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NOTE ON SIMPLE STABLE MAPS OF 3-MANIFOLDS INTO SURFACES

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1. Introduction

Let $f : M \to N$ be a C^∞-stable map of a closed orientable 3-manifold M into a surface N (possibly non-orientable, open, or with boundary). It is known that the local singularities of f consist of three types: definite fold points, indefinite fold points, and cusp points (for example, see [1,7,8]). Burlet and de Rham have studied smooth maps with only definite fold points as the singularities, which are called special generic maps (see [1]). Saeki has studied simple stable maps, which are characterized as stable maps with no cusp points such that every connected component of the fiber $f^{-1}(x)$ contains at most one singular point for all $x \in N$(see [7,8]). Note that special generic maps which are stable are simple stable.

Simple stable maps have been studied by Saeki (see [7,8]). For example, it is known that a closed orientable 3-manifold admits a simple stable map if and only if it is a graph manifold (see [7, Theorem 3.1]). In [8, section 6], Saeki has studied simple stable maps of homology 3-spheres into the plane $\mathbf{R}^2$ whose singular sets have a few connected components.

For a simple stable map $f : M \to N$, the singular set $S(f)$ is a link in M. The components of the link $S(f)$, each of which is called a fold, are classified into definite ones and indefinite ones: the former consists of only definite fold points, and the latter indefinite fold points. The indefinite folds are further classified into two types, (I) and (II), according to the behavior of f in a neighborhood of a given fold. This classification is important in this paper.

It is already known that $f|_{S(f)}$ is an immersion with normal crossings for a simple stable map f. In this paper, we count the number of its crossings and show that this number is congruent modulo 2 to the number of indefinite folds of type (II) in $S(f)$. Note that in [2, Propositions A and C(b)], Chess has considered (not necessarily simple) stable maps without cusps of odd dimensional manifolds into the plane and has obtained the same result as ours for such maps. Here we prove the result using a method totally different from [2]. We also note that our result is applicable also to maps into arbitrary surfaces.

In general, it is important to study the behavior of $f|_{S(f)}$ of a stable map f in the global study of differentiable maps. For example, for a stable map $f : M \to N$ of an n-manifold M ($n \geq 3$) into a 3-manifold N, the number modulo 2 of triple points of $f|_{S(f)}$ has been studied in [9].

This paper is organized as follows. In section 2, we recall some basic definitions and facts about simple stable maps and introduce the notion of the Stein factorization, which is an important tool for the investigation of simple stable maps. In section 3, we introduce a new notion of the "generalized" rotation number for families of immersed oriented circles in a surface (possibly non-orientable or with boundary). This notion is a natural generalization of the well-known rotation number for families of immersed oriented circles in the plane $\mathbf{R}^2$. A similar notion has been defined for immersions of closed n-manifolds (possibly disconnected) into oriented $(n+1)$-manifolds in [4], where a theorem including Theorem 3.16 in the present paper is also proved. In this paper, we define the generalized rotation number using a different method, including the case where the target surfaces are non-orientable. The generalized rotation number of a family of immersed oriented circles is closely related to the number of its crossings, which is an important tool in this paper. The referee has given the author a suggestion for simplifying the proofs of Theorems 3.2 and 3.3 (the case where the surface is orientable and closed). That was a great help for the author to prove the other cases. In section 4, we prove the main result, using the notion of the generalized rotation number. In section 5, we give some examples of immersions of branched surfaces into (unbranched) surfaces (for the definition, see [8]), from which we can construct simple stable maps. These examples show that the number of crossings of $f|_{S(f)}$ for a simple stable map f can be even and odd; i.e., the both possibilities are realized.

Throughout the paper, all manifolds and maps are differentiable of class C^∞ and all the homology groups are with integral coefficients, unless otherwise indicated. For a manifold X, we denote by id_X the identity map of X. We denote by the symbol "$\cong$" a diffeomorphism between manifolds.

2. Preliminaries

We give a brief definition of simple stable maps (for a definition of stable maps, see [1,5]).

DEFINITION 2.1. Let M be an orientable closed 3-manifold and N a surface (possibly non-orientable, open, or with boundary). A map $f : M \to N$ is *simple stable* if it satisfies the following local and global conditions: For all $p \in M$, there exist local coordinates (u, x, y) centered at p and (X, Y) centered at $f(p)$ such that f has one of the following forms:

(L_0) $X \circ f = u,$ $Y \circ f = x$ (p : *regular* point),
(L_1) $X \circ f = u,$ $Y \circ f = x^2 + y^2$ (p : *definite fold* point),
(L_2) $X \circ f = u,$ $Y \circ f = x^2 - y^2$ (p : *indefinite fold* point),

and

(G_1) $f|_{S(f)}$ is an immersion with normal crossings, where $S(f)$ is the singular set of f; i.e., $S(f)$ is the set of the points of M where the rank of the differential df is strictly less than 2, and

(G_2) for all $p \in S(f)$, the connected component of $f^{-1}(f(p))$ containing p has only the point p as singular points.

We set $S_0(f) = \{$definite fold points of $f\}$ and $S_1(f) = \{$indefinite fold points of $f\}$. Clearly we have $S(f) = S_0(f) \cup S_1(f)$. Note that $S(f)$, $S_0(f)$, and $S_1(f)$ are smooth links in M. We call a component of $S_0(f)$ a *definite fold* and a component of $S_1(f)$ an *indefinite fold.*

In the rest of this section we will summarize some known results. For precise proofs, see [5].

Let us observe the behavior of a simple stable map f in a neighborhood of $S(f)$. First, let $l \subset S(f)$ be a definite fold. We set $D = \{(x,y) \in \mathbf{R}^2 | x^2+y^2 \leq 1\}$ and define $r : D \to [0,1]$ by $r(x,y) = x^2 + y^2$. Then the behavior of f in a neighborhood of l is right-left equivalent (for a definition, see [7,8]) to the map $j \circ (r \times \mathrm{id}_{S^1}) : D \times S^1 \to N$, where $j : [0,1] \times S^1 \to N$ is an immersion. Note that l corresponds to $\{c\} \times S^1$, where $c = (0,0) \in D$ (see Figure 2.1).

Second, let $l \subset S(f)$ be an indefinite fold. Let X be the 2-sphere with three open holes. We consider a Morse function $h : X \to \mathbf{R}$ as in Figure 2.2. Note that a is the unique critical point of h.

We set $V_i = X \times_{\varphi_i} S^1 = X \times [0,1]/(x,0) \sim (\varphi_i(x), 1)$ $(i = 1,2)$, where $\varphi_1 = \mathrm{id}_X$ and $\varphi_2 : X \to X$ is the rotation of π around the vertical line passing through a (see Figure 2.2). Note that $h \circ \varphi_i = h$. Then we define $g_i = h \times_{\varphi_i} \mathrm{id}_{S^1} : V_i \to \mathbf{R} \times S^1$,

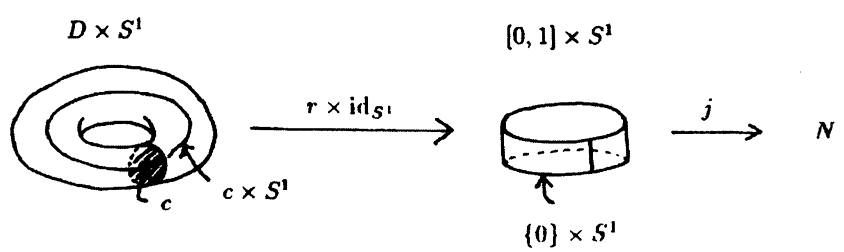
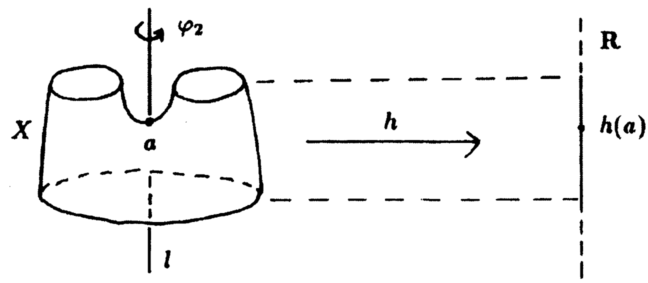


Figure 2.1



φ_2: **rotation of π around the vertical line l**

Figure 2.2

which is induced from the map $h \times \mathrm{id}_{[0,1]} : X \times [0,1] \to \mathbf{R} \times [0,1]$ through the identifications $(x,0) \sim (\varphi_i(x),1)$ $(x \in X)$ and $(y,0) \sim (y,1)$ $(y \in \mathbf{R})$. Note that g_i is well-defined. Then the behavior of f in a neighborhood of l is right-left equivalent to either $j \circ g_1 : V_1 \to N$ or $j \circ g_2 : V_2 \to N$, where $j : \mathbf{R} \times S^1 \to N$ is an immersion. Note that l corresponds to $\{a\} \times S^1$ in both cases.

By the observation above, indefinite folds are classified into two types. We say that an indefinite fold l is of *type* (**I**) if f is equivalent to $j \circ g_1$ on some neighborhood of l, and of *type* (**II**) if it is equivalent to $j \circ g_2$. This classification is important in this paper. We put $S_1^{(\mathrm{I})}(f) = \{$points of indefinite folds of type (**I**)$\}$ and $S_1^{(\mathrm{II})}(f) = \{$points of indefinite folds of type (**II**)$\}$. Clearly we have $S_1(f) = S_1^{(\mathrm{I})}(f) \cup S_1^{(\mathrm{II})}(f)$.

Now we recall the notion of the Stein factorization of a stable map $f : M \to N$. For p and $p' \in M$, we define $p \sim_f p'$ if $f(p) = f(p')$ and p and p' are in the same connected component of $f^{-1}(f(p)) = f^{-1}(f(p'))$. Let $W_f(= M/\sim_f)$ be the quotient space of M under this equivalence relation and we denote by $q_f : M \to W_f$ the quotient map. By the definition of the relation, we have a unique map $\bar{f} : W_f \to N$ such that $f = \bar{f} \circ q_f$. The quotient space W_f or the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {}_{q_f}\searrow & & \nearrow_{\bar{f}} \\ & W_f & \end{array}$$

is called the *Stein factorization* of f [1, 5, 6]. In general, W_f is not a manifold. However, if $f : M \to N$ is a simple stable map, then W_f turns out to be a *branched surface* (for the definition, see [7,8]). Points of W_f are classified into *inner, boundary,* and *branching* points. See Figure 2.3.

We set $\Sigma_0(f) = \{$boundary points of $W_f\}$, $\Sigma_1(f) = \{$branching points of $W_f\}$, and $\Sigma(f) = \Sigma_0(f) \cup \Sigma_1(f)$. The set $\Sigma(f)$ (or $\Sigma_0(f)$, $\Sigma_1(f)$) is a disjoint union of simple closed curves in W_f. If l is a connected component of $\Sigma_0(f)$, then the regular neighborhood $N(l)$ of l in W_f is homeomorphic to $[0,1] \times S^1$, where $\{0\} \times S^1$ corresponds to l. Let l be a connected component of $\Sigma_1(f)$. Set $Y = \{r\exp(\sqrt{-1}\theta) \in$

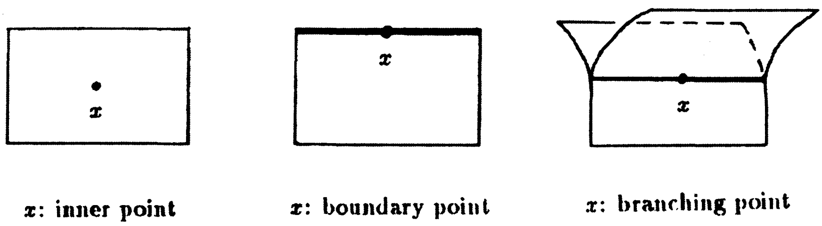


Figure 2.3

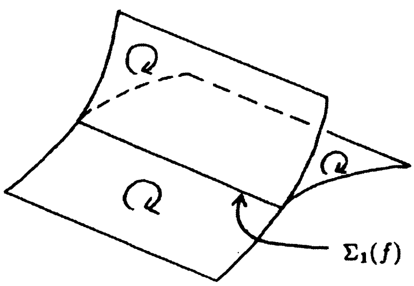


Figure 2.4

$\mathbf{C} | 0 \le r \le 1, \theta = 0, \pm 2\pi/3\}$ and $\omega = 0$ ($\in Y$). Let $\tau : Y \to Y$ be the restriction of the complex conjugation of $\mathbf{C}$ to Y. Note that τ is a homeomorphism. Then the regular neighborhood $N(l)$ of l in W_f is homeomorphic to either $Y \times S^1$ or $Y \times_\tau S^1 = Y \times [0,1]/(y,0) \sim (\tau(y), 1)$, where $\{\omega\} \times S^1$ corresponds to l in both cases. We say that a connected component l of $\Sigma_1(f)$ is of *type* (**I**) if $N(l)$ is homeomorphic to $Y \times S^1$, and of *type* (**II**) if it is homeomorphic to $Y \times_\tau S^1$. We set $\Sigma_1^{(\mathbf{I})}(f) = \{$points of components of $\Sigma_1(f)$ of type (**I**)$\}$, and $\Sigma_1^{(\mathbf{II})}(f) = \{$points of components of $\Sigma_1(f)$ of type (**II**)$\}$. It is easy to observe that $q_f(S(f)) = \Sigma(f)$, $q_f(S_i(f)) = \Sigma_i(f)$ $(i = 1, 2)$, $q_f(S_1^{(\mathbf{I})}(f)) = \Sigma_1^{(\mathbf{I})}(f)$, and $q_f(S_1^{(\mathbf{II})}(f)) = \Sigma_1^{(\mathbf{II})}(f)$. Moreover, $q_f|_{S(f)} : S(f) \to \Sigma(f)$ is a homeomorphism. That is, definite (or indefinite) fold points of M correspond to boundary (resp. branching) points of W_f, and indefinite folds of type (**I**) (or (**II**)) correspond to connected components of $\Sigma_1(f)$ of type (**I**) (resp. (**II**)).

We can introduce a natural "differentiable structure" on W_f (see [7]). Then the map $\bar{f} : W_f \to N$ turns out to be an immersion and $q_f|_{S(f)} : S(f) \to W_f$ an embedding.

Orientability of branched surfaces is also defined. A branched surface W_f is *orientable* if $W_f - \Sigma_1(f)$, which is a surface in an ordinary sense, can be oriented so that a neighborhood of $\Sigma_1(f)$ is oriented as in Figure 2.4.

In the above argument, we have always considered branched surfaces W_f which are the quotient spaces of simple stable maps f. For a general branched surface W

(see [7,8]), points of W are similarly classified into inner, boundary, and branching points as in Figure 2.3. We define $\Sigma_0(W) = \{\text{boundary points of } W\}$, $\Sigma_1(W) = \{\text{branching points of } W\}$, and $\Sigma(W) = \Sigma_0(W) \cup \Sigma_1(W)$. The sets $\Sigma_1^{(\mathbf{I})}(W)$ and $\Sigma_1^{(\mathbf{II})}(W)$ ($\subset \Sigma_1(W)$) are defined similarly. Orientability of W is also defined similarly.

3. Generalized rotation number

In this section, we define generalized rotation numbers for families of immersed oriented circles in surfaces. For a family L of immersed oriented circles in a manifold X, we denote by $[L]$ the homology class in $H_1(X)$ represented by L.

First we define the generalized rotation number $r(L)$ of a family L of immersed oriented circles in an oriented closed surface. Let F_g be the oriented closed surface of genus g, and we fix a Riemannian metric on F_g. Let $T_1 F_g$ be the unit tangent sphere bundle over F_g, i.e., $T_1 F_g = \{v \in T_x F_g | x \in F_g, |v| = 1\}$, and α a fiber over a point of F_g. Let L be a union of n immersed oriented circles in F_g. We call n the number of *transverse components* of L and we denote n by $\sharp_t L$. We denote by $\mathbf{v}(L)$ the unit vector field along L such that at each point of L, $\mathbf{v}(L)$ is tangent to L and is consistent with the orientation of L. Note that α (or $\mathbf{v}(L)$) can be considered as a circle (resp. a family of circles) in $T_1 F_g$. We induce the orientation of α (or $\mathbf{v}(L)$) from that of F_g (resp. L).

REMARK 3.1. Let $l_1, ..., l_{2g}$ be immersed oriented circles in F_g such that $[l_1], ..., [l_{2g}]$ are generators of $H_1(F_g)$. Then $[\mathbf{v}(l_1)], ..., [\mathbf{v}(l_{2g})], [\alpha]$ are generators of $H_1(T_1 F_g)$. Note that $[\alpha]$ is of order $2-2g$ and that $H_1(T_1 F_g)$ is isomorphic to $H_1(F_g) \oplus \mathbf{Z}/(2-2g)\mathbf{Z}$. If $p : T_1 F_g \to F_g$ denotes the bundle projection, then $\ker(p_* : H_1(T_1 F_g) \to H_1(F_g))$ is isomorphic to $\mathbf{Z}/(2-2g)\mathbf{Z}$ and is generated by $[\alpha]$.

Theorem 3.2. *Let F_g, L, and α be as above. If $[L] = 0$ in $H_1(F_g)$, then there exists an integer $r(L)$ such that $[\mathbf{v}(L)] = r(L) \cdot [\alpha]$ in $H_1(T_1 F_g)$. The integer $r(L)$ is uniquely determined modulo $2-2g$. We call $r(L) \in \mathbf{Z}/(2-2g)\mathbf{Z}$ the (generalized) rotation number of the family L of immersed oriented circles.*

Proof. Let $p : T_1 F_g \to F_g$ be the projection of the unit tangent sphere bundle. Then it is clear that $[L] = 0$ in $H_1(F_g)$ if and only if $p_*([\mathbf{v}(L)]) = 0$ in $H_1(T_1 F_g)$. From Remark 3.1, the result follows immediately. □

It is easily shown that $r(L)$ changes its sign if F_g changes its orientation. Note that $r(L)$ does not depend on the choice of a Riemannian metric on F_g and that $r(L)$ is a regular homotopy invariant of L.

Theorem 3.3. *Let L be a family of immersed oriented circles in F_g. We suppose that L has only normal crossings and we denote by x the number of its crossings. If*

$[L] = 0$ *in* $H_1(F_g)$, *then*

$$x \equiv r(L) - \sharp_t L \pmod 2 .$$

Proof. We transform L in a neighborhood of each crossing as in Figure 3.1.

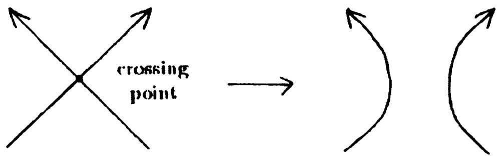


Figure 3.1

Note that $[L]$ and $[\mathbf{v}(L)]$ do not change under this transformation. Moreover, both x and $\sharp_t L$ change their parity at each time of the transformation. Thus we have only to prove the theorem in the case where L is a disjoint union of oriented simple closed curves. To complete the proof, we show the following lemma.

Lemma 3.4. *Let L be a disjoint union of oriented simple closed curves in* F_g. *If* $[L] = 0$ *in* $H_1(F_g)$, *then there exists a decomposition* $L = L_1 \cup \cdots \cup L_m$ $(1 \leq m < \infty)$ *such that* (1) *each* L_i *is a disjoint union of oriented simple closed curves,*
(2) $L_i \cap L_j = \emptyset$ *for* $i \neq j$, *and*
(3) *for each* i, *there exists a compact connected codimension*-0 *submanifold* A_i *of* F_g *such that* $\partial A_i = L_i$ *as oriented* 1-*dimensional manifolds, where the orientation of* A_i *is induced from that of* F_g.

Proof. We fix a point $x_0 \in F_g - L$. For each $x \in F_g - L$, we define $n(x)$ as follows. Let $\gamma \subset F_g$ be a piecewise smooth oriented path from x_0 to x which transversely intersects L at finitely many points. Each intersection point $p \in L \cap \gamma$ has its own sign (either $+1$ or -1) according to the orientations of L, γ, and F_g. Then we define $n(x)$ to be the sum of all the signs of $L \cap \gamma$. If $L \cap \gamma = \emptyset$, then we set $n(x) = 0$. This definition does not depend on the choice of γ. If γ' is another choice, then consider the oriented closed curve $\gamma - \gamma'$ as in Figure 3.2. Since $[L] = 0$ in $H_1(F_g)$, the intersection number $(\gamma - \gamma') \cdot L$ is equal to 0, which means that $n(x)$ obtained from γ is equal to that obtained from γ'. It is then obvious that $n(x)$ is constant on each connected component of $F_g - L$. Thus each connected component X of $F_g - L$ has its own integer $n(X)$. Moreover, if X_1 and X_2 are connected components of $F_g - L$ next to each other (i.e., $\overline{X_1} \cap \overline{X_2} \neq \emptyset$), then $n(X_1) - n(X_2) = \pm 1$. The number $\sharp(F_g - L)$ of connected components of $F_g - L$ is finite, so there exists a connected component X such that $n(X)$ is maximum. We set $A_1 = \overline{X}$ and $L_1 = \partial A_1$. Then the orientation of L_1 as the oriented boundary of A_1 coincides with that as a

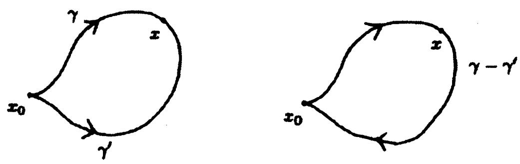


Figure 3.2

component of the family L of oriented simple closed curves, since the sign of each intersection point of ∂A_1 with a path going out of A_1 is always equal to -1. Thus, we have obtained L_1 and A_1 as in the lemma. If $L' = L - L_1$ is not empty, then $[L'] = [L]-[\partial A_1] = 0$ in $H_1(F_g)$. Thus we can continue the argument on L' and obtain L_i and A_i inductively, and the conditions (1), (2), and (3) are obviously satisfied. $\square$

We complete the proof of Theorem 3.3 as follows. Let $L = L_1 \cup \cdots \cup L_m$ and $A_1, ..., A_m \subset F_g$ be as in Lemma 3.4. For each L_i, $[\mathbf{v}(L_i)] = \chi(A_i)\cdot[\alpha]$ in $H_1(T_1F_g)$, where $\chi(A_i)$ denotes the Euler characteristic of A_i. This is shown by Poincaré-Hopf's theorem (or see [3, Proposition 3]). Thus $[\mathbf{v}(L)] = (\sum_{i=1}^m \chi(A_i)) \cdot [\alpha]$. If A_i is an orientable compact surface of genus g_i with λ_i boundary components, then $\chi(A_i) = 2 - 2g_i - \lambda_i$. Thus

$$\begin{aligned} r(L) - \sharp_t L &\equiv \sum_{i=1}^m \chi(A_i) - \sharp_t L \pmod{2-2g} \\ &= \sum_{i=1}^m (2 - 2g_i - \lambda_i) - \sharp_t L \\ &\equiv \sum_{i=1}^m \lambda_i - \sharp_t L \pmod 2 \\ &= 0\,. \end{aligned}$$

This completes the proof of Theorem 3.3. $\square$

Next, we define the generalized rotation number $r(L)$ for a family L of immersed oriented circles in an oriented compact surface with *nonempty boundary*. Let $F_{g,k}$ be the oriented compact surface of genus g with k ($k \geq 1$) boundary components and $L \subset F_{g,k}$ a family of immersed oriented circles. Let $T_1F_{g,k}$ be the unit tangent sphere bundle and α a fiber over a point of $F_{g,k}$. Let $\mathbf{v}(L)$ be the unit vector field along L such that at each point of L, $\mathbf{v}(L)$ is tangent to L and is consistent with the orientation of L. Note that α (or $\mathbf{v}(L)$) can be considered as a circle (resp. a family of circles) in $T_1F_{g,k}$. We orient α and $\mathbf{v}(L)$ as in the previous case.

REMARK 3.5. Let $l_1, ..., l_{2g+k-1}$ be immersed oriented circles in $F_{g,k}$ such that $[l_1], ..., [l_{2g+k-1}]$ are generators of $H_1(F_{g,k})$. Then $[\mathbf{v}(l_1)], ..., [\mathbf{v}(l_{2g+k-1})], [\alpha]$ are generators of $H_1(T_1F_{g,k})$. Note that $[\alpha]$ is of infinite order and that $H_1(T_1F_{g,k})$ is isomorphic to $H_1(F_{g,k}) \oplus \mathbf{Z}$. If $p : T_1F_{g,k} \to F_{g,k}$ denotes the bundle projection, then it is easily observed that $\ker(p_* : H_1(T_1F_{g,k}) \to H_1(F_{g,k}))$ is isomorphic to $\mathbf{Z}$ and is generated by $[\alpha]$.

Theorem 3.6. *Let L be a family of immersed oriented circles in $F_{g,k}$. If $[L] = 0$ in $H_1(F_{g,k})$, then there exists a unique integer $r(L)$ such that $[\mathbf{v}(L)] = r(L) \cdot [\alpha]$ in $H_1(T_1F_{g,k})$. We call $r(L) \in \mathbf{Z}$ the (generalized) rotation number of the family L of immersed oriented circles.*

We can prove the above theorem by an argument similar to that in the proof of Theorem 3.2, so we omit the proof here. It is easily shown that $r(L)$ changes its sign if $F_{g,k}$ changes its orientation. Note that $r(L)$ does not depend on the choice of a Riemannian metric on $F_{g,k}$ and that $r(L)$ is a regular homotopy invariant of L.

Theorem 3.7. *Let L be a family of immersed oriented circles in $F_{g,k}$ with only normal crossings. We denote by $\sharp_t L$ the number of its transverse components and by x the number of its crossings. If $[L] = 0$ in $H_1(F_{g,k})$, then*

$$x \equiv r(L) - \sharp_t L \pmod 2 .$$

Proof. Let $i : F_{g,k} \to F_{g'}$ be an orientation-preserving embedding ($g \leq g'$). By the embedding i, we can consider L to be a family of immersed oriented circles in $F_{g'}$. Thus, by Theorem 3.3,

$$x \equiv r(i(L)) - \sharp_t L \pmod 2 .$$

To complete the proof, we show that the rotation number $r(L) \in \mathbf{Z}$ is congruent modulo 2 to the rotation number $r(i(L)) \in \mathbf{Z}/(2-2g')\mathbf{Z}$. Let $p : T_1F_{g,k} \to F_{g,k}$ denote the bundle projection and $p_* : H_1(T_1F_{g,k}) \to H_1(F_{g,k})$ the induced homomorphism. Furthermore, let $p' : T_1F_{g'} \to F_{g'}$ denote the bundle projection and $p'_* : H_1(T_1F_{g'}) \to H_1(F_{g'})$ the induced homomorphism. Let $T_1 i : T_1F_{g,k} \to T_1F_{g'}$ denote the map induced from the embedding i, and we consider the induced homomorphism $(T_1 i)_* : H_1(T_1F_{g,k}) \to H_1(T_1F_{g'})$. Then it is easily shown that $(T_1 i)_*(\ker(p_*)) = \ker(p'_*)$ and the restriction of $(T_1 i)_*$ to $\ker(p_*)$ is equivalent to the natural projection of $\mathbf{Z}$ onto $\mathbf{Z}/(2-2g')\mathbf{Z}$. Then it follows that $r(L)$ is equal to $r(i(L))$ if considered as elements of $\mathbf{Z}/(2-2g')\mathbf{Z}$. Thus $r(L)$ is congruent modulo 2 to $r(i(L))$, which completes the proof. □

Let us consider the case where L is a family of immersed oriented circles in a *non-orientable* closed surface. Let N_g be the non-orientable closed surface of non-orientable genus g and $L \subset N_g$ a family of immersed oriented circles. Let T_1N_g be the unit tangent sphere bundle and α a fiber over a point of N_g. Let $\mathbf{v}(L)$ be the unit vector field along L such that at each point of L, $\mathbf{v}(L)$ is tangent to L and is consistent with the orientation of L. Note that α (or $\mathbf{v}(L)$) can be considered as a circle (resp. a family of circles) in T_1N_g. We induce the orientation of $\mathbf{v}(L)$ from that of L. We choose the orientation of α arbitrarily.

REMARK 3.8. If k is a non-negative integer, then $N_{2k+1} \cong F_k \sharp N_1$ and $N_{2k+2} \cong F_k \sharp N_2$. That is, we obtain non-orientable closed surfaces as follows. Let a be the oriented boundary of the oriented surface $F_{k,1}$. Let M be the compact Möbius band and b its boundary. Furthermore, let K be the Klein bottle with one open hole and c its boundary. Then N_{2k+1} is obtained from $F_{k,1}$ and M by gluing a and b, and N_{2k+2} is obtained from $F_{k,1}$ and K by gluing a and c (see Figure 3.3).

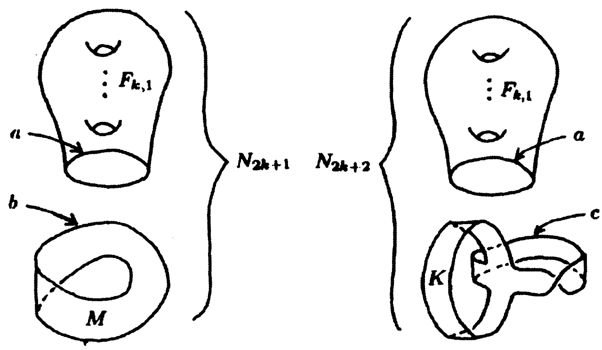


Figure 3.3

REMARK 3.9. Note that $H_1(T_1N_{2k+1})$ is isomorphic to $\mathbf{Z}^{2k} \oplus \mathbf{Z}/4\mathbf{Z}$ and $H_1(T_1 N_{2k+2})$ to $\mathbf{Z}^{2k+1} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. This is observed as follows. We consider $F_{k,1}, M \subset N_{2k+1}$ as in Remark 3.8. Let $l_1, ..., l_{2k}$ be immersed oriented circles in $F_{k,1}$ such that $[l_1], ..., [l_{2k}]$ are generators of $H_1(F_{k,1})$. Let m be an immersed oriented circle in M such that $[m]$ is a generator of $H_1(M)$ (for example, we choose the center circle of the Möbius band M). Then $[l_1], ..., [l_{2k}], [m]$ are generators of $H_1(N_{2k+1})$ and $[\mathbf{v}(l_1)], ..., [\mathbf{v}(l_{2k})], [\mathbf{v}(m)]$ are generators of $H_1(T_1N_{2k+1})$. Note that $[m] \in H_1(N_{2k+1})$ is of order 2, $[\mathbf{v}(m)] \in H_1(T_1N_{2k+1})$ is of order 4, and $[\alpha] = 2[\mathbf{v}(m)]$ in $H_1(T_1N_{2k+1})$. Similarly, we consider $F_{k,1}, K \subset N_{2k+2}$. Let $l_1, ..., l_{2k}$ be as above and m_1 and m_2 embedded oriented circles in K as in Figure 3.4, where the regular neighborhood $N(m_1)$ of m_1 is diffeomorphic to the Möbius band and the regular neighborhood

$N(m_2)$ of m_2 to the annulus. Note that m_1 and m_2 are generators of $H_1(K)$. Then $[l_1], ..., [l_{2k}], [m_1], [m_2]$ are generators of $H_1(N_{2k+2})$ and $[\mathbf{v}(l_1)], ..., [\mathbf{v}(l_{2k})], [\mathbf{v}(m_1)], [\mathbf{v}(m_2)], [\alpha]$ are generators of $H_1(T_1 N_{2k+2})$. Note that $[m_2] \in H_1(N_{2k+2})$ is of order 2 and that $[\mathbf{v}(m_2)], [\alpha] \in H_1(T_1 N_{2k+2})$ are of order 2. Note that $[\alpha]$ is of order 2 in both cases. If $p : T_1 N_g \to N_g$ denotes the bundle projection, then $\ker(p_* : H_1(T_1 N_g) \to H_1(N_g))$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ and is generated by $[\alpha]$.

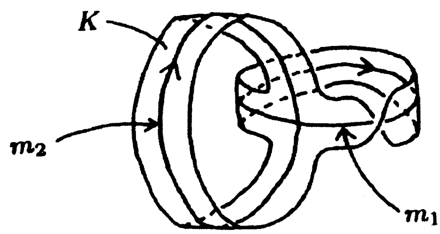


Figure 3.4

Theorem 3.10. *Let L be a family of immersed oriented circles in N_g. If $[L] = 0$ in $H_1(N_g)$, then there exists an integer $r(L)$ such that $[\mathbf{v}(L)] = r(L) \cdot [\alpha]$ in $H_1(T_1 N_g)$. The integer $r(L)$ is uniquely determined modulo 2. We call $r(L) \in \mathbf{Z}/2\mathbf{Z}$ the (generalized) rotation number of the family L of immersed oriented circles.*

We can prove the above theorem by an argument similar to that in the proof of Theorem 3.2, so we omit the proof here. Note that the definition of $r(L)$ does not depend on the choice of a Riemannian metric on N_g nor on the choice of the orientation of α and that $r(L)$ is a regular homotopy invariant of L.

Theorem 3.11. *Let L be a family of immersed oriented circles in N_g with only normal crossings. We denote by $\sharp_t L$ the number of its transverse components and by x the number of its crossings. If $[L] = 0$ in $H_1(N_g)$, then*

$$x \equiv r(L) - \sharp_t L \pmod 2 .$$

Proof. First, we consider the case $g = 2k+1$. Let $F_{k,1}, M \subset N_{2k+1}$ be as in Remark 3.8.

Case 1. The family L is contained in $F_{k,1}$.

Since the homomorphism $H_1(F_{k,1}) \to H_1(N_{2k+1})$ induced by the inclusion is injective, we see that L is null homologous in $F_{k,1}$. Then the result follows from Theorem 3.7.

Set $\gamma = F_{k,1} \cap M = \partial F_{k,1} = \partial M$.

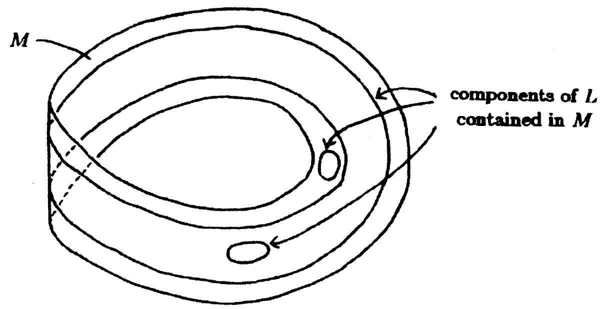


Figure 3.5

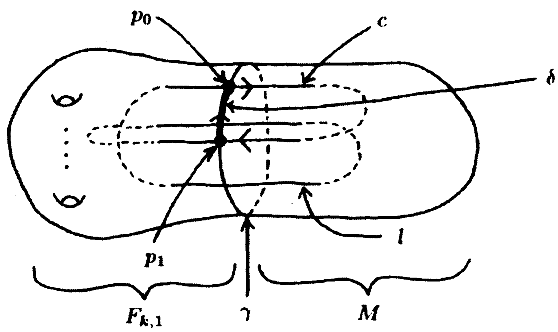


Figure 3.6

Case 2. The family L is not contained in $F_{k,1}$ and $L \cap \gamma = \emptyset$.

By applying the transformation as in Figure 3.1 to L, we may assume that L is a disjoint union of oriented simple closed curves. Let us first show that the regular neighborhood $N(l)$ of each component l of L is diffeomorphic to the annulus. For each point $x \in N_{2k+1} - L$, we can define $n(x) \in \mathbf{Z}/2\mathbf{Z}$ as in the proof of Lemma 3.4, using the intersection number modulo 2. Suppose that $N(l)$ is not diffeomorphic to the annulus. Then $N(l)$ is diffeomorphic to the Möbius band. Let p and p' be distinct points of $N(l) - l$. Since $N(l) - l$ is connected, we have $n(p) = n(p')$. On the other hand, there exists a smooth path in $N(l)$ connecting p and p' which intersects l transversely at one point. This implies that $n(p) - n(p') = 1$, which is a contradiction. Thus $N(l)$ is diffeomorphic to the annulus. Then it is not difficult to show, using standard arguments, that each component l of L contained in M either bounds a disk embedded in M or is boundary parallel (see Figure 3.5). Thus, by isotopy, we may assume that L is contained in $F_{k,1}$, which reduces this case to Case 1 above.

Case 3. The family L is not contained in $F_{k,1}$ and $L \cap \gamma \neq \emptyset$.

By an isotopy of L, we may assume that γ does not pass through the crossings of L and that γ and L intersect transversely at finitely many points.

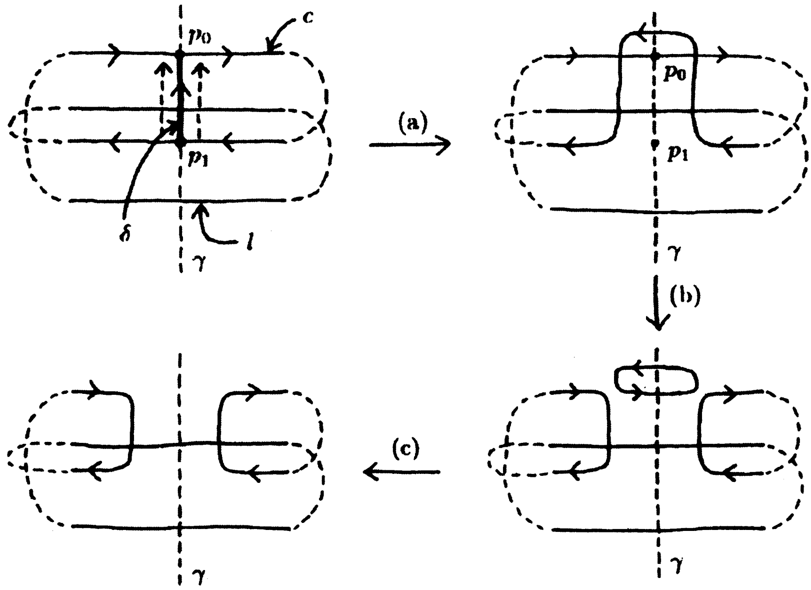


Figure 3.7

Let l be a component of L which intersects γ. We consider the transformation as follows. Let c be a component of $l \cap M$ and we denote by p_0 and p_1 the end points of c. We assume that c is oriented from p_0 to p_1. Let δ be an arc embedded in γ connecting p_0 and p_1, where we assume that it is oriented from p_1 to p_0 (see Figure 3.6). Then we consider the transformation of l as in Figure 3.7 in a neighborhood of δ. It is not difficult to observe that the difference $x - (r(L) - \sharp_t L)$ does not change modulo 2 under this transformation and that the number of intersection points of L with γ decreases by 2. Thus, iterating such a transformation finitely many times, we may assume that $L \cap \gamma = \emptyset$, which reduces this case to Case 2 above. This completes the proof in the case where $g = 2k + 1$.

In the case where $g = 2k + 2$, let a_1 and a_2 be the oriented boundary components of the oriented surface $F_{k,2}$. Let M_i $(i = 1, 2)$ be the compact Möbius band and b_i its boundary. Then N_{2k+2} is obtained from $F_{k,2}$, M_1 and M_2 by gluing a_i and b_i $(i = 1, 2)$(see Figure 3.8). Note that $N_{2k+2} \cong N_1 \sharp F_k \sharp N_1$. If we set $\gamma = \partial F_{k,2} = \partial M_1 \cup \partial M_2$, then we can prove the theorem in this case by a similar argument. $\square$

Let us consider the case where L is a family of immersed oriented circles in a *non-orientable* compact surface with *nonempty* boundary. Let $N_{g,k}$ be the non-orientable compact surface of genus g with k $(k \geq 1)$ boundary components and $L \subset N_{g,k}$ a family of immersed oriented circles. Let $T_1 N_{g,k}$ be the unit tangent sphere bundle and α a fiber over a point of $N_{g,k}$. Let $\mathbf{v}(L)$ be the unit vector field along L such

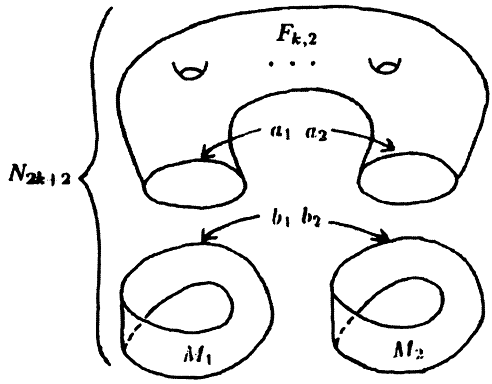


Figure 3.8

that at each point of L, $\mathbf{v}(L)$ is tangent to L and is consistent with the orientation of L. Note that α (or $\mathbf{v}(L)$) can be considered as a circle (resp. a family of circles) in $T_1N_{g,k}$. We induce the orientation of $\mathbf{v}(L)$ from that of L. We choose the orientation of α arbitrarily. Note that $H_1(T_1N_{g,k})$ is isomorphic to $H_1(N_{g,k}) \oplus \mathbf{Z}/2\mathbf{Z}$ and that $[\alpha] \in H_1(T_1N_{g,k})$ is of order 2. If $p : T_1N_{g,k} \to N_{g,k}$ denotes the bundle projection, then $\ker(p_* : H_1(T_1N_{g,k}) \to H_1(N_{g,k}))$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ and is generated by $[\alpha]$.

Theorem 3.12. *Let L be a family of immersed oriented circles in $N_{g,k}$. If $[L] = 0$ in $H_1(N_{g,k})$, then there exists an integer $r(L)$ such that $[\mathbf{v}(L)] = r(L) \cdot [\alpha]$ in $H_1(T_1N_{g,k})$. The integer $r(L)$ is uniquely determined modulo 2. We call $r(L) \in \mathbf{Z}/2\mathbf{Z}$ the (generalized) rotation number of the family L of immersed oriented circles.*

Theorem 3.13. *Let L be a family of immersed oriented circles in $N_{g,k}$ with only normal crossings. We denote by $\sharp_t L$ the number of its transverse components and by x the number of its crossings. If $[L] = 0$ in $H_1(N_{g,k})$, then*

$$x \equiv r(L) - \sharp_t L \pmod 2 .$$

The proofs of the theorems above are obvious now (see the proofs of Theorems 3.6 and 3.7), so we omit them here. Note that the definition of $r(L)$ does not depend on the choice of a Riemannian metric on $N_{g,k}$ nor on the choice of the orientation of α and that $r(L)$ is a regular homotopy invariant of L.

Let us consider the case where L is a family of immersed oriented circles in the oriented plane $\mathbf{R}^2$. Let $T_1\mathbf{R}^2$ be the unit tangent sphere bundle and α a fiber over a point of $\mathbf{R}^2$. Let $\mathbf{v}(L)$ be the unit vector field along L such that at each point of L, $\mathbf{v}(L)$ is tangent to L and is consistent with the orientation of L. Note that α (or

$\mathbf{v}(L)$) can be considered as a circle (resp. a family of circles) in $T_1\mathbf{R}^2$. We induce the orientation of α (or $\mathbf{v}(L)$) from that of $\mathbf{R}^2$ (resp. L). Note that $H_1(T_1\mathbf{R}^2)$ is isomorphic to $\mathbf{Z}$ and is generated by $[\alpha]$.

Theorem 3.14. *Let L be a family of immersed oriented circles in the oriented plane $\mathbf{R}^2$. Then there exists a unique integer $r(L)$ such that $[\mathbf{v}(L)] = r(L) \cdot [\alpha]$ in $H_1(T_1\mathbf{R}^2)$. We call $r(L) \in \mathbf{Z}$ the rotation number of the family L of immersed oriented circles.*

The proof is obvious. Moreover, it is easily shown that $r(L)$ changes its sign if the plane $\mathbf{R}^2$ changes its orientation. Note that the definition of $r(L)$ does not depend on the choice of a Riemannian metric on $\mathbf{R}^2$ and that $r(L)$ is a regular homotopy invariant of L.

Theorem 3.15. *Let L be a family of immersed oriented circles in the oriented plane $\mathbf{R}^2$ with only normal crossings. We denote by $\sharp_t L$ the number of its transverse components and by x the number of its crossings. Then*

$$x \equiv r(L) - \sharp_t L \pmod 2 .$$

Theorem 3.14 coincides with the definition of the (well-known) rotation number, and Theorem 3.15 is also well-known. Thus, the results in this section can be considered as generalizations of the results known for the usual rotation number.

We have the following consequence of the above arguments. Let X be an oriented compact surface with boundary ∂X. We induce the orientation of ∂X from that of X.

Theorem 3.16. *Let F_g be the oriented closed surface of genus g and $f : X \to F_g$ an orientation-preserving immersion. Then the rotation number $r(f(\partial X))$ of $f(\partial X)$ is congruent modulo $2 - 2g$ to $\chi(X)$, where $\chi(X)$ denotes the Euler characteristic of X.*

Proof. Let $p : T_1X \to X$ denote the bundle projection and $p_* : H_1(T_1X) \to H_1(X)$ the induced homomorphism. Let $p' : T_1F_g \to F_g$ denote the bundle projection and $p'_* : H_1(T_1F_g) \to H_1(F_g)$ the induced homomorphism. Furthermore, let $T_1f : T_1X \to T_1F_g$ denote the map induced from the immersion f, and we consider the induced homomorphism $(T_1f)_* : H_1(T_1X) \to H_1(T_1F_g)$. Then it is easily seen that $(T_1f)_*(\ker(p_*)) = \ker(p'_*)$ and the restriction of $(T_1f)_*$ to $\ker(p_*)$ is equivalent to the natural projection of $\mathbf{Z}$ onto $\mathbf{Z}/(2-2g)\mathbf{Z}$. It follows that $r(\partial X)$ is equal to $r(f(\partial X))$ if they are considered as elements of $\mathbf{Z}/(2-2g)\mathbf{Z}$. By Poincaré-Hopf's theorem, we have $r(\partial X) = \chi(X)$, which completes the proof. □

We obtain the following theorems by similar arguments.

Theorem 3.17. *Let $F_{g,k}$ be the oriented compact surface of genus g with k ($k \geq 1$) boundary components and $f : X \to F_{g,k}$ an orientation-preserving immersion. Then the rotation number $r(f(\partial X))$ of $f(\partial X)$ is equal to $\chi(X)$.*

Theorem 3.18. *Let N be a non-orientable compact surface (possibly with boundary) and $f : X \to N$ an immersion. Then the rotation number $r(f(\partial X))$ of $f(\partial X)$ is congruent modulo 2 to $\chi(X)$.*

Theorem 3.19. *Let $f : X \to \boldsymbol{R}^2$ be an orientation-preserving immersion. Then the rotation number $r(f(\partial X))$ of $f(\partial X)$ is equal to $\chi(X)$.*

4. Main theorem

In this section, we present and prove the main theorems of this paper. Let $f : M \to N$ be a simple stable map of an orientable closed 3-manifold M into a compact surface N (possibly with boundary or non-orientable).

Theorem 4.1. *If the Stein factorization W_f is orientable, then the number of crossings of the immersion $f|_{S(f)}$ has the same parity as $\sharp S_1^{(II)}(f)$, where $\sharp$ denotes the number of connected components.*

Let W be an orientable branched surface and $h : W \to N$ an immersion.

Theorem 4.2. *The number of crossings of the immersion $h|_{\Sigma(W)}$ has the same parity as $\sharp\Sigma_1^{(II)}(W)$.*

Theorem 4.1 is easily proved from Theorem 4.2 (note that $f(S(f)) = \bar{f}(\Sigma(W_f))$ and that $\sharp S_1^{(\mathrm{II})}(f) = \sharp\Sigma_1^{(\mathrm{II})}(W_f))$. To prove Theorem 4.2, we show the following propositions.

Proposition 4.3. *Let W be an oriented branched surface. If we orient $\Sigma(W)$ as in Figure 4.1, then $[\Sigma(W)] = 0$ in $H_1(W)$.*

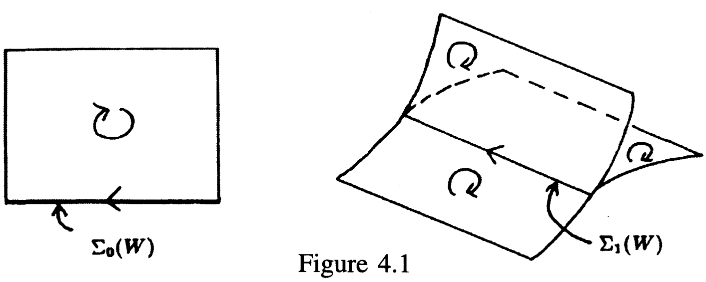


Figure 4.1

By Proposition 4.3, we can define the generalized rotation number $r(h(\Sigma(W)))$ of $h(\Sigma(W))$.

Proposition 4.4. *The generalized rotation number $r(h(\Sigma(W)))$ of $h(\Sigma(W))$ is congruent to $\sharp\Sigma(W) + \sharp\Sigma_1^{(II)}(W)$ modulo 2.*

Theorem 4.2 is easily obtained from Theorems 3.3, 3.7, 3.11, 3.13 and Proposition 4.4. Note that for $L = h(\Sigma(W))$, we have $\sharp_t L = \sharp_t h(\Sigma(W)) = \sharp\Sigma(W)$.

Proof of Proposition 4.3. We prove the proposition by induction on $\sharp\Sigma_1(W)$. When $\sharp\Sigma_1(W) = 0$, W is an orientable compact surface and the proposition is obvious. When $\sharp\Sigma_1(W) > 0$, we choose a connected component l of $\Sigma_1(W)$. We consider the branched surface $W - \mathrm{int}N(l) = W'$, where $N(l)$ denotes the regular neighborhood of l in W.

(i) The case $l \subset \Sigma_1^{(I)}(W)$. In this case $N(l) \cong Y \times S^1$ and $\sharp\partial N(l) = 3$, where $\partial N(l)$ denotes the set of the boundary points of $N(l)$. Let a, b, and c be the connected components of $\partial N(l)$ (see Figure 4.2(i)). We consider the orientation of W' induced from that of W. We choose the orientations of a, b and c as the oriented boundary of W'. Then $[l] = -[a] = [b] = [c]$ in $H_1(W)$. Note that $\sharp\Sigma_0(W') = \sharp\Sigma_0(W) + 3$ and $\sharp\Sigma_1(W') = \sharp\Sigma_1(W) - 1$. Then

$$\begin{aligned}[\Sigma(W)] &= [\Sigma(W')] - [a] - [b] - [c] + [l] \\ &= [\Sigma(W')] \\ &= 0 \quad (\text{in } H_1(W)),\end{aligned}$$

where the last equality follows from the induction hypothesis.

(ii) The case $l \subset \Sigma_1^{(II)}(W)$. In this case $N(l) \cong Y \times_\tau S^1$ and $\sharp\partial N(l) = 2$. Let a and b be the connected components of $\partial N(l)$ (see Figure 4.2(ii)). We consider the orientation of W' induced from that of W as in case (i). We choose the orientations of a and b as the oriented boundary of W'. Then $[l] = -[a]$ and $2[l] = [b]$ in $H_1(W)$. Note that $\sharp\Sigma_0(W') = \sharp\Sigma_0(W) + 2$ and $\sharp\Sigma_1(W') = \sharp\Sigma_1(W) - 1$. Then

$$\begin{aligned}[\Sigma(W)] &= [\Sigma(W')] - [a] - [b] + [l] \\ &= [\Sigma(W')] \\ &= 0 \quad (\text{in } H_1(W)),\end{aligned}$$

where the last equality follows from the induction hypothesis. This completes the proof of Proposition 4.3. □

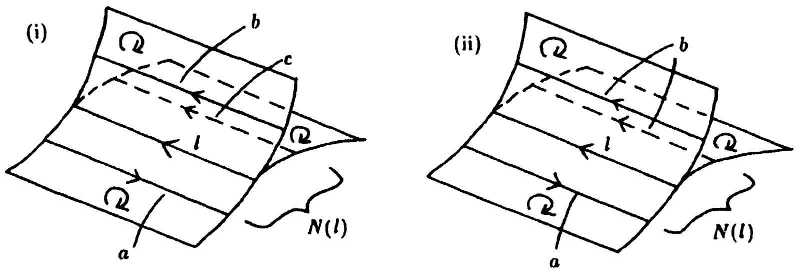


Figure 4.2

Proof of Proposition 4.4. We prove the proposition also by induction on $\sharp\Sigma_1(W)$. When $\sharp\Sigma_1(W) = 0$, W is an orientable compact surface and the proposition is easily proved by using Theorems 3.16, 3.17, and 3.18. When $\sharp\Sigma_1(W) > 0$, we choose a connected component l of $\Sigma_1(W)$. We consider the branched surface W' as above and the immersion $h' : W' \to N$, which is the restriction of h to W'.

(i) The case $l \subset \Sigma_1^{(\mathbf{I})}(W)$. In this case, we put $\partial N(l) = a \cup b \cup c$ and orient a, b and c as in the proof of Proposition 4.3. Clearly we have $[h(l)] = -[h(a)] = [h(b)] = [h(c)]$ in $H_1(N)$ and $[\mathbf{v}(h(l))] = -[\mathbf{v}(h(a))] = [\mathbf{v}(h(b))] = [\mathbf{v}(h(c))]$ in $H_1(T_1N)$. Furthermore, we have

$$[\mathbf{v}(h'(\Sigma(W')))] = (\sharp\Sigma(W') + \sharp\Sigma_1^{(\mathbf{II})}(W') + 2k) \cdot [\alpha]$$

for some integer k by our induction hypothesis. Thus we have

$$\begin{aligned}
[\mathbf{v}(h(\Sigma(W)))] &= [\mathbf{v}(h(\Sigma(W')))] - [\mathbf{v}(h(a))] - [\mathbf{v}(h(b))] - [\mathbf{v}(h(c))] + [\mathbf{v}(h(l))] \\
&= [\mathbf{v}(h'(\Sigma(W')))] \\
&= (\sharp\Sigma(W') + \sharp\Sigma_1^{(\mathbf{II})}(W') + 2k) \cdot [\alpha] \\
&= (\sharp\Sigma(W) + 2 + \sharp\Sigma_1^{(\mathbf{II})}(W) + 2k) \cdot [\alpha] \\
&= (\sharp\Sigma(W) + \sharp\Sigma_1^{(\mathbf{II})}(W) + 2(k+1)) \cdot [\alpha].
\end{aligned}$$

Note that $\sharp\Sigma(W') = \sharp\Sigma(W) + 2$ and $\sharp\Sigma_1^{(\mathbf{II})}(W') = \sharp\Sigma_1^{(\mathbf{II})}(W)$. Therefore we have $r(h(\Sigma(W))) \equiv \sharp\Sigma(W) + \sharp\Sigma_1^{(\mathbf{II})}(W) \mod 2$.

(ii) The case $l \subset \Sigma_1^{(\mathbf{II})}(W)$. In this case, we put $\partial N(l) = a \cup b$ and orient a and b as in the proof of Proposition 4.3. Clearly we have $[h(l)] = -[h(a)]$ and $2[h(l)] = [h(b)]$ in $H_1(N)$ and $[\mathbf{v}(h(l))] = -[\mathbf{v}(h(a))]$ and $2[\mathbf{v}(h(l))] = [\mathbf{v}(h(b))]$ in $H_1(T_1N)$. Furthermore, we have

$$[\mathbf{v}(h'(\Sigma(W')))] = (\sharp\Sigma(W') + \sharp\Sigma_1^{(\mathbf{II})}(W') + 2k) \cdot [\alpha]$$

for some integer k by our induction hypothesis. Thus we have

$$\begin{aligned}
[\mathbf{v}(h(\Sigma(W)))] &= [\mathbf{v}(h(\Sigma(W')))] - [\mathbf{v}(h(a))] - [\mathbf{v}(h(b))] + [\mathbf{v}(h(l))] \\
&= [\mathbf{v}(h'(\Sigma(W')))] \\
&= (\sharp\Sigma(W') + \sharp\Sigma_1^{(\mathbf{II})}(W') + 2k)\cdot[\alpha] \\
&= (\sharp\Sigma(W) + 1 + \sharp\Sigma_1^{(\mathbf{II})}(W) - 1 + 2k)\cdot[\alpha] \\
&= (\sharp\Sigma(W) + \sharp\Sigma_1^{(\mathbf{II})}(W) + 2k)\cdot[\alpha].
\end{aligned}$$

Note that $\sharp\Sigma(W') = \sharp\Sigma(W) + 1$ and $\sharp\Sigma_1^{(\mathbf{II})}(W') = \sharp\Sigma_1^{(\mathbf{II})}(W) - 1$. Thus we have $r(h(\Sigma(W))) \equiv \sharp\Sigma(W) + \sharp\Sigma_1^{(\mathbf{II})}(W) \bmod 2$. This completes the proof of Proposition 4.4 and hence Theorems 4.1 and 4.2. □

We can extend Theorem 4.1 to the case where the target surface N is open as follows. Note that Theorem 4.2 can be extended in the same way.

Theorem 4.5. *Let $f : M \to N$ be a simple stable map of an orientable closed 3-manifold into a surface (possibly non-orientable, open, or with boundary). If the Stein factorization W_f is orientable, then the number of crossings of the immersion $f|_{S(f)}$ has the same parity as $\sharp S_1^{(II)}(f)$.*

Proof. We have only to prove the theorem in the case where N is open. Since M is compact, $f(M)$ is contained in a certain compact surface N' embedded in N. Therefore, we may consider f to be a simple stable map $f : M \to N'$, which reduces this case to Theorem 4.1. □

We have the following consequence of the above results.

Corollary 4.6. *Let $f : M \to N$ be a simple stable map and suppose that the Stein factorization W_f is orientable. If it is special generic (see section* 1 *or* [1]*) or full-definite (see* [8, *section* 5]*), then the number of crossings of $f|_{S(f)}$ is even.*

The proof is obvious, since we have $\sharp S_1^{(\mathbf{II})}(f) = 0$ in the above situations.

Corollary 4.7. *If W_f is orientable and $\sharp S_1^{(II)}(f)$ is odd, then the immersion $f|_{S(f)}$ cannot be an embedding.*

REMARK 4.8. Let $f : M \to N$ be a simple stable map. If N is orientable, then the Stein factorization W_f is orientable, since an orientation of W_f is induced from that of N by the immersion $\bar{f} : W_f \to N$. When N is not orientable, W_f is not necessarily orientable. For example, let $S^2 = \{(x,y,z) \in \mathbf{R}^3 | x^2+y^2+z^2 = 1\}$ be the unit 2-sphere and $h : S^2 \to [-2,2]$ the Morse function which maps (x,y,z) to z. Let $\varphi : S^2 \to S^2$ be the map defined by $\varphi(x,y,z) = (x,-y,-z)$, and define $\gamma : [-2,2] \to [-2,2]$ by $\gamma(q) = -q$. Then the map $h \times \mathrm{id}_I : S^2 \times I \to [-2,2] \times I$ induces a simple stable map $f : S^2 \times_\varphi S^1 \to [-2,2] \times_\gamma S^1$, where $I = [0,1]$, $S^2 \times_\varphi S^1 = S^2 \times I/(p,0) \sim (\varphi(p),1)$, and $[-2,2] \times_\gamma S^1 = [-2,2] \times I/(q,0) \sim (\gamma(q),1)$. Note that the source manifold $S^2 \times_\varphi S^1$ is diffeomorphic to $S^2 \times S^1$ and the target manifold $[-2,2] \times_\gamma S^1$ to the Möbius band. It is easily observed that the Stein factorization W_f is diffeomorphic to the Möbius band, whose boundary coincides with $\Sigma(f)$. Since $\Sigma(f)$ is not null homologous in W_f, we cannot define the generalized rotation number $r(\bar{f}(\Sigma(f)))$ of $\bar{f}(\Sigma(f))$.

The author does not know the answer to the following question.

QUESTION 4.9. Can we extend Theorem 4.1 (or 4.5) to the case where W_f is non-orientable? If not, can we construct a simple stable map $f : M \to N$ such that (W_f is non-orientable and) the number of crossings of the immersion $f|_{S(f)}$ is not congruent modulo 2 to $\sharp S_1^{(\mathrm{II})}(f)$?

5. Immersions of branched surfaces

In this section, we give some examples of immersions of branched surfaces into (unbranched) surfaces. Let $h : W \to N$ be an immersion of a branched surface W into a surface N such that $h|_{\Sigma(W)}$ is an immersion with normal crossings. Let $p : M \to W$ be a fold map (for a definition, see [8]) of a closed orientable 3-manifold into W. Then it is known that $h \circ p : M \to N$ is a simple stable map (see [7,8]). For each example of $h : W \to N$ in this section, we can construct a closed orientable 3-manifold M, a fold map $p : M \to W$, and hence a simple stable map $h \circ p : M \to N$. This fact has been proved by Mata-Lorenzo (for a sketch of the proof, see [6, section 3]). Therefore, in this section we will give examples of $h : W \to N$ only. These examples show that the number of crossings of $f|_{S(f)}$ for a simple stable map f can be even and odd; i.e., the both possibilities are realized.

REMARK 5.1. A fold map $p : M \to W$ is a projection in the sense that M can be considered as the total space of an S^1-bundle with singular fibers, whose base space is W. Note that M is not uniquely determined for a given branched surface W. Nevertheless, there is some relationship between the topologies of M and W (see [6]).

EXAMPLE 5.2. Let W_1 be the orientable branched surface whose associated graph (for the definition, see [8]) is as in Figure 5.1(a). This branched surface is obtained as follows. Let X and Y be the closed orientable surfaces of genera 4 and 5 respectively (i.e., $X \cong F_4, Y \cong F_5$) and A and B compact surfaces of genus 3 with one boundary component embedded in X and Y respectively. Then the branched surface W_1 is obtained from X and Y by identifying A and B (i.e., $W_1 = X \cup Y/A \sim B$). If we define the immersions f and g of X and Y respectively into F_2 as in Figure 5.1(b), then f and g induce the immersion $h : W_1 \to F_2$. Note that $f|_A$ is equivalent to $g|_B$.

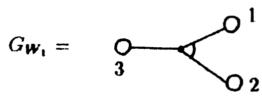


Figure 5.1(a)

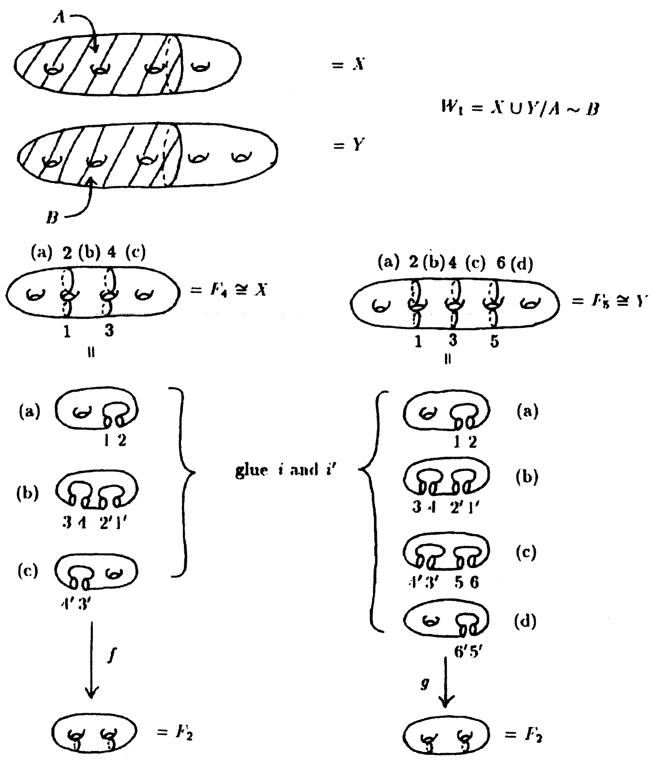


Figure 5.1(b)

EXAMPLE 5.3. Let W_2 be the orientable branched surface whose associated graph is as in Figure 5.2(a). This branched surface is obtained as follows. Let $X = F_{2,2}$ be the oriented compact surface of genus 2 with oriented boundary components l_1 and l_2, and $N(l_i)$ $(i = 1, 2)$ the regular neighborhood of l_i in X. Let $Y = F_{1,1}$ be the oriented compact surface of genus 1 with one oriented boundary component l_3 and $N(l_3)$ the regular neighborhood of l_3 in Y. Let $m_1 : l_1 \to l_2$ and $m_2 : l_1 \to l_3$ be orientation-reversing diffeomorphisms. Then we obtain the branched surface $W_2 = X \cup Y/m_1, m_2$ by using the gluing maps m_1 and m_2 such that $N(l_2)$ and $N(l_3)$ are on the same side of the branching. Note that W_2 contains the oriented closed surface X' $(= X/m_1)$ of genus 3 and that W_2 can be obtained from X' and Y by attaching l_3 to l_1 ($\subset X'$). We define immersions $f : X' \to F_2$ and $g : Y \to F_2$ as in Figure 5.2(b), which induce the immersion $h : W_2 \to F_2$. Note that $f|_{l_1}$ $(= f|_{l_2})$ is equivalent to $g|_{l_3}$.

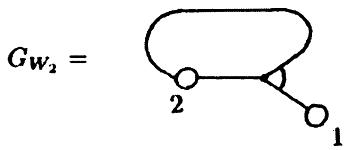


Figure 5.2(a)

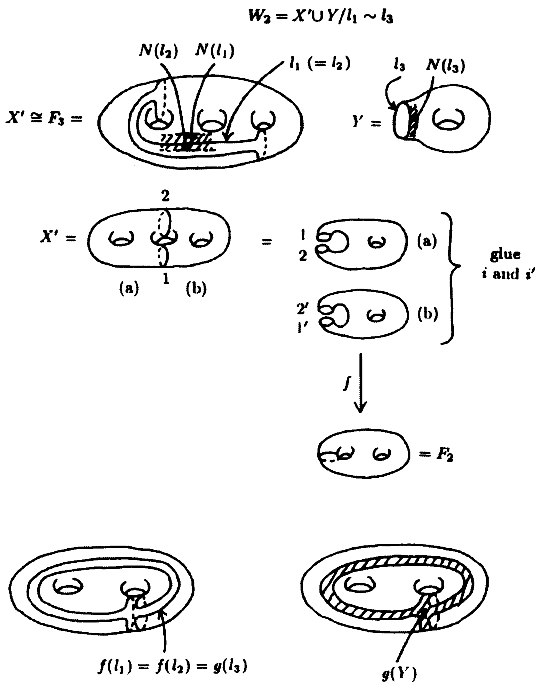


Figure 5.2(b)

EXAMPLE 5.4. Let W_3 be the orientable branched surface whose associated graph is as in Figure 5.3(a). This branched surface is obtained as follows. Let X be the oriented disk with boundary l_1 and Y the oriented annulus with boundary components l_2 and l_3. Let $f : l_2 \to l_1$ be an orientation-reversing double cover. Then the branched surface W_3 is obtained from X and Y by gluing l_1 and l_2 by f (i.e., $W_3 = X \cup Y/f$). We can construct an immersion $h : W_3 \to \mathbf{R}^2$ as in Figure 5.3(b). Note that $h|_{\Sigma(W_3)}$ is an immersion with a normal crossing.

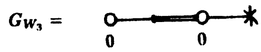


Figure 5.3(a)

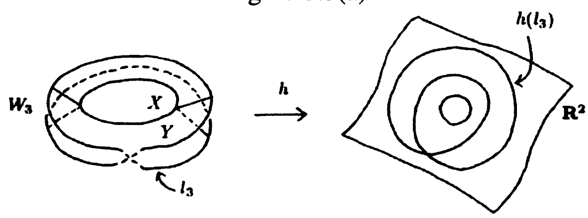


Figure 5.3(b)

EXAMPLE 5.5. Let W_4 be the orientable branched surface whose associated graph is as in Figure 5.4(a). This branched surface is obtained as follows. Let X and Y be oriented annuli with boundary components l_1, l_2, m_1 and m_2. Let $f_i : l_i \to m_i$ $(i = 1, 2)$ be orientation-reversing double covers. Then the branched surface W_4 is obtained from X and Y by using the gluing maps f_1 and f_2 (i.e., $W_4 = X \cup Y / f_1, f_2$). In Figure 5.4(b), $h|_Y : Y \to h(Y)$ is a diffeomorphism and $h|_X : X \to h(X)$ is a double cover of an annulus. The map $c : F_1 \to F_1$ is a double covering map such that $c|_\gamma : \gamma \to c(\gamma) \cong S^1$ is a double cover and that $c|_{h(m_i)}$ is an immersion with a normal crossing $(i = 1, 2)$, where γ is a simple closed curve on F_1 as depicted in Figure 5.4(b). Then the maps h and h' $(= c \circ h)$ are immersions of W_4 into F_1. Note that $h'|_{\Sigma(W_4)}$ is an immersion with normal crossings.

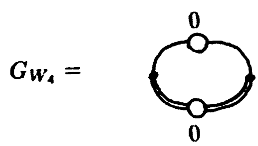


Figure 5.4(a)

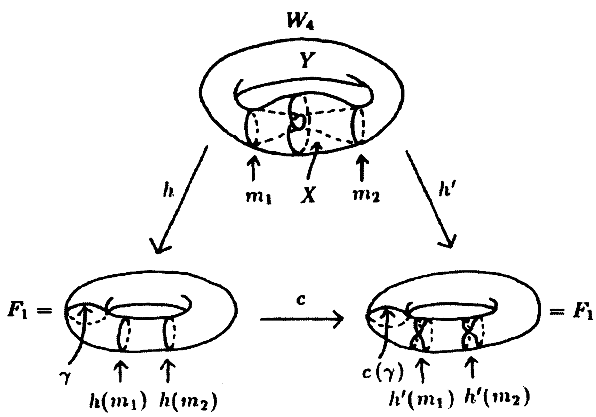


Figure 5.4(b)

REMARK 5.6. In the examples above, $\sharp\Sigma_1^{(\mathrm{II})}(W_1) = \sharp\Sigma_1^{(\mathrm{II})}(W_2) = 0$, $\sharp\Sigma_1^{(\mathrm{II})}(W_3) = 1$, and $\sharp\Sigma_1^{(\mathrm{II})}(W_4) = 2$. Note that Theorem 4.2 can be easily checked for all these examples.

REMARK 5.7. Let W_5 be the orientable branched surface whose associated graph is as in Figure 5.5. This branched surface is obtained as follows. Let X_i $(i = 1, 2)$ be the oriented compact surface of genus g_i with one boundary component l_i. Then the branched surface W_5 is obtained from X_1 and X_2 by using an orientation-reversing double cover $f : l_1 \to l_2$ (that is, $W_5 = X_1 \cup X_2/f$). However, W_5 can never be immersed into any compact or open surfaces. This is shown as follows. If there exists an immersion h of W_5 into a compact surface N, then h induces immersions of X_1 and X_2. Then the generalized rotation number $r(h(l_i))$ of $h(l_i)$ must be odd by Theorems 3.16, 3.17, and 3.18, since $\chi(X_i)$ is odd $(i = 1, 2)$. On the other hand, we have $[\mathbf{v}(h(l_1))] = -2[\mathbf{v}(h(l_2))]$ in $H_1(T_1N)$ and hence $r(h(l_1)) = -2r(h(l_2))$, which is a contradiction. If there exists an immersion h of W_5 into an open surface N, then there exists a compact surface $N' \subset N$ such that $h(W_5) \subset N'$, which reduces this case to the case where N is compact.

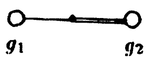
$G_{W_5} =$ 

Figure 5.5

The example constructed in Remark 5.7 is a generalization of Saeki's example [8, Example 3.10]. He has considered the case where X_1 and X_2 are the 2-disks. The argument there is quite similar to the above one.

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References

[1] O.Burlet and G.de Rham: *Sur certaines applications génériques d'une variété close à 3 dimensions dans le plan,* l'Enseign Math. **20** (1974), 275–292

[2] D.Chess: *A Poincaré-Hopf type theorem for the de Rham invariant* Bull. Amer. Math. Soc. **3** (1980), 1031–1035

[3] G.K.Francis: *Assembling compact Riemann surfaces with given boundary curves and branch points on the sphere* Illinois J. Math. **20** (1976), 198–217

[4] A.Gramain: *Une généralisation du degré normal d'une immersion en codimension 1* C. R. Acad. Sc. Paris, Série A **266** (1968), 1223–1225

[5] H.Levine: *Classifying immersions into $\boldsymbol{R}^4$ over stable maps of 3-manifolds into $\boldsymbol{R}^2$* Lect. Notes in Math. 1157, Springer, Berlin (1985)

[6] W.Motta, P.Porto JR, and O.Saeki: *Stable maps of 3-manifolds into the plane and their quotient spaces* Proc. London Math. Soc. (3) **71** (1995), 158–174

[7] O.Saeki: *Simple stable maps of 3-manifolds into surfaces* Topology **35** no.3 (1996), 671–698
[8] O.Saeki:*Simple stable maps of 3-manifolds into surfaces II* J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **40** (1993), 73–124
[9] O.Saeki:*Studying the topology of Morin singularities from a global viewpoint* Math. Proc. Camb. Phil. Soc. **117** (1995), 223–235

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