$ and any $k, m = 1, \ldots, n$, we have

(6.2) $$\eta^{r+1} \notin p^{\mathbb{Z}},$$

$$\eta^{\pm r} x_k / x_m \notin p^{\mathbb{Z}}, \qquad \eta^{\pm r} y_k / y_m \notin p^{\mathbb{Z}}, \qquad \eta^{-r} x_k / y_m \notin p^{\mathbb{Z}}, \qquad k \neq m,$$

$$\eta^{-s} x_m / y_m \notin p^{\mathbb{Z}}, \qquad s = 0, \ldots, \ell_m - 1.$$

Comparing with (2.1), here we impose weaker conditions for the ratios x_k / y_m.

Proposition 6.1. *Assume that nonzero parameters η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ satisfy conditions (6.2). Then for any $\mathfrak{l} \in \bar{\mathcal{Z}}_\ell^n$, $\mathfrak{m} \in \mathcal{Z}_\ell^n$ the hypergeometric integrals $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ and $I'(W_{\mathfrak{l}}', w_{\mathfrak{m}}')$ are well defined and are holomorphic functions of the parameters.*

Proof. Assume that $|\eta| > 1$ and $|x_m| < 1$, $|y_m| > 1$, $m = 1, \ldots, n$. Then

$$I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) = \frac{1}{(2\pi i)^\ell \ell!} \int\limits_{\mathbb{T}^\ell} W_{\mathfrak{l}}(t) \, w_{\mathfrak{m}}(t) \, \Phi(t) \, (dt/t)^\ell,$$

cf. (2.8) and (2.12). Observe that the integrand $W_{\mathfrak{l}}(t) \, w_{\mathfrak{m}}(t) \, \Phi(t)$ is a symmetric function of $t_1, \ldots, t_\ell$. Since the integration contour $\mathbb{T}^\ell$ is invariant with respect to permutations of the variables $t_1, \ldots, t_\ell$, we can drop the summation in the definition of the function $W_{\mathfrak{l}}$, cf. (3.1), multiplying the result of the integration by $\ell!$:

$$I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) = \frac{1}{(2\pi i)^\ell} \int\limits_{\mathbb{T}^\ell} P_{\mathfrak{l}}(t) \, w_{\mathfrak{m}}(t) \, \Phi(t) \, (dt/t)^\ell,$$

$$P_{\mathfrak{l}}(t) = \prod_{m=1}^{n} \prod_{s=1}^{\mathfrak{l}_m} \frac{\theta(\eta)}{\theta(\eta^s)} \prod_{m=1}^{n} \prod_{a \in \Gamma_m} \left(\frac{\theta(\eta^{2a-2} \alpha_m^{-1} t_a / x_m)}{\theta(t_a / x_m)} \prod_{1 \leqslant l < m} \frac{\theta(t_a / y_l)}{\theta(t_a / x_l)} \right)$$

where $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \ldots, \mathfrak{l}^m\}$ and $\alpha_m = \alpha \prod_{1 \leqslant l < m} x_l / y_l$, $m = 1, \ldots, n$.

Under assumptions (6.2) singularities of the integrand $P_{\mathfrak{l}}(t) \, w_{\mathfrak{m}}(t) \, \Phi(t)$ are at most at the following hyperplanes:

(6.3) $$t_a = 0, \qquad t_a = p^{-s} \eta t_b, \qquad 1 \leqslant a < b \leqslant \ell,$$

$$t_a = p^s x_j, \qquad t_a = p^s y_k, \qquad a \in \{1 + \mathfrak{l}^{m-1}, \ldots, \mathfrak{l}^m\}, \qquad 1 \leqslant j \leqslant m \leqslant k \leqslant \ell,$$

$s \in \mathbb{Z}_{\geqslant 0}$. The number of edges of configuration (6.3) and dimensions of the edges are always the same provided that the parameters are nonzero and assumptions

(6.2) hold. Therefore, the topology of the complement in $\mathbb{C}^\ell$ of the union of the hyperplanes (6.3) does not change under the assumptions of Proposition 6.1. The rest of the proof is similar to the proof of Theorem 5.7 in [12].

The proof for the hypergeometric integral $I'(W'_\mathfrak{l}, w'_\mathfrak{m})$ is similar to the proof for the hypergeometric integral $I(W_\mathfrak{l}, w_\mathfrak{m})$ given above. Proposition 6.1 is proved.

$\square$

REMARK. More detailed results for a similar problem concerning the multidimensional hypergeometric integrals of Mellin-Barnes type are obtained in [8].

Theorem 6.2. *Assume that nonzero parameters* η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ *satisfy conditions (6.2) and* $x_m = \eta^{\ell_m} y_m$ *if* $\ell_m < \ell$. *Then*

$$(6.4) \qquad \sum_{\mathfrak{m} \in \bar{\mathcal{Z}}_\ell^n} M_\mathfrak{m} \, I'(W'_\mathfrak{l}, w'_\mathfrak{m}) \, I(W_\mathfrak{n}, w_\mathfrak{m}) = \delta_{\mathfrak{l}\mathfrak{n}} N_\mathfrak{l}, \qquad \mathfrak{l}, \mathfrak{n} \in \bar{\mathcal{Z}}_\ell^n,$$

where the coefficients $M_\mathfrak{m}, N_\mathfrak{l}$ *are defined by (6.1). Moreover, if* $N_\mathfrak{m} \neq 0$ *for all* $\mathfrak{m} \in \bar{\mathcal{Z}}_\ell^n$, *then*

$$(6.5) \qquad \sum_{\mathfrak{m} \in \bar{\mathcal{Z}}_\ell^n} N_\mathfrak{m}^{-1} \, I'(W'_\mathfrak{m}, w'_\mathfrak{l}) \, I(W_\mathfrak{m}, w_\mathfrak{n}) = \delta_{\mathfrak{l}\mathfrak{n}} M_\mathfrak{l}^{-1}, \qquad \mathfrak{l}, \mathfrak{n} \in \bar{\mathcal{Z}}_\ell^n.$$

Proof. Formula (6.4) follows from formula (6.1), because all the terms in (6.1) are well defined under the assumptions of the theorem, see Proposition 6.1, and $M_\mathfrak{m} = 0$ unless $\mathfrak{m} \in \bar{\mathcal{Z}}_\ell^n$.

Writing down relation (6.4) in the matrix form: $I' M I^t = N$, for matrices I, I', M, N with entries

$$I_{\mathfrak{l}\mathfrak{m}} = I(W_\mathfrak{l}, w_\mathfrak{m}), \qquad I'_{\mathfrak{l}\mathfrak{m}} = I'(W'_\mathfrak{l}, w'_\mathfrak{m}), \qquad M_{\mathfrak{l}\mathfrak{m}} = \delta_{\mathfrak{l}\mathfrak{m}} M_\mathfrak{l}, \qquad N_{\mathfrak{l}\mathfrak{m}} = \delta_{\mathfrak{l}\mathfrak{m}} N_\mathfrak{l},$$

we immediately get that $I^t N^{-1} I' = M^{-1}$, which is the same as formula (6.5). $\square$

We call relations (6.4) and (6.5) the *restricted hypergeometric Riemann identities.*

It is possible to introduce restricted versions of the hypergeometric spaces, the Shapovalov pairings and the hypergeometric pairings, and reformulate Theorem 6.2 similarly to Theorem 2.9. This will be done elsewhere.

7. Bilinear identities for basic hypergeometric series

In this section we consider the hypergeometric Riemann identity in the one-dimensional case. That is, all over this section we assume that $\ell = 1$.

Let $_n\varphi_{n-1}(a_1,\ldots,a_n;b_1,\ldots,b_{n-1};z)$ be the basic hypergeometric series [5]:

$$_n\varphi_{n-1}(a_1,\ldots,a_n;b_1,\ldots,b_{n-1};z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_n)_k}{(b_1)_k \ldots (b_{n-1})_k (p)_k} z^k \,,$$

$$(u)_k = \prod_{s=0}^{k-1}(1-p^s u)\,.$$

For any $k = 1,\ldots,n$ we define functions f_k, f'_k, F_k, F'_k by the formulae:

$$f_k(t) = \frac{t}{y_k} \prod_{m=1}^{n} \frac{y_k - x_m}{t - x_m} \prod_{\substack{m=1 \\ m\neq k}}^{n} \frac{t - y_m}{y_k - y_m}\,, \qquad f'_k(t) = \frac{y_k}{t - y_k}\,,$$

$$F_k(t) = (p)_\infty^2 \frac{\theta(\widetilde{\alpha}^{-1}t/y_k)}{\theta(\widetilde{\alpha}^{-1})} \prod_{m=1}^{n} \frac{(px_m/y_k)_\infty}{\theta(t/x_m)} \prod_{\substack{m=1 \\ m\neq k}}^{n} \frac{\theta(t/y_m)}{(py_m/y_k)_\infty}\,, \quad \widetilde{\alpha} = \alpha \prod_{m=1}^{n} x_m/y_m\,,$$

$$F'_k(t) = (p)_\infty^2 \frac{\theta(\alpha t/y_k)}{\theta(\alpha)\,\theta(t/y_k)} \prod_{m=1}^{n} (y_k/x_m)_\infty \prod_{\substack{m=1 \\ m\neq k}}^{n} (y_k/y_m)_\infty^{-1}\,.$$

The functions $\{f_m\}_{m=1}^n$, $\{f'_m\}_{m=1}^n$, $\{F_m\}_{m=1}^n$, $\{F'_m\}_{m=1}^n$ form bases in the respective hypergeometric spaces $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}_{ell}[\alpha]$, $\mathcal{F}'_{ell}[\alpha]$, and these bases are biorthonormal with respect to the Shapovalov pairings:

$$S(f'_l,f_m) = \delta_{lm}\,, \qquad S_{ell}(F'_l,F_m) = -\delta_{lm}\,.$$

The hypergeometric integrals $I(F_l,f_m)$ and $I'(F'_l,f'_m)$ can be expressed via the basic hypergeometric series $_n\varphi_{n-1}$. For instance,

$$I(F_1,f_1) = {_n\varphi_{n-1}}(x_1 y_1^{-1},\ldots,x_n y_1^{-1};y_2 y_1^{-1},\ldots,y_n y_1^{-1};\widetilde{\alpha}^{-1})\,,$$

$$I(F_1,f_k) = -\widetilde{\alpha}^{-1} y_1 y_k^{-1} (y_1 - py_k)^{-1} \prod_{\substack{m=1 \\ m\neq k}}^{n} (y_m - y_k)^{-1} \prod_{m=1}^{n} (x_m - y_k)$$

$$\times\ {_n\varphi_{n-1}}(px_1 y_1^{-1},\ldots,px_n y_1^{-1};py_2 y_1^{-1},\ldots,p^2 y_k y_1^{-1},\ldots,py_{n-1} y_1^{-1};\widetilde{\alpha}^{-1})\,,$$

$$I'(F'_1,f'_1) = {_n\varphi_{n-1}}(y_1 x_1^{-1},\ldots,y_1 x_n^{-1};y_1 y_2^{-1},\ldots,y_1 y_n^{-1};\alpha^{-1})\,,$$

$$I'(F'_1,f'_k) = \widetilde{\alpha}^{-1} y_k y_1^{-1} (y_k - py_1)^{-1} \prod_{m=2}^{n} (y_m - y_1)^{-1} \prod_{m=1}^{n} (x_m - y_1)$$

$$\times\ {_n\varphi_{n-1}}(py_1 x_1^{-1},\ldots,py_1 x_n^{-1};py_1 y_2^{-1},\ldots,p^2 y_1 y_k^{-1},\ldots,py_1 y_{n-1}^{-1};\alpha^{-1})\,,$$

$k = 2, \ldots, n$. General formulae for $I(F_l, f_m)$, $I'(F_l', f_m')$ can be obtained by a suitable change of indices.

In the one-dimensional example in question, Theorem 2.9 is equivalent to each of the next formulae:

$$(7.1) \qquad \sum_{m=1}^{n} I(F_k, f_m) \, I'(F_l', f_m') = \delta_{kl}, \qquad \sum_{m=1}^{n} I(F_m, f_k) \, I'(F_m', f_l') = \delta_{kl}.$$

These formulae deliver bilinear identities for the basic hypergeometric series. They read as follows:

$${}_n\varphi_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; z) \, {}_n\varphi_{n-1}(a_1^{-1}, \ldots, a_n^{-1}; b_1^{-1}, \ldots, b_{n-1}^{-1}; \tilde{z})$$

$$= 1 + \sum_{m=1}^{n-1} \frac{z^2 A_0 A_m}{(1 - p b_m)(p - b_m)} \, {}_n\varphi_{n-1}(p a_1, \ldots, p a_n; p b_1, \ldots, p^2 b_m, \ldots, p b_{n-1}; z)$$

$$\times \, {}_n\varphi_{n-1}(p a_1^{-1}, \ldots, p a_n^{-1}; p b_1^{-1}, \ldots, p^2 b_m^{-1}, \ldots, p b_{n-1}^{-1}; \tilde{z}),$$

$${}_n\varphi_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; z)$$

$$\times \, {}_n\varphi_{n-1}(p b_1 a_1^{-1}, \ldots, p b_1 a_n^{-1}; p^2 b_1, p b_1 b_2^{-1}, \ldots, p b_1 b_{n-1}^{-1}; \tilde{z})$$

$$= \, {}_n\varphi_{n-1}(p a_1, \ldots, p a_n; p^2 b_1, p b_2, \ldots, p b_{n-1}; z)$$

$$\times \, {}_n\varphi_{n-1}(b_1 a_1^{-1}, \ldots, b_1 a_n^{-1}; b_1, b_1 b_2^{-1}, \ldots, b_1 b_{n-1}^{-1}; \tilde{z})$$

$$+ \sum_{m=2}^{n-1} \frac{z A_m (1 - p b_1)}{(1 - p b_m)(b_m - p b_1)} \, {}_n\varphi_{n-1}(p a_1, \ldots, p a_n; p b_1, \ldots, p^2 b_m, \ldots, p b_{n-1}; z)$$

$$\times \, {}_n\varphi_{n-1}(p b_1 a_1^{-1}, \ldots, p b_1 a_n^{-1}; p b_1, p b_1 b_2^{-1}, \ldots, p^2 b_1 b_m^{-1}, \ldots, p b_1 b_{n-1}^{-1}; \tilde{z}),$$

$${}_n\varphi_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; z) \, {}_n\varphi_{n-1}(a_1^{-1}, \ldots, a_n^{-1}; b_1^{-1}, \ldots, b_{n-1}^{-1}; \tilde{z})$$

$$= 1 + \sum_{m=1}^{n-1} \frac{z^2 A_0 A_m}{(1 - p b_m)(p - b_m)}$$

$$\times \, {}_n\varphi_{n-1}(p b_m^{-1} a_1, \ldots, p b_m^{-1} a_n; p b_m^{-1} b_1, \ldots, p^2 b_m^{-1}, \ldots, p b_m^{-1} b_{n-1}; z)$$

$$\times \, {}_n\varphi_{n-1}(p b_m a_1^{-1}, \ldots, p b_m a_n^{-1}; p b_m b_1^{-1}, \ldots, p^2 b_m, \ldots, p b_m b_{n-1}^{-1}; \tilde{z}),$$

$${}_n\varphi_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; z) \, {}_n\varphi_{n-1}(p a_1^{-1}, \ldots, p a_n^{-1}; p^2 b_1^{-1}, p b_2^{-1}, \ldots, p b_{n-1}^{-1}; \tilde{z})$$

$$= \, {}_n\varphi_{n-1}(p b_1^{-1} a_1, \ldots, p b_1^{-1} a_n; p^2 b_1^{-1}, p b_1^{-1} b_2, \ldots, p b_1^{-1} b_{n-1}; z)$$

$$\times \, {}_n\varphi_{n-1}(b_1 a_1^{-1}, \ldots, b_1 a_n^{-1}; b_1, b_1 b_2^{-1}, \ldots, b_1 b_{n-1}^{-1}; \tilde{z})$$

$$+ \sum_{m=2}^{n-1} \frac{z\, A_m\, (p - b_1)}{(p - b_m)\,(b_1 - p b_m)}$$

$$\times\ {}_n\varphi_{n-1}(p b_m^{-1} a_1, \ldots, p b_m^{-1} a_n; p b_m^{-1} b_1, \ldots, p^2 b_m^{-1}, \ldots, p b_m^{-1} b_{n-1}; z)$$

$$\times\ {}_n\varphi_{n-1}(p b_m a_1^{-1}, \ldots, p b_m a_n^{-1}; p b_m, p^2 b_m b_1^{-1}, p b_m b_2^{-1}, \ldots, p b_m b_{n-1}^{-1}; \tilde{z})$$

where

$$\tilde{z} = z\, \frac{\prod a_m}{\prod^\bullet b_m}\,, \qquad A_0 = \frac{\prod (1 - a_m)}{\prod^\bullet (1 - b_m)}\,, \qquad A_k = \frac{\prod (a_m - b_k)}{(1 - b_k)\, \prod^\bullet_{m \neq k} (b_m - b_k)}\,,$$

$k = 1, \ldots, n - 1$. Here and below $\prod$ stands for $\prod_{m=1}^{n}$ and $\prod^\bullet$ stands for $\prod_{m=1}^{n-1}$.

Proposition 7.1. $\det\big[I(F_l, f_m) \big]_{l,m=1}^{n} = \dfrac{(\alpha^{-1})_\infty}{(\alpha^{-1} \prod y_m / x_m)_\infty}$.

Proof. Let $\mathfrak{e}(m) = (0, \ldots, \underset{m\text{-th}}{1}, \ldots, 0)$, $m = 1, \ldots, n$. Let $w_m(t) = w_{\mathfrak{e}(m)}(t; x; y; \eta)$ and $W_m(t) = W_{\mathfrak{e}(m)}(t; \alpha; x; y; \eta)$ be the weight functions. We have that

$$w_m(y_l) = 0\,, \qquad W_m(y_l) = 0\,, \qquad 1 \leqslant l < m \leqslant n\,,$$

and

$$f_m(y_l) = 0\,, \qquad F_m(y_l) = 0\,, \qquad 1 \leqslant l \neq m \leqslant n\,.$$

Therefore,

$$\det\big[I(F_l, f_m) \big]_{l,m=1}^{n} = \prod_{m=1}^{n} \frac{f_m(y_m)\, F_m(y_m)}{w_m(y_m)\, W_m(y_m)}\ \det\big[I(W_l, w_m) \big]_{l,m=1}^{n}$$

and the claim follows from Proposition B.2 for $\ell = 1$. $\qquad\qquad\qquad\square$

Calculating a matrix inverse to the matrix $\big[I'(F_l', f_m') \big]_{l,m=1}^{n}$ in two different ways using either formulae (7.1) or Proposition 7.1, we obtain that

$$I(F_l, f_m) = \frac{(-1)^{l+m}\, (\alpha^{-1})_\infty}{(\alpha^{-1} \prod y_m / x_m)_\infty}\ \det\big[I'(F_j', f_k') \big]_{\substack{j,k=1 \\ j \neq l \\ k \neq m}}^{n}\,, \qquad l, m = 1, \ldots, n\,.$$

For $n = 2$ these relations are equivalent to

$$\ _2\varphi_1(a_1, a_2; b; z)\,(z)_\infty = \ _2\varphi_1(b a_1^{-1}, b a_2^{-1}; b; z a_1 a_2 b^{-1})\,(z a_1 a_2 b^{-1})_\infty$$

and for $n = 3$ they give

$$_3\varphi_2(a_1, a_2, a_3; b_1, b_2; z) \frac{(z)_\infty}{(\tilde{z})_\infty}$$

$$= {}_3\varphi_2(b_1 a_1^{-1}, b_1 a_2^{-1}, b_1 a_3^{-1}; b_1, b_1 b_2^{-1}; \tilde{z}) \, {}_3\varphi_2(b_2 a_1^{-1}, b_2 a_2^{-1}, b_2 a_3^{-1}; b_2, b_2 b_1^{-1}; \tilde{z})$$

$$- z^2 \frac{(a_1 - b_1)(a_2 - b_1)(a_3 - b_1)(a_1 - b_2)(a_2 - b_2)(a_3 - b_2)}{(1 - b_1)(1 - b_2)(b_1 - b_2)^2 (b_1 - pb_2)(pb_1 - b_2)}$$

$$\times \, {}_3\varphi_2(pb_1 a_1^{-1}, pb_1 a_2^{-1}, pb_1 a_3^{-1}; pb_1, p^2 b_1 b_2^{-1}; \tilde{z})$$

$$\times \, {}_3\varphi_2(pb_2 a_1^{-1}, pb_2 a_2^{-1}, pb_2 a_3^{-1}; pb_2, p^2 b_2 b_1^{-1}; \tilde{z}) \,,$$

$$_3\varphi_2(pa_1, pa_2, pa_3; p^2 b_1, pb_2; z) \frac{(z)_\infty}{(\tilde{z})_\infty}$$

$$= {}_3\varphi_2(pb_1 a_1^{-1}, pb_1 a_2^{-1}, pb_1 a_3^{-1}; p^2 b_1, pb_1 b_2^{-1}; \tilde{z}) \, {}_3\varphi_2(b_2 a_1^{-1}, b_2 a_2^{-1}, b_2 a_3^{-1}; b_2, b_2 b_1^{-1}; \tilde{z})$$

$$+ z \frac{(1 - pb_1)(a_1 - b_2)(a_2 - b_2)(a_3 - b_2)}{(1 - b_2)(1 - pb_2)(b_1 - b_2)(pb_1 - b_2)}$$

$$\times \, {}_3\varphi_2(pb_1 a_1^{-1}, pb_1 a_2^{-1}, pb_1 a_3^{-1}; pb_1, p^2 b_1 b_2^{-1}; \tilde{z})$$

$$\times \, {}_3\varphi_2(pb_2 a_1^{-1}, pb_2 a_2^{-1}, pb_2 a_3^{-1}; p^2 b_2, pb_2 b_1^{-1}; \tilde{z})$$

where $\tilde{z} = z a_1 a_2 a_3 b_1^{-1} b_2^{-1}$. Notice that $(z)_\infty/(\tilde{z})_\infty = {}_1\varphi_0(z/\tilde{z}; \tilde{z}) = {}_1\varphi_0(\tilde{z}/z; z)^{-1}$.

The rational version of the hypergeometric Riemann identity in the one-dimensional case gives similar formulae for the generalized hypergeometric function $_n F_{n-1}$. They can be obtained from the formulae for the basic hypergeometric series after the standard substitution $a_m = p^{\alpha_m}$, $b_m = p^{\beta_m}$ in the limit $p \to 1$ which degenerates $_n\varphi_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; z)$ to $_n F_{n-1}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_{n-1}; z)$.

Appendix A. Nondegeneracy of the elliptic Shapovalov pairing

Let $A = \alpha \eta^{1-\ell} \prod_{m=1}^{n} x_m$. Let $\mathcal{E}[A]$ be the space of holomorphic functions on $\mathbb{C}^\times$ such that $f(pu) = A(-u)^{-n} f(u)$. It is easy to see that $\dim \mathcal{E}[A] = n$, say by Fourier series.

Let $\omega = \exp(2\pi i/n)$. Fix complex numbers ξ and ζ such that $\xi^n = p$ and $\zeta^n = -A^{-1}$. Set

$$\vartheta_l(u) = u^{l-1} \prod_{m=1}^{n} \theta(-\zeta\xi^{l-1}\omega^m u), \qquad l = 1, \dots, n.$$

Lemma A.1. *The functions* $\vartheta_1, \dots, \vartheta_n$ *form a basis in the space* $\mathcal{E}[A]$.

Proof. Clearly, $\vartheta_l \in \mathcal{E}[A]$ for any $l = 1, \dots, n$. Moreover, $\vartheta_l(\omega u) = \omega^{l-1}\vartheta_l(u)$, that is the functions $\vartheta_1, \dots, \vartheta_n$ are eigenfunctions of the translation operator with distinct eigenvalues. Hence, they are linearly independent. $\qquad\square$

For any $\mathfrak{l} \in \mathcal{Z}_\ell^n$ let $G_\mathfrak{l}(t; \alpha; x; y; \eta)$ be the following function:

(A.1) $G_\mathfrak{l}(t; \alpha; x; y; \eta) =$

$$= \frac{1}{\mathfrak{l}_1! \dots \mathfrak{l}_n!} \prod_{m=1}^{n} \prod_{a=1}^{\ell} \frac{1}{\theta(t_a/x_m)} \prod_{1 \leqslant a < b \leqslant \ell} \frac{\theta(t_a/t_b)}{\theta(\eta t_a/t_b)} \sum_{\sigma \in \mathbf{S}_\ell} \prod_{m=1}^{n} \prod_{a \in \Gamma_m} \vartheta_m(t_{\sigma_a}).$$

Here $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \dots, \mathfrak{l}^m\}$, $m = 1, \dots, n$.

Lemma A.2. *The functions* $G_\mathfrak{l}(t; \alpha; x; \eta)$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, *form a basis in the elliptic hypergeometric space* $\mathcal{F}_{ell}[\alpha; x; \eta; \ell]$.

Proof. The elliptic hypergeometric space $\mathcal{F}_{ell}[\alpha; x; \eta; \ell]$ is naturally isomorphic to the ℓ-th symmetric power of the space $\mathcal{E}[A]$ — the space of symmetric functions in $t_1, \dots, t_\ell$ which considered as functions of one variable t_a belong to $\mathcal{E}[A]$ for any $a = 1, \dots, \ell$. The isomorphism reads as follows:

$$f(t_1, \dots, t_\ell) \mapsto f(t_1, \dots, t_\ell) \prod_{m=1}^{n} \prod_{a=1}^{\ell} \theta(t_a/x_m) \prod_{1 \leqslant a < b \leqslant \ell} \frac{\theta(\eta t_a/t_b)}{\theta(t_a/t_b)}, \quad f \in \mathcal{F}_{ell}[\alpha; x; \eta].$$

Now the proposition follows from Lemma A.1. $\qquad\square$

Let $W_\mathfrak{l}$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, be the elliptic weight functions. Define a matrix $Q(\alpha; x; y; \eta)$ by the rule:

$$W_\mathfrak{l}(t; \alpha; x; y; \eta) = \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} Q_{\mathfrak{l}\mathfrak{m}}(\alpha; x; y; \eta) \, G_\mathfrak{m}(t; \alpha; x; \eta), \qquad \mathfrak{l} \in \mathcal{Z}_\ell^n.$$

Set

(A.2) $\qquad d(n, m, \ell, s) = \sum_{\substack{i,j \geqslant 0 \\ i+j < \ell \\ i-j=s}} \binom{m-1+i}{m-1} \binom{n-m-1+j}{n-m-1}$

Proposition A.3([11]).

$$\det Q(\alpha;x;y;\eta) \;=\; \Xi \prod_{s=1-\ell}^{\ell-1} \prod_{m=1}^{n-1} \theta\Big(\eta^{s+\ell-1}\alpha^{-1}\prod_{1\leqslant l\leqslant m} y_l/x_l\Big)^{d(n,m,\ell,s)}$$

$$\times \prod_{m=1}^{n} y_m^{(m-n)\binom{n+\ell-1}{n}} \prod_{s=0}^{\ell-1} \prod_{1\leqslant l<m\leqslant n} \theta(\eta^s y_l/x_m)^{\binom{n+\ell-s-2}{n-1}}$$

where

(A.3)
$$\Xi \;=\; \Big[(p)_\infty^{1-n^2} \prod_{m=1}^{n-1} \Big(\frac{\theta(\omega^m)}{\omega^m-1}\Big)^{n-m} \Big]^{\binom{n+\ell-1}{n}}.$$

Let $S_{ell} = S_{ell}[\alpha;x;y;\eta]$ be the elliptic Shapovalov pairing.
Let $G_\mathfrak{l} = G_\mathfrak{l}(t;\alpha;x;\eta)$ and $G'_\mathfrak{l} = G_\mathfrak{l}(t;\alpha^{-1};y;\eta^{-1})$.

Proposition A.4.

$$\det\big[S_{ell}(G'_\mathfrak{l},G_\mathfrak{m})\big]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}^n_\ell} \;=\;$$

$$= \Xi^{-2}\,(-1)^{n(n-1)/2\,\cdot\,\binom{n+\ell-1}{n}}\, \eta^{n(3-n)/2\,\cdot\,\binom{n+\ell-1}{n+1}} \prod_{m=1}^{n} x_m^{(n-1)\binom{n+\ell-1}{n}}$$

$$\times \prod_{s=0}^{\ell-1} \Big[\frac{\theta(\eta)^n\,\theta(\eta^s\alpha^{-1})\,\theta\big(\eta^{s+2-2\ell}\alpha\prod x_m/y_m\big)}{\theta'(1)^n\,\theta(\eta^{s+1})^n\,\prod\prod\theta(\eta^{-s}x_l/y_m)} \Big]^{\binom{n+\ell-s-2}{n-1}}.$$

Here Ξ *is given by* (A.3), $\prod$ *stands for* $\prod\limits_{m=1}^{n}$ *and* $\prod\prod$ *stands for* $\prod\limits_{l=1}^{n}\prod\limits_{m=1}^{n}$.

Proof. By Lemma 3.5 we have that

$$\det\big[S_{ell}(G'_\mathfrak{l},G_\mathfrak{m})\big]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}^n_\ell} \;=\; \big(\det Q(\alpha;x;y;\eta)\,\det Q(\alpha^{-1};y;x;\eta^{-1})\big)^{-1}$$

$$\times \prod_{\mathfrak{l}\in\mathcal{Z}^n_\ell}\prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1} \frac{\eta^s\,\theta(\eta)\,\theta(\eta^s\alpha^{-1}_{\mathfrak{l},m})\,\theta(\eta^{1-s-\mathfrak{l}_m}\alpha_{\mathfrak{l},m}x_m/y_m)}{\theta'(1)\,\theta(\eta^{s+1})\,\theta(\eta^{-s}x_m/y_m)}\;.$$

To get the final answer we use Proposition A.3 and simplify the triple product changing the order of the products and applying Lemma A.5 several times. $\qquad\square$

Lemma A.5. *The following identity holds:*

$$\sum_{a=0}^{l} \binom{j+a}{j}\binom{j+k+a}{k}\binom{l+m-a}{m} = \binom{j+k}{k}\binom{j+k+l+m+1}{j+k+m+1}.$$

The statement can be proved by induction with respect to l and m.

Appendix B. Three determinant formulae

For any $\mathfrak{l} \in \mathcal{Z}_\ell^n$ let $g_\mathfrak{l}(t; x; y; \eta)$ be the following function:

(B.1) $g_\mathfrak{l}(t; x; y; \eta) =$

$$= \frac{1}{\mathfrak{l}_1! \dots \mathfrak{l}_n!} \prod_{m=1}^n \prod_{a=1}^\ell \frac{1}{t_a - x_m} \prod_{1 \leqslant a < b \leqslant \ell} \frac{t_a - t_b}{\eta t_a - t_b} \sum_{\sigma \in \mathbf{S}_\ell} \prod_{m=1}^n \prod_{a \in \Gamma_m} t_{\sigma_a}^m .$$

Here $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \dots, \mathfrak{l}^m\}$, $m = 1, \dots, n$. The functions $g_\mathfrak{l}(t; x; \eta)$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, form a basis in the trigonometric hypergeometric space $\mathcal{F}[x; \eta; \ell]$.

Let $w_\mathfrak{l}$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, be the trigonometric weight functions. Define a matrix $X(x; y; \eta)$ by the rule:

$$w_\mathfrak{l}(t; x; y; \eta) = \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} X_{\mathfrak{l}\mathfrak{m}}(x; y; \eta) g_\mathfrak{m}(t; x; \eta), \qquad \mathfrak{l} \in \mathcal{Z}_\ell^n .$$

Proposition B.1([10], [11]).

$$\det X(x; y; \eta) = \prod_{s=0}^{\ell-1} \prod_{1 \leqslant l < m \leqslant n} (\eta^s y_l - x_m)^{\binom{n+\ell-s-2}{n-1}} .$$

Let $W_\mathfrak{l}$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, be the elliptic weight functions and let $I = I[\alpha; x; y; \eta]$ be the hypergeometric pairing.

Proposition B.2([11]).

$$\det \left[I(W_\mathfrak{l}, w_\mathfrak{m}) \right]_{\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_\ell^n} = \eta^{-n \binom{n+\ell-1}{n+1}} \prod_{s=1-\ell}^{\ell-1} \prod_{m=1}^{n-1} \theta \Big(\eta^{s+\ell-1} \alpha^{-1} \prod_{1 \leqslant l \leqslant m} y_l / x_l \Big)^{d(n,m,\ell,s)} \times$$

$$\prod_{s=0}^{\ell-1} \left[\frac{(\eta^{-1})_\infty^n (\eta^s \alpha^{-1})_\infty (p \eta^{s+2-2\ell} \alpha \prod x_m / y_m)_\infty}{(\eta^{-s-1})_\infty^n (p)_\infty^{2n-1} \prod (\eta^{-s} x_m / y_m)_\infty} \prod_{1 \leqslant l < m \leqslant n} \frac{(\eta^s y_l / x_m)_\infty}{(\eta^{-s} x_l / y_m)_\infty} \right]^{\binom{n+\ell-s-2}{n-1}} .$$

Here $\prod$ *stands for* $\prod_{m=1}^n$ *and the exponents* $d(n, m, \ell, s)$ *are given by* (A.2).

Let functions $G_\mathfrak{l}$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, be given by (A.1).

Corollary B.3([11]).

$$\det \left[I(G_\mathfrak{l}, g_\mathfrak{m}) \right]_{\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_\ell^n} = \Xi^{-1} \eta^{-n(n+1)/2 \cdot \binom{n+\ell-1}{n+1}}$$

$$\times \prod_{s=0}^{\ell-1} \left[\frac{(\eta^{-1})_\infty^n (\eta^s \alpha^{-1})_\infty (p \eta^{s+2-2\ell} \alpha \prod x_m / y_m)_\infty}{(\eta^{-s-1})_\infty^n (p)_\infty^{2n-1+n(n-1)/2} \prod \prod (\eta^{-s} x_l / y_m)_\infty} \right]^{\binom{n+\ell-s-2}{n-1}}$$

Here Ξ is given by (A.3), $\prod$ stands for $\prod_{m=1}^{n}$ and $\prod\prod$ stands for $\prod_{l=1}^{n}\prod_{m=1}^{n}$.

Proof. The statement follows from Propositions A.3, B.1 and B.2. $\square$

Appendix C. The Jackson integrals via the hypergeometric integrals

Consider the hypergeometric integral $Int\,[x;y;\eta](f\widetilde{\Phi})$, cf. (2.10), for a function $f(t_1,\ldots,t_\ell)$ of the form

$$(B.2) \qquad f(t_1,\ldots,t_\ell) = P(t_1,\ldots,t_\ell)\,\Theta(t_1,\ldots,t_\ell)\prod_{1\leqslant a<b\leqslant\ell}(t_a/t_b)_\infty$$

where $P(t_1,\ldots,t_\ell)$ is a symmetric polynomial of degree at most M in each of the variables $t_1,\ldots,t_\ell$ and $\Theta(t_1,\ldots,t_\ell)$ is a symmetric holomorphic function on $\mathbb{C}^{\times\ell}$ such that

$$(B.3) \qquad \Theta(t_1,\ldots,pt_a,\ldots,t_\ell) = A\,(-t_a)^{-n}\,\Theta(t_1,\ldots,t_\ell)$$

for some constant A. The hypergeometric integrals which appear in the definition of the hypergeometric pairings, see (2.12), fit this case for $M = n - 1$ and A determined by α,η, $x_1,\ldots,x_n$, $y_1,\ldots,y_n$.

For any $x \in \mathbb{C}^n$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, $\mathbf{s} \in \mathbb{Z}^\ell$, let $x \triangleright (\mathfrak{l},\mathbf{s})\,[\eta] \in \mathbb{C}^{\times\ell}$ be the following point:

$$x \triangleright (\mathfrak{l},\mathbf{s})\,[\eta] = (p^{s_1+\cdots+s_{\mathfrak{l}_1}}\eta^{1-\mathfrak{l}_1}x_1, p^{s_2+\cdots+s_{\mathfrak{l}_1}}\eta^{2-\mathfrak{l}_1}x_1, \ldots, p^{s_{\mathfrak{l}_1}}x_1,$$

$$p^{s_{\mathfrak{l}_1+1}+\cdots+s_{\mathfrak{l}_1+\mathfrak{l}_2}}\eta^{1-\mathfrak{l}_2}x_2, \ldots, p^{s_{\mathfrak{l}_1+\mathfrak{l}_2}}x_2, \ldots, p^{s_{\mathfrak{l}-\mathfrak{l}_n+1}+\cdots+s_\ell}\eta^{1-\mathfrak{l}_n}x_n, \ldots, p^{s_\ell}x_n).$$

For instance, if $\mathbf{s} = (0,\ldots,0)$, then $x \triangleright (\mathfrak{l},\mathbf{s})\,[\eta] = x \triangleright \mathfrak{l}\,[\eta]$, cf. (2.5).

Proposition B.4. Let the parameters η, $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ be generic. Let a function $f(t_1,\ldots,t_\ell)$ have the form (B.2), (B.3). Assume that

$$\Big|p^n A\prod_{m=1}^{n}x_m^{-1}\Big| < \min\left(1,|\eta|^{1-\ell}\right).$$

Then the sum below is convergent and

$$\frac{1}{\ell!}\,Int\,[x;y;\eta](f\widetilde{\Phi}) = \sum_{\mathbf{m}\in\mathcal{Z}_\ell^n}\sum_{\mathbf{s}\in\mathbb{Z}_{\geqslant 0}^\ell}\mathrm{Res}\left(t_1^{-1}\ldots t_\ell^{-1}f(t)\,\widetilde{\Phi}(t;x;y;\eta)\right)\Big|_{t=x\triangleright(\mathbf{m},\mathbf{s})[\eta]}.$$

Similarly, if $\Big|p^M A\prod_{m=1}^{n}y_m^{-1}\Big| > \max\left(1,|\eta|^{\ell-1}\right)$, then the sum below is convergent and

$$\frac{1}{\ell!}\,Int\,[x;y;\eta](f\widetilde{\Phi}) = (-1)^\ell\sum_{\mathbf{m}\in\mathcal{Z}_\ell^n}\sum_{\mathbf{s}\in\mathbb{Z}_{\leqslant 0}^\ell}\mathrm{Res}\left(t_1^{-1}\ldots t_\ell^{-1}f(t)\,\widetilde{\Phi}(t;x;y;\eta)\right)\Big|_{t=y\triangleright(\mathbf{m},\mathbf{s})[\eta^{-1}]}.$$

The proof is similar to the proof of Theorem F.1 in [11]. The sums in Proposition B.4 coincide with the symmetric A-type Jackson integrals, see for example [2].

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